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SYSTEM OF CONFOCAL HYPERQUADRICS IN SPACE OF n DIMENSIONS

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1. INTRODUCTION

In geometry of three dimensions we have the well-known THEOREM. *Given a system of confocal quadrics*

$$\frac{x_1^2}{a_1^2-\lambda}+\frac{x_2^2}{a_2^2-\lambda}+\frac{x_3^2}{a_3^2-\lambda}=1,$$

then (1) *through any point there pass three quadrics of the system;* (2) *two quadrics of the system touch a given line;* (3) *one quadric of the system touches a given plane.*

The purpose of this paper is to extend this theorem to space of n dimensions.

2. TANGENT LINEAR SPACE

Consider a variety V_m of m dimensions in S_n ; then the locus of the tangents at a point P to the ∞^{m-1} curves on V_m through P is a linear space S_{m-1} of m-l dimensions called the tangent linear space of m-l dimensions, or simply tangent S_{m-1}, to the variety V_m at a point P. *Any linear space S_k of k dimensions through P and in the tangent S_{m-1} to V_m at P is defined to be the tangent S_k to V_m at P.*

Consider, in S_n, a hyperquadric H_n given by the equation

$$\sum_{i=1}^{n}\frac{x_i^2}{a_i^2}=1 \qquad (1)$$

and a linear space S_k, given by the system of equations

$$\sum_{i=1}^{n} b_{ji}x_i=1 \qquad (j=1,2,\cdots,n-k) \qquad (2)$$

Let us find the necessary and sufficient condition that the given S_k should be a tangent S_k of the given hyperquadric H_n.

We can easily show that the tangent hyperplane to H_n at a point $(y_1, \cdots\cdots y_n)$ is given by the equation

$$\sum_{i=1}^{n} \frac{y_i\, x_i}{a_i^2} = 1.$$

Hence the hyperplane

$$\sum_{i=1}^{n} c_i\, x_i = 1$$

is tangent to the hyperquadric H_n if, and only if,

$$\sum_{i=1}^{n} a_i^2\, c_i^2 = 1.$$

The given S_k lies in a tangent hyperplane of the given H_n if, and only if, there exists a set of constants λ_i $(j = 1, 2, \ldots, n\text{-}k)$ not all zero such that the equation

$$(3) \qquad \sum_{i=1}^{n} \left(\sum_{j=1}^{n-k} b_{ji}\, \lambda_j \right) x_i = \sum_{j=1}^{n-k} \lambda_j$$

represents a tangent hyperplane of H_n; and we have, therefore, the necessary and sufficient condition

$$(4) \qquad \sum_{j=1}^{n-k} \sum_{m=1}^{n-k} \left(-1 + \sum_{i=1}^{n} a_i^2\, b_{ji}\, b_{mi} \right) \lambda_j\, \lambda_m = 0,$$

which shows that there are ∞^{n-k-2} tangent hyperplanes to the given H_n through a given S_k. In case the given S_k is a tangent S_k to the given H_n and $n\text{-}k \leq 3$, the order infinity is decreased by a unity and we have

$$(5) \qquad | A_{jm} | = 0,$$

where $|A_{jm}|$ represents a symmetric determinant of order n-k whose elements A_{jm} are given by

(6) $$A_{jm} = -1 + \sum_{i=1}^{n} a_i^2 b_{ji} b_{mi} \ (j, m = 1, 2, \cdots, n-k).$$

If n-k=2, the equation (4) takes the form

$$A_{11} \lambda_1^2 + 2A_{12} \lambda_1 \lambda_2 + A_{22} \lambda_2^2 = 0.$$

There are evidently two tangent hyperplanes through the given S_k; in case the given S_k is a tangent S_k of the given H_n, the tangent hyperplane to H_n through S_k is uniquely determined and we have

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{vmatrix} = 0.$$

If n-k=1, the given S_k is evidently a hyperplane in S_n ; in case the given S_k is a tangent S_k of the given H_n, we have

$$A_{11} = 0.$$

Recapitulating the results we have the

THEOREM. *The S_k given by the system of equations* (2) *is tangent to the hyperquadric* (1) *if, and only if,* $|A_{jm}| = 0$.

3. SYSTEM OF CONFOCAL HYPERQUADRICS

Consider a system of confocal hyperquadrics given by the equation

(7) $$\sum_{i=1}^{n} \frac{x_i^2}{a_i^2 - \lambda} = 1,$$

where λ is a parameter, and an S_k given by the system of equations (2). A hyperquadric of the system (7), corresponding to a particular value λ_0 of λ is tangent to the given S_k if, and only if, λ_0 is a root of the equation

(8) $$|B_{jm}| = 0,$$

where $|B_{jm}|$ represents a symmetric determinant of order n-k whose elements B_{jm} are given by

(9) $$B_{jm} = -1 + \sum_{i=1}^{n} (a_i^2 - \lambda)\, b_{ji}\, b_{mi} \qquad (j, m = 1, 2, \cdots, n-k).$$

Since the equation (8) is of the (n-k)th degree in λ and whose n-k roots are all real, we have the

THEOREM. *Of the* ∞' *hyperquadrics in the system* (7), *n-k of them touch a given* S_k ($k = 0, 1, 2, \ldots, n\text{-}1$). *The n-k values of* λ *corresponding to these n-k hyperquadrics are given by the equation* (8).

ON THE INTENSITY DISTRIBUTION BETWEEN THE COMPTON LINES.

By Y. H. Woo (吳有訓)

National Tsing Hua University, Peiping.

On attempting to examine the intensity of scattered x-rays, a noticeable effect is the change of the intensity of the modified ray relative to that of the unmodified line with the atomic number of the secondary radiator. The photographs of Ross[1] and the ionization curves of Compton and the writer[2] lead to the conclusion that for a given angle of scattering the ratio of the intensity of the modified line to the unmodified decreases as the atomic number of the scatterer is increased. This conclusion has been confirmed by the recent experiments of Allison and Duane[3] and the writer[4]. The purpose of the present paper is to present more precise experimental data than has previously been given regarding this question of the variation in the intensity ratio of the modified to the unmodified line with the atomic number of the scattering element. In view of the possibility of adapting the apparatus employed here to the investigation of the secondary x-radiation in another field[5], the design of the apparatus will be described somewhat in detail.

Apparatus and Method

For the quantitative measurement of the intensity distribution of the scattered x-rays it is clearly desirable to employ

1 P. A. Ross, Phys. Rev. 22, 525 (1923)

2 Compton and Woo, Proc. Nat. Acad. Sci. 10, 271 (1924) Y. H. Woo, Proc. Nat. Acad. Sci. II, 123 (1925)

3 Allison and Duane, Phys. Rev. 26, 300 (1925)

4 Y. H. Woo, Phys. Rev. 27, 119 (1926)

5 Cf. Y. H. Woo, Phys. Rev. 28, 427 (1926)

a spectroscopic method. On account of the low intensity of the scattered x-radiation the apparatus had to be devised in such a manner as to secure the maximum intensity in the beam whose wave-length was measured. The arrangement of the apparatus was essentially the same as that employed by the writer[4] for measuring the intensity ratio of the Compton lines as a function of the scattering angle and is diagrammatically shown in Fig. I.

The x-ray tube was set up with its axis vertical. The primary rays from the target T fell upon the secondary radiator R, which was placed in line with the collimator. The lead screen, L_1, disposed between the tube and the collimator, shielded the slit system from the direct rays from the target. Since the screen, L_1, was mounted on an insulating support, it was possible to place it close to the x-ray tube without danger of puncture. The lead plate, L_2, prevented stray radiation from passing through the collimator. The lead box as indicated in the figure surrounded the x-ray tube and the scatterer and had at the back a door, that might be opened or closed. The rays from the secondary radiator R, after passing through the collimator, struck the calcite crystal of the Bragg spectrometer, which reflected some of them into the ionization chamber. A sensitive Compton electrometer measured the ionization in the usual way. The ionization chamber, well protected from stray radiation by lead, was filled with vapor of ethyl bromide.

The collimator, similar to that first designed by Soller[6], was constructed with a pile of sheets of lead foil separated by strips of lead so as to form twenty five different slits side by side. Each slit in the collimator was 0.4 mm. broad and 2 cm. high and had a length of about 17 cm. The detail of this multiple slit system is shown in Fig. I(B).

The composite beam from the Soller collimator was I cm. wide necessitated therefore a crystal face 10.7 cm. long in order

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6 W. Soller, Phys. Rev. 23, 272 (1924)

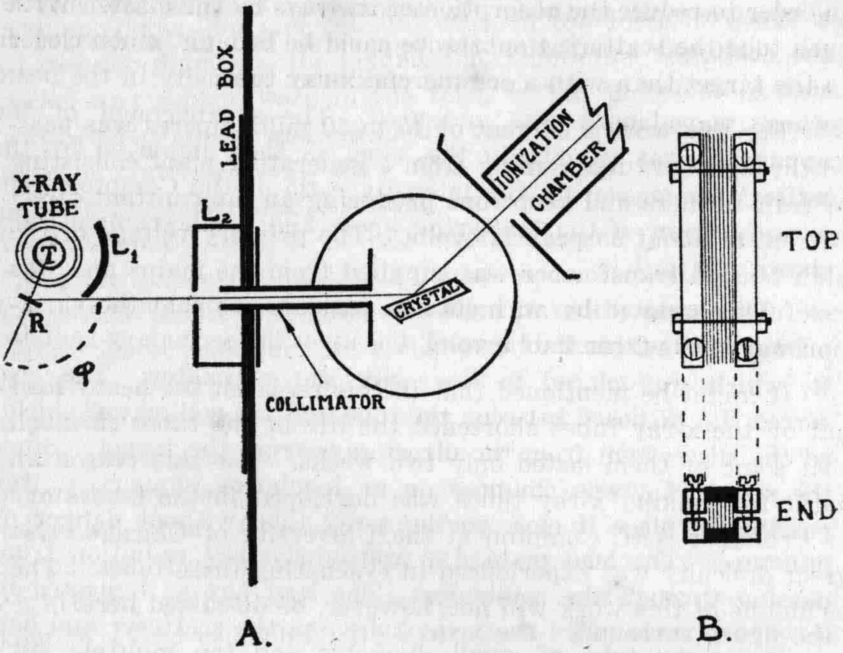

FIG. I

to reflect the whole beam at an angle of incidence of five degrees and twenty minutes. (The critical angle of reflection of the Ag K_α rays is 5° 18′). A carefully polished face on a calcite crystal about II cm. long was employed. This face was not shown to give spurious reflection by analyzing the Ag K_α rays scatterered from a silver radiator.

The x-ray tube was of the type described by Compton[7]. It had a water-cooled copper anode containing a button of silver on which the cathode rays from a Coolidge cathode fell. The target was sealed into a glass tube of approximately uniform diameter. This tube was about 3 cm. in diameter and its walls were made somewhat thinner opposite the target than elsewhere

7 A. H. Compton, Phys. Rev. 22, 409 (1923)

in order to reduce the absorption of the rays by the glass. With such tube the scattering substance could be brought much closer to the target than with a commercial x-ray tube.

For this work a current of 40 to 50 milliamperes was passed through the tube, coming from a generating plant consisting of transformers and kenotrons producing an intermittent direct current at about 65 peak kilovolts. The primary voltage of the high tension transformer was supplied from the mains and was carefully regulated by an induction regulator so that the variation was kept within half a volt.

It might be mentioned that in this research the heavy load put on the x-ray tubes shortened the life of the tubes so much that some of them lasted only two weeks. For this reason an outfit for making x-ray tubes was developed in the laboratory of Professor A.H. Compton at the University of Chicago. No great difficulty was experienced in evacuating these tubes. The technique of this work will not, however, be discussed here.

The x-ray tube of small diameter and the multiple slit system very greatly increased the intensity of the secondary radiation. This has two distinct advantages. Firstly, the spectrometer possesses a much better resolving power and secondly, small secondary radiators could be employed. While the importance of the former is too well known to require comment, the latter enabled to select for examination radiation from a portion of the scatterer in which the range of the scattering angle was small and is therefore of considerable importance in determining the sharpness of the modified line in the Compton effect.

With these devices, the apparatus is satisfactory for researches in the field of the secondary x-radiation by the ionization method.

Experimental Results

In the present experiments fifteen chemical elements of atomic numbers ranging from 3 to 29 were employed as the secondary radiators. While the samples of lithium, carbon, sodium, potassium and calcium were in the form of cylinders of one cm. diameter, the magnesium, aluminium, sulphur, iron, nickel and copper were in the form of flat plates of one cm. width. The silicon and chromium were in the form of bloocks of comparatively large size. The boron and beryllium used were in the amorphous form and were held in very thin walled, waxed paper cylinders of one cm. diameter. The scattering from the paper cylinder was found to be so feeble, that no measurable effect was observed when it was used as the scatterer. These radiators were clamped in turn approximately 4 cm. from the centre of the focal spot of the x-ray tube, in such a position that they would scatter into the multiple slit system at an angle equal to 120° with the primary beam. The amorphous boron contains 4 or 5 per cent of oxygen and also had traces of silica. The beryllium was found to have traces of zirconium. During the course of the experiments the samples of lithium, sodium and potassium were freshly prepared and kept in a lead cell filled with hydrogen[4]. The other scattering elements presumably had impurities only in relatively small amounts.

As is well known, the investigation of the scattering by heavy elements is made difficult by the extremely low intensity of the scattered radiation from these elements. In this work the intensity of the scattered radiation from chromium, iron, nickel and copper was so small, that reliable measurements were obtained only with great difficulty. Experiments performed to examine the scattering from zinc were tried several times without success, though the x-ray tube was operated at about 65 kilovolts and 60 milliamperes.

Two experiments were performed with each of the fifteen samples as the secondary radiator. One complete set of the experiments with various scatterers is shown in Fig. 2, in which the curves represent the ionization current in arbitrary units as a function of the glancing angle of incidence of the rays on

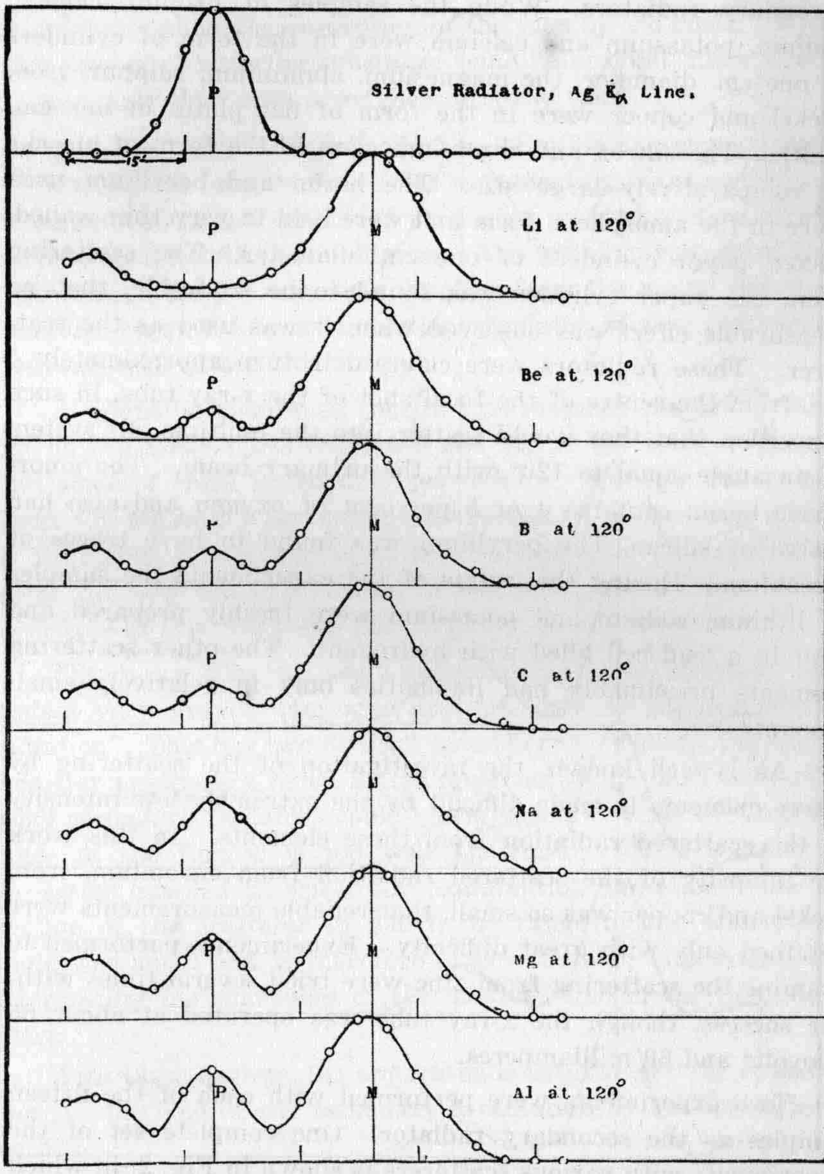

Fig. II (A). Intensity distribution of x-rays scattered from various radiators at 120°. P marks the position of the primary Ag Kα line, M the position of the modified peak calculated from Compton's theory.

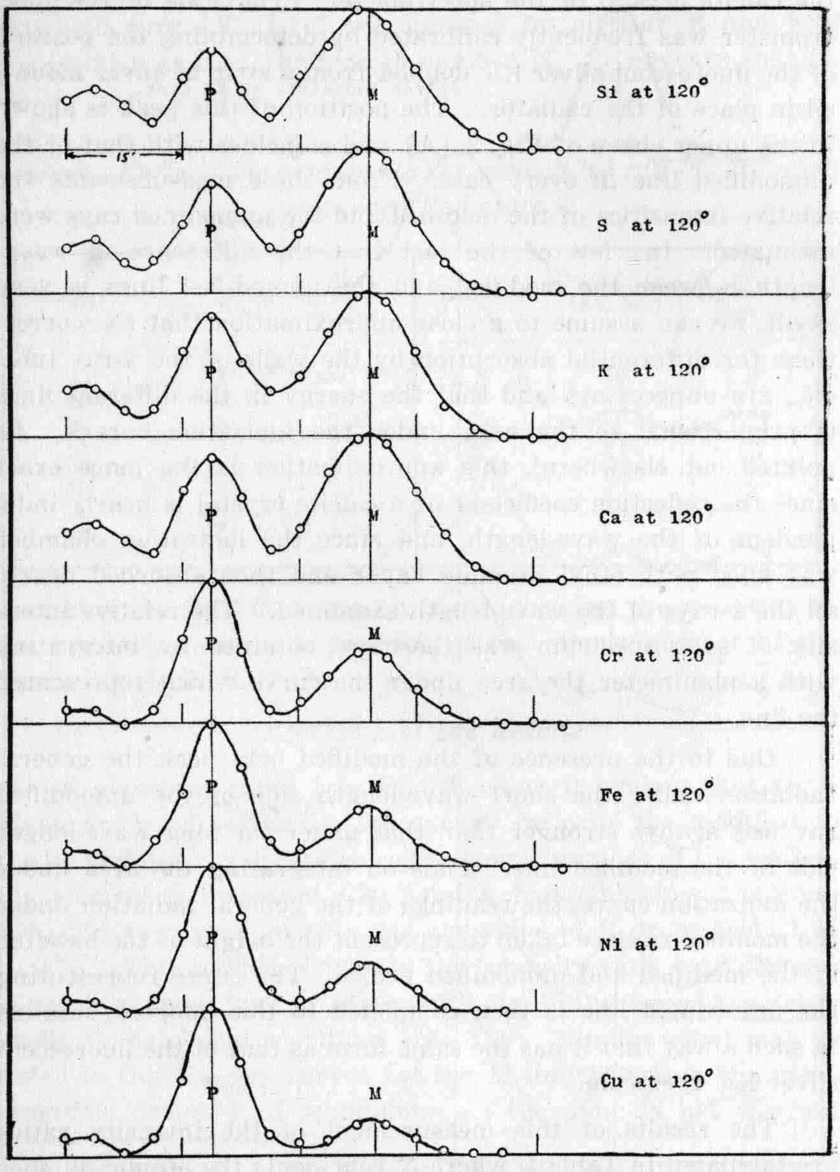

Fig. II (B). Intensity distribution of x-rays scattered from various radiators at 120°. P marks the position of the primary Ag Kα line, M the position of the modified peak calculated from Compton's theory.

the calcite crystal of the spectrometer. The scale of the spectrometer was frequently calibrated by determining the position of the fluorescent silver $K\alpha$ doublet from a strip of silver mounted in place of the radiator. The position of this peak is shown in the upper curve of Fig. 2 (A) and coincides with that of the unmodified line in every case. From these measurements the relative intensities of the modified and the unmodified rays were estimated. In view of the fact that the difference in wave-length between the modified and the unmodified lines is very small, we can assume to a close approximation that the corrections for differential absorption by the walls of the x-ray tube, etc., are unnecessary and that the energy in the different lines is proportional to the area under the ionization curves. As pointed out elsewhere[4], this approximation is the more exact since the reflection coefficient of a calcite crystal is nearly independent of the wave-length, and since the ionization chamber was filled with ethyl bromide vapor and thus absorbed nearly all the x-rays of the wave-length examined. The relative intensity of each spectrum was therefore obtained by integrating with a planimeter the area under the curve which represented the line.

Due to the presence of the modified beta peak the general radiation under the short wave-length side of the unmodified ray was always stronger than that under the long wave-length side of the modified line. Thus on integrating the area under the ionization curve, the readings of the general radiation under the modified ray are taken to represent the height of the baseline of the modified and unmodified peaks. The curve representing the unmodified line is then completed to this adopted baseline in such a way that it has the same form as that of the fluorescent silver $K\alpha$ spectrum.

The results of this measurement of the intensity ratios are tabulated in Table I, where N represents the atomic number of the scattering element and R the ratio of the intensity expressed in arbitrary units of the modified line to that of the unmodified. These data are shown in graphical form in Fig. 3, in which curves I and 2 are obtained by plotting R and I/R respectively against the atomic number N. It appears that in

Intensity Ratio of the modified to the unmodified ray for different secondary Radiators

Radiator	N (At. No.)	R (Intensity Ratio)	Radiator	N (At. No.)	R (Intensity Ratio)
Li	3	∞ (?)	S	16	1.91
Be	4	8.72	K	19	1.72
B	5	7.02	Ca	20	1.71
C	6	5.48	Cr	24	.75
Na	11	3.04	Fe	26	.51
Mg	12	2.78	Ni	28	.40
Al	13	2.61	Cu	29	.21
Si	14	2.33			

each case the data fit a smooth curve, showing the variation of the intensity ratio as a function of the atomic number of the scattering element.

Discussion of the Results

A glance at curve I in Fig. 3 makes it evident that for a given angle of scattering the energy ratio of the modified to the unmodified ray decreases as the atomic number of the scattering element is increased. In passing from the elment calcium (N=20) to chromium (N=24) the value of R drops from 1.71 to 0.75. This sudden change in the intensity ratio may perhaps connect in some way or other with the filling up of the inner shells at the elment scandium (N=21)[8]. Similar effect may be noted in the Moseley curves for the M limits[9] and in the paramagnetic property of scandium. As the data do not warrant

8 Cf. McLennan, McLay and Grayson, Proc. Roy. Soc. 112, 77 (1926). I am indebted to Dr. C. S. Yeh for calling my attension to this point.

9 Bohr and Coster, Zeits. f. Phys. 12, 342 (1923)

an attempt to ascertain an empirical relation about this point, we just mention it here.

It is of interest that the portion of curve 2 in Fig. 3 corresponding to elements of atomic numbers ranging from 4 (beryllium) to 20 (calcium) is nearly a straight line. This indicates that for the scattering of x-rays by light elements the ratio of the intensity of the unmodified ray to that of the modified line increases linearly with the atomic number of the radiator.

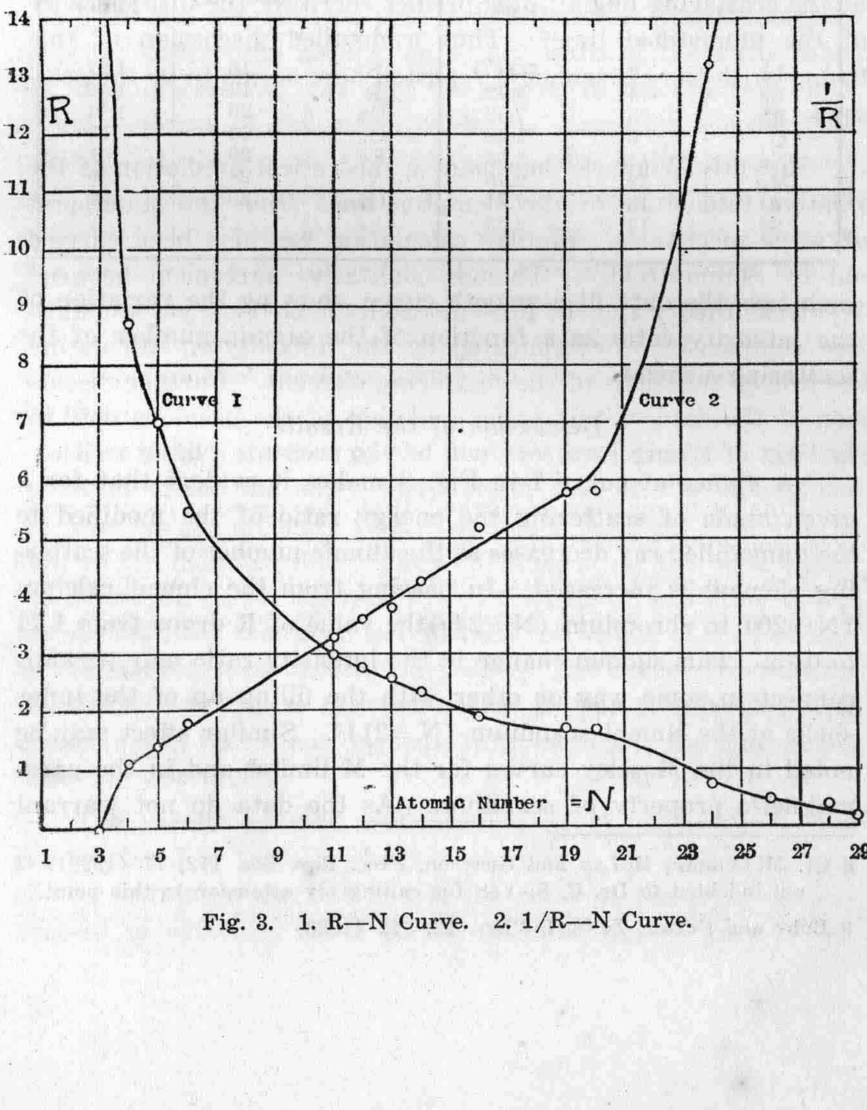

Fig. 3. 1. R—N Curve. 2. 1/R—N Curve.

Jauncey[10] has developed a theory of the intensity of the modified ray in the Compton effect. The theory requires for a given angle of scattering a decrease of the intensity of the modified line relative to that of the unmodified as the atomic number of the scatterer increases. This agrees qualitatively with the present results. Recent experiments reported by the writer have shown, however, that in its details the theory as developed by Jauncey does not accurately describe the relative prominence of the modified and unmodified lines as a function of the scattering angle[11], nor predict correctly the disapperance of the unmodified line[12]. Thus a detailed discussion of this theory with the experiments reported here seems to be unnecessary.

Recently Wentzel[13] has made a theoretical prediction of the relative intensities of the Compton lines from the standpoint of wave mechanics. Similar calculation has also been carried out by Sommerfeld.[14]. Though qualitative agreement between Wentzel's theory and the present results is obvious, yet a quantitive comparison is complicated by the difficulty in calculating the proper functions of the scattering element. Further discussion of the data of this paper and that of the previous one[9] in the light of Wentzel's theory will be reserved for a later article.

Finally it may be remarked that the present results and those reported before[9] also show that for a definite scatterer the intensity of the modified line relative to that of the unmodified ray increases as the wave-length of the primary x-rays de-

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10 G. E. M. Jauncey, Phys. Rev. 25 314 and 723 (1925)

11 Y. H. Woo, Phys. Rev. 27 119 (1926). Also cf. G. E. M. Jauncey, Phys. Rev. 27, 678 (1926)

12 Y. H. Woo, Phys. Rev. 28, 426 (1926). Also cf. Jauncey and Boyd, Phys. Rev. 28, 620 (1926)

13 G. Wentzel, Zeit. f. Phys. 43, I and 779 (1927)

14. A. Sommerfeld, Wellenmechanischer Ergaenzungsband, p. 257 (1929)

creases. This is in qualitative agreement with Wentzel's theory mentioned above.

This research was done in Ryerson physical Laboratory at the University of Chicago and a preliminary report has appeared elsewhere. The writer wishes to express his thanks to Prof. A. H. Compton for his kind assistance, suggestions and advice throughout the progress of this work.

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15 Y. H. Woo, Phys. Rev. 28, 426 (1926)