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CONTENTS

ON THE CLASSES OF SEMI-HOMEOMORPHIC SPACES AND SEMI-TOPOLOGICAL PROPERTIES

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I. INTRODUCTION

The concepts of semi-homeomorphisms and semi-topological properties were introduced by S. G. Crossley in 1972. Since then, a series of papers concerning them have been published[1-3]. In this note we shall discuss the structure of the class $[\mathscr{U}]$ consisting of all topologies on a set X which has the same semi-open sets as $(X, \mathscr{U})$. At first, we shall give two new constructural forms of the finest topology in $[\mathscr{U}]$. Then we shall turn to deal with the conditions for the existence of the coarsest topology in $[\mathscr{U}]$. Furthermore, some necessary and sufficient conditions for two topological spaces being semi-homeomorphic will be obtained and some new semi-topological properties will be given. In addition, the results in Refs. [1-3] can be simplified by using the results obtained.

Now, we introduce the main definitions. For the other unimportant definitions, see Refs. [4—7]. We do not assume that regular and normal spaces are T_1-spaces.

A set S is said semi-open if there exists an open set U such that $U\subset S\subset U^{-}$[1]. A set P is said regular open if $P^{-0}=P$[5]. Let f be a one-to-one mapping of a space X onto a space Y and, for every $S\subset X$, S is semi-open in X iff $f(S)$ is semi-open in Y, then f is said to be a semi-homeomorphic mapping from X onto Y, and X and Y are said to be semi-homeomorphic[1]. A property which is preserved by semi-homeomorphic mappings is called semi-topological property[1]. $(X, \mathscr{U})$ and $(Y, \mathscr{W})$ are semi-homeomorphic iff there is a topology $\mathscr{V}$ on X such that $(X, \mathscr{V})$ and $(X, \mathscr{U})$ have the same collections of semi-open sets and the spaces $(X, \mathscr{V})$ and $(Y, \mathscr{W})$ are homeomorphic. Hence, we always think that every pair of semi-homeomorphic spaces are defined on the same set.

In the following, $(X, \mathscr{U})$ will always be a topological space. let $A\subset X$ and $\mathscr{V}$ be a topology on X. Denote by $A^{-}_{\mathscr{V}}$ and $A^{0}_{\mathscr{V}}$ the closure and the interior of the set A in the space $(X, \mathscr{V})$ respectively. We shall abbreviate $(A^{-}_{\mathscr{V}})^{0}_{\mathscr{V}}$ and $(A^{0}_{\mathscr{V}})^{-}_{\mathscr{V}}$ to $A^{-0}_{\mathscr{V}}$ and $A^{0-}_{\mathscr{V}}$ respectively. Assume that $\mathscr{S}_{\mathscr{V}}$, $\mathscr{N}_{\mathscr{V}}$, $\mathscr{D}_{\mathscr{V}}$ and $\mathscr{R}_{\mathscr{V}}$ are collections of all semi-open sets, nowhere dense sets, dense sets and regular open sets of $(X, \mathscr{V})$ respectively. If $\mathscr{V}=\mathscr{U}$, the subscript $\mathscr{V}$ will be omitted. Let $[\mathscr{U}]=\{\mathscr{V}\mid \mathscr{S}_{\mathscr{V}}=\mathscr{S}\}$. For convenience, we shall use the same symbol $[\mathscr{U}]$ to express the family $\{(X, \mathscr{V})\mid \mathscr{S}_{\mathscr{V}}=\mathscr{S}\}$. The meaning of $[\mathscr{U}]$ is clear from the context.

Proposition. *Property P is a semi-topological property iff for every* $(X, \mathscr{U})$, $(X, \mathscr{U})$ *having property P implies that every space of* $[\mathscr{U}]$ *has property P.*

II. The Construction of the Finest Topology

Theorem 1. $F(\mathscr{U}) = \{V \in \mathscr{S} \mid \forall S \in \mathscr{S},\ V \cap S \in \mathscr{S}\}$

is the finest topology in $[\mathscr{U}]$.

Lemma 1. *If* $S \in \mathscr{S}$, *then* $S^{-0} \setminus S \in \mathscr{N}$.

Lemma 2. *If* $V \in F(\mathscr{U})$, *then* $V \subset V^{-0}$.

The Proof of Theorem 1. It is easy to see that $F(\mathscr{U})$ is a topology on X. And every topology of $[\mathscr{U}]$ is coarser than the topology $F(\mathscr{U})$ because the intersection of an open set and a semi-open set is semi-open[2]. Hence, we have only to show that $\mathscr{S}_{F(\mathscr{U})} = \mathscr{S}$. For every $S \in \mathscr{S}_{F(\mathscr{U})}$, there exists $V \in F(\mathscr{U}) \subset \mathscr{S}$ such that $V \subset S \subset V^-_{F(\mathscr{U})}$ and hence $V^0 \subset S \subset V^{0-}$, i.e. $S \in \mathscr{S}$. Thus $\mathscr{S}_{F(\mathscr{U})} \subset \mathscr{S}$. To prove that $\mathscr{S} \subset \mathscr{S}_{F(\mathscr{U})}$, it is sufficient that $U^- \subset U^-_{F(\mathscr{U})}$ holds for every $U \in \mathscr{U}$. For every $x \in U^-$ and every $W \in F(\mathscr{U})$ such that $W \ni x$, by virtue of Lemma 2 we have $W^{-0} \supset W \ni x$, hence $W^{-0} \cap U$ is a nonempty open set in $(X, \mathscr{U})$, and by Lemma 1, we have $W \cap U = (W^{-0} \cap U) \setminus (W^{-0} \setminus W) \neq \emptyset$. Thus $x \in U^-_{F(\mathscr{U})}$.

Corollary 1. *Let* $\mathscr{T}$ *be a family of some subsets of a set X which is closed with respect to arbitrary union and* ϕ, $X \in \mathscr{T}$, *then* $\mathscr{T}$ *is the set of all semi-open sets of some topology on X iff* $\mathscr{T}$ *satisfies the following conditions:*

$\forall S \in \mathscr{T},\ \exists V \in \mathscr{T},\ \forall A,\ B \in \mathscr{T}\ (A \cap V \in \mathscr{T})$ *and if* $B \cap S \in \mathscr{T}$, $B \cap S \neq \emptyset$ *then* $B \cap V \neq \emptyset$.

Theorem 2. $F(\mathscr{U}) = \{V \in \mathscr{S} \mid V \subset V^{-0}\}$ *is the finest topology in* $[\mathscr{U}]$.

III. Conditions for the Existence of the Coarsest Topology

Lemma 3. *If* $\mathscr{V} \in [\mathscr{U}]$, *then* $\forall S \in \mathscr{S}$,

$$S^-_{\mathscr{V}} = S^-, \quad S^{-0}_{\mathscr{V}} = S^{-0}.$$

Corollary 2. $\forall \mathscr{V} \in [\mathscr{U}]$, $\mathscr{R}_{\mathscr{V}} = \mathscr{R}$.

Corollary 3. *The semi-regularizations of all the spaces of* $[\mathscr{U}]$ *are one and the same.*

Corollary 4. *The collections of the clopen subsets of all the spaces of* $[\mathscr{U}]$ *are one and the same.*

Theorem 3. *If there is a semi-regular space in* $[\mathscr{U}]$, *then the semi-regular space is the coarsest topology in* $[\mathscr{U}]$.

Theorem 4. *There exists at most one topological space in* $[\mathscr{U}]$ *which is semi-regular.*

It is shown by example that none of regularity, normality, T_3, $T_{3\frac{1}{2}}$, and T_4, compactness etc. is a semi-topological property. Theorem 4 explains the fact theoretically.

Lemma 4. *If $(X, \mathscr{U})$ is T_1 and $X \ni x_0$, then the space $(X, \mathscr{V})$ with topology generated by $\mathscr{B} = \{U \in \mathscr{U} \mid U \not\ni x_0$ or $U = U^{-0}\}$ as a base belongs to $[\mathscr{U}]$.*

This lemma implies

Theorem 5. *If $(X, \mathscr{U})$ is T_1, then $\{S^{-0} \mid S \in \mathscr{S}\} = \{U^{-0} \mid U \in \mathscr{U}\}$ is a base for the space $(X, \cap [\mathscr{U}])$, where $\cap [\mathscr{U}] = \bigcap_{\mathscr{V} \in [\mathscr{U}]} \mathscr{V}$.*

Corollary 5. *If $(X, \mathscr{U})$ is T_1 and $[\mathscr{U}]$ contains the coarsest topology $\mathscr{V}$, then $\{S^{-0} \mid S \in \mathscr{S}\} = \{U^{-0} \mid U \in \mathscr{U}\}$ is a base for $\mathscr{V}$ and $(X, \mathscr{V})$ is a semi-regular space.*

Remark 1. In Theorem 5 the assumption that X is a T_1-space cannot be replaced by the assumption that X is a T_0-space.

Remark 2. The fact that $(X, \mathscr{U})$ is a T_1-space, even if $(X, \mathscr{U})$ is a Urysohn space, does not imply the existence of the coarsest topology.

Example 1. Let X be the set of all real numbers and Q the set of all rational numbers. For every $x \in X$, define $\mathscr{B}(x) = \left\{\{x\} \cup \left[\left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap Q\right] \,\middle|\, n = 1, 2, \cdots\right\}$ as a local base at point x. The topological space generated by the neighbourhood system $\{\mathscr{B}(x)\}_{x \in X}$ is a Urysohn space but the coarsest topology in $[\mathscr{U}]$ does not exist.

Theorem 6. *A sufficient condition for the existence of the coarsest topology in $[\mathscr{U}]$ is that every nonempty open set of $(X, \mathscr{U})$ includes a nonempty regular open set. If $(X, \mathscr{U})$ is T_1, then this condition is also necessary.*

Theorem 6 will be proved in the next section.

IV. Some Semi-Topological Properties

Lemma 5. $\mathscr{N} = \{N \subset X \mid \forall D \in \mathscr{D},\ N' \cap D \in \mathscr{D}\}$.

Theorem 7. *Let $\mathscr{U}, \mathscr{V}$ be two topologies on X, then the following statements are equivalent:*

(i) $\mathscr{S} = \mathscr{S}_{\mathscr{V}}$;

(ii) $\mathscr{R} = \mathscr{R}_{\mathscr{V}}$ *and* $\mathscr{D} = \mathscr{D}_{\mathscr{V}}$;

(iii) $\mathscr{R} = \mathscr{R}_{\mathscr{V}}$ *and* $\mathscr{N} = \mathscr{N}_{\mathscr{V}}$.

The following examples show that none of the facts $\mathscr{R} = \mathscr{R}_{\mathscr{V}}$, $\mathscr{D} = \mathscr{D}_{\mathscr{V}}$ or $\mathscr{N} = \mathscr{N}_{\mathscr{V}}$ implies $\mathscr{S} = \mathscr{S}_{\mathscr{V}}$.

Example 2. Let X be an infinite set and $\mathscr{U}$ the finite-complement topology on X and $\mathscr{V}$ the anti-discrete topology on X, then $\mathscr{R} = \mathscr{R}_{\mathscr{V}} = \{\phi, X\}$, but $\mathscr{S} \neq \mathscr{S}_{\mathscr{V}}$.

Example 3. Let X be the set of all real numbers and $\mathscr{U}$ be the natural topology on X. Let $(X, \mathscr{V})$ be the Sorgenfrey line[7]. Then $\mathscr{D} = \mathscr{D}_{\mathscr{V}}$ and $\mathscr{N} = \mathscr{N}_{\mathscr{V}}$, but $\mathscr{S} \neq \mathscr{S}_{\mathscr{V}}$.

The Proof of Theorem 6. Sufficiency. By Ref. [5] and the assumption, we have directly shown that the space $(X, \mathscr{V})$ with topology generated by $\{S^{-0} | S \in \varphi\}$ as a base satisfies that $\mathscr{R} = \mathscr{R}_{\mathscr{V}}$ and $\mathscr{D} = \mathscr{D}_{\mathscr{V}}$.

Necessity. Let $(X, \mathscr{U})$ be a T_1-space. By Theorem 7 and Corollary 5, we know that if $\mathscr{V}$ is the coarsest topology in $[\mathscr{U}]$, then $\{S^{-0} | S \in \varphi\} = \{U^{-0} | U \in \mathscr{U}\}$ is a base for the space $(X, \mathscr{V})$ and $\mathscr{D} = \mathscr{D}_{\mathscr{V}}$. Hence, every nonempty open set of $(X, \mathscr{U})$ includes a nonempty regular open set.

Theorem 8. *T_1^*, almost regular, T_2, extremal disconnective and S-closed are all semi-topological properties.*

Theorem 9. *H-closed is a semi-topological property.*

Theorem 10. *The property of a space having cellularity equal to m is a semi-topological property.*

The author wishes to express his gratitude to the reviewer for the giving of Corollary 1.

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ESTIMATE OF COMPLETE TRIGONOMETRIC SUMS

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Let q be an integer >1 and $f(x)=a_kx^k+\cdots+a_1x+a_0$ be a polynomial of degree k with integral coefficients such that $(a_1, \cdots, a_k, q)=1$. By a complete trigonometric sum we mean a sum of the form

$$S(q, f(x))=\sum_{x=1}^{q} e^{2\pi i f(x)/q}.$$

Since for $k=1$, $S(q, f(x))=0$, and the case $k=2$ can be settled by the theory of Gaussian sums, we only consider $k\geqslant 3$.

In 1940, Prof. Hua[1] first proved that

$$S(q, f(x))=O(q^{1-\frac{1}{k}+\varepsilon}), \tag{1}$$

where the constant implied by "O" depends only on k and ε. The main order $1-\frac{1}{k}$ is the best possible. Afterwards, some mathematicians are interested in the improvements of the constant implied by "O". In 1977, Prof. Chen[2] and С. Б. Стечкин[3] proved respectively that

$$|S(q, f(x))|\leqslant e^{6.1k}q^{1-\frac{1}{k}}, \quad (k\geqslant 3),$$

and

$$|S(q, f(x))|\leqslant B(k)q^{1-\frac{1}{k}},$$

where

$$B(k)\leqslant \exp\left\{k+O\left(\frac{k}{\log k}\right)\right\}, \quad k\to\infty.$$

Recently, Lu Minggao[4] has proved that

$$|S(q, f(x))|\leqslant e^{3k}q^{1-\frac{1}{k}}, \quad (k\geqslant 3).$$

Our object in this note is to prove the following

Theorem. *Let $f(x)=a_kx^k+\cdots+a_1x+a_0$ be a polynomial with integral cfficients such that $(a_1, \cdots, a_k, q)=1$. Then for $k\geqslant 3$ we have*

$$|S(q,\ f(x))| \leqslant e^{2k} q^{1-\frac{1}{k}}.$$

Lemma 1[2]. *Let k be an integer $\geqslant 3$ and $f(x) = a_k x^k + \cdots + a_1 x + a_0$ be a polynomial with integral coefficients such that $(a_1, \cdots, a_k, p) = 1$, where p is a prime $> k$. Then for $l \geqslant 1$, we have*

$$|S(p^l,\ f(x))| p^{-l\left(1-\frac{1}{k}\right)} \leqslant \begin{cases} 1, & p > (k-1)^{\frac{2k}{k-2}}; \\ (k-1) p^{-\frac{1}{2}+\frac{1}{k}}, & (k-1)^2 < p \leqslant (k-1)^{\frac{2k}{k-2}}; \\ p^{\frac{1}{k}}, & (k-1)^{\frac{k}{k-2}} < p \leqslant (k-1)^2; \\ (k-1) p^{\frac{3}{k}-1}, & k < p \leqslant (k-1)^{\frac{k}{k-2}}. \end{cases}$$

Lemma 2. *Let k be an integer $\geqslant 3$ and p be a prime $\leqslant k$. Again let $f(x) = a_k x^k + \cdots + a_1 x + a_0$ be a polynomial with integral coefficients such that $(a_1, \cdots, a_k, p) = 1$. Then for $l \geqslant 1$, we have*

$$|S(p^l,\ f(x))| \leqslant (k-1) k^{\frac{2}{k}} p^{\frac{2}{k}-1} p^{l\left(1-\frac{1}{k}\right)}.$$

Proof. We define t by $p^t \| (ka_k, \cdots, 2a_2, a_1)$. Let $\mu_1, \cdots, \mu_r$ be the different zeros modulo p of the congruence $p^{-t} f'(x) \equiv 0 \pmod p$, $(0 \leqslant x < p)$, and let $m_1, \cdots, m_r$ be their multiplicities. Putting $m_1 + \cdots + m_r = m$, we get obviously $m \leqslant k-1$.

(i) $l \leqslant 2t$. It is obvious that the lemma is true.

(ii) $l = 2t+1$. If $t = 0$, then the lemma is true. If $t \geqslant 1$, we have the substitution that $x = y + p^{l-t-1} z$, where y and z run independently through the values

$$y = 1, \cdots, p^{l-t-1}, \quad z = 0, \cdots, p^{t+1} - 1.$$

Thus we have

$$S(p^l,\ f(x)) = \sum_{y=1}^{p^{l-t-1}} e_{p^l}(f(y)) \sum_{z=0}^{p^{t+1}-1} e_p\left(z \frac{f'(y)}{p^t} + \frac{1}{2} z^2 f''(y)\right).$$

If either p is an odd prime and $t \geqslant 1$, or $p = 2$ and $t \geqslant 2$, then $2p | f''(y)$, $(1 \leqslant y \leqslant p^{l-t-1})$. Hence we have

$$\begin{aligned} |S(p^l,\ f(x))| &\leqslant \sum_{j=1}^{r} \Bigg| \sum_{\substack{y=1 \\ y \equiv \mu_j \pmod p}}^{p^l} e_{p^l}(f(y)) \Bigg| \leqslant r p^{l-1} \\ &\leqslant (k-1) k^{\frac{2}{k}} p^{\frac{1}{k}-1} p^{l\left(1-\frac{1}{k}\right)}. \end{aligned}$$

If $p = 2$ and $t = 1$, then $l = 3$. Hence we have

$$|S(p^l,\ f(x))| \leqslant p^l = 2^{\frac{3}{k}} p^{l\left(1-\frac{1}{k}\right)} \leqslant (k-1) k^{\frac{2}{k}} p^{\frac{1}{k}-1} p^{l\left(1-\frac{1}{k}\right)}.$$

(iii) $l \geqslant 2(t+1)$. We make a substitution as (ii), then we have

$$S(p^l,\ f(x)) = \sum_{j=1}^{r} \sum_{\substack{y=1 \\ y \equiv \mu_j \pmod p}}^{p^l} e_{p^l}(f(y)) = \sum_{j=1}^{r} S_{\mu_j, p^l}. \tag{2}$$

If $p^{\sigma_j}\|(f(py+\mu_j)-f(\mu_j))$, then we put $g_{\mu_j}(y)=p^{-\sigma_j}(f(py+\mu_j)-f(\mu_j))$.

We now apply the method of induction on l to proving that if $l\geqslant 2(t+1)$, then

$$|S(p^l,\ f(x))|\leqslant mk^{\frac{2}{k}}p^{\frac{2}{k}-1}\ p^{l\left(1-\frac{1}{k}\right)}. \tag{3}$$

For any $f(x)$ satisfying the conditions of the lemma, when $l=2(t+1)$, it follows from (2) that (3) holds.

Let t_j satisfy $p^{t_j}\|g'_{\mu_j}(y)$. For $l>2(t+1)$, let

$$\mathscr{A}_1=\{j:\ l\geqslant\sigma_j+2t_j+2\},$$

and

$$\mathscr{A}_2=\{j:\ l=\sigma_j+2t_j+1\},$$

$$\mathscr{A}_3=\{j:\ l\leqslant\sigma_j+2t_j\}.$$

Putting $M_i=\sum\limits_{j\in\mathscr{A}_i}m_j(i=1,\ 2,\ 3)$, we have $\sum\limits_{i=1}^{3}M_i=m$. It is easily seen that

$$|S_{\mu_j,p^l}|=p^{\sigma_j-1}|S(p^{l-\sigma_j},\ g_{\mu_j}(y))|. \tag{4}$$

By the inductive hypothesis and (4), we have

$$\sum_{j\in\mathscr{A}_1}|S_{\mu_j,p^l}|=\sum_{j\in\mathscr{A}_1}p^{\sigma_j-1}|S(p^{l-\sigma_j},\ g_{\mu_j}(y))|$$
$$\leqslant M_1k^{\frac{2}{k}}p^{\frac{2}{k}-1}\ p^{l\left(1-\frac{1}{k}\right)}. \tag{5}$$

For $S(p^{l-\sigma_j},\ g_{\mu_j}(y))$, we apply the method similar to (ii), then we have

$$\sum_{j\in\mathscr{A}_2}|S_{\mu_j,p^l}|=\sum_{j\in\mathscr{A}_2}p^{\sigma_j-1}|S(p^{l-\sigma_j},\ g_{\mu_j}(y))|$$
$$\leqslant M_2\ k^{\frac{2}{k}}p^{\frac{2}{k}-1}\ p^{l\left(1-\frac{1}{k}\right)}. \tag{6}$$

Finally, it is obvious that

$$\sum_{j\in\mathscr{A}_3}|S_{\mu_j,p^l}|=\sum_{j\in\mathscr{A}_3}\left|\sum_{y=1}^{p^{l-1}}e_{p^l}(f(\mu_j+py))\right|$$
$$\leqslant M_3k^{\frac{2}{k}}p^{\frac{2}{k}-1}\ p^{l\left(1-\frac{1}{k}\right)}. \tag{7}$$

It follows from (5)—(7) that (3) holds, hence the lemma is true.

In view of Lemmas 1 and 2, and by computation, we complete the proof of the theorem.

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COMPARISON OF LINEAR MODELS ON AN ESTIMABLE SUBSPACE

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I. INTRODUCTION

A linear model denoted by $l = L(X\beta, \sigma^2 I_n)$ is a structure

$$y = X\beta + e, \quad E(e) = 0, \quad \mathrm{Cov}(e) = \sigma^2 I_n, \tag{1.1}$$

where y is an $n \times 1$ vector of observations, X is an $n \times p$ design matrix, β is a $p \times 1$ vector of parameters, e is an $n \times 1$ vector of random errors, and σ^2 is a known error variance.

For two linear models $l_i = L(X_i\beta,\ \sigma_i^2 I_{n_i})$ $i = 1,\ 2$, with common parameter β, their comparison might be made in terms of the variances of BLUE (the Best Linear Unbiased Estimator) of all linear functions $c'\beta$ which are estimable in $l_i (i = 1,\ 2)$. For a matrix A, denote its column space by $\mu(A)$. Under the condition $\mu(X_2') \subset \mu(X_1')$ comparison of the models (1.1) is studied in [1]. In view of application, however, the condition $\mu(X_2') \subset \mu(X_1')$ is too stringent. One often would like to compare two linear models based on the performance of BLUE of some specially chosen estimable functions. The present note is devoted to this topic. We first give some characterizations for three types of relationships of linear models with nuisance parameters (defined in Definitions 1 and 2 below) in terms of the variances of BLUE of the estimable functions in a subspace $\mu(A) \subset \mu(X_1') \cap \mu(X_2')$, and as a special case, then, analogous characterizations for models (1.1) are obtained.

II. MAIN RESULTS

In this section we first consider the following linear models with nuisance parameters

$$l_1 = L(X_1\beta + Z_1\nu,\ \sigma_1^2 I_{n_1}):\ \ y_1 = X_1\beta + Z_1\nu + e_1,\ E(e_1) = 0,\ \ \mathrm{Cov}(e_1) = \sigma_1^2 I_{n_1}, \tag{2.1}$$

$$l_2 = L(X_2\beta + Z_2\delta,\ \sigma_2^2 I_{n_2}):\ \ y_2 = X_2\beta + Z_2\delta + e_2,\ E(e_2) = 0,\ \ \mathrm{Cov}(e_2) = \sigma_2^2 I_{n_2}, \tag{2.2}$$

where $(X_i,\ Z_i)$, $i = 1, 2$ are design matrices, $\begin{pmatrix}\beta \\ \nu\end{pmatrix}$ and $\begin{pmatrix}\beta \\ \delta\end{pmatrix}$ are unknown vectors of parameters, ν and δ are nuisance parameters. σ_i^2 is known.

Let $\mu(A) \subset \mu\begin{pmatrix}X_1' \\ Z_1'\end{pmatrix} \cap \mu\begin{pmatrix}X_2' \\ Z_2\end{pmatrix}$, and denote by $\hat\beta_i$ the least squares solution of β

under l_i, $i=1,2$.

Definition 1. If for any $c\in\mu(A)$, $\mathrm{Var}(c'\hat\beta_1)\leqslant\mathrm{Var}(c'\hat\beta_2)$, we say that l_1 is at least as good as l_2 with respect to β on $\mu(A)$, and write $l_1\geqslant l_2$, $(\beta, \mu(A))$. Further, if there exists $c_0\in\mu(A)$ such that the strict inequality holds, we say that l_1 is strictly better than l_2 with respect to β on $\mu(A)$, and write $l_1>l_2$, $(\beta, \mu(A))$.

Definition 2. If $l_1\geqslant l_2$, $(\beta, \mu(A))$ and $l_2\geqslant l_1$, $(\beta, \mu(A))$, we say that l_1 is equivalent to l_2 with respect to β on $\mu(A)$, and write $l_1\cong l_2$, $(\beta, \mu(A))$.

We partition X_i, Z_i, into $X_i=\begin{pmatrix}X_{i1}\\X_{i2}\end{pmatrix}$, $Z_i=\begin{pmatrix}Z_{i1}\\Z_{i2}\end{pmatrix}$, $i=1, 2$. Since ν and δ are nuisance parameters, we are only interested in the estimable functions with form $c'\beta$ for comparison of (2.1) and (2.2). Suppose that we are to compare the models in terms of the variances of BLUE of $c'\beta$, where c lies in the subspace $\mu(X'_{11})=\mu(X'_{21})$. Observe that $c'\beta$, $c\in\mu(X'_{11})$ is estimable in l_i iff $c\in\mu(X'_{i1}Z^{\perp}_{i1})$, where $A^{\perp}$ is a matrix with maximum rank such that $A'A^{\perp}=0$. Therefore the estimable subspace used can be taken as $\mu(A)=\mu(X'_{11}Z^{\perp}_{11})=\mu(X'_{21}Z^{\perp}_{21})$. It is easy to show that $l_1\geqslant l_2$, $(\beta, \mu(A))$ iff

$$(Z^{\perp}_{21})'X_{21}M_2^-X'_{21}Z^{\perp}_{21}\geqslant(Z^{\perp}_{21})'X_{21}M_1^-X'_{21}Z^{\perp}_{21}, \tag{2.3}$$

where $M_i=\sigma_i^2(X'_iX_i-X'_iZ_i(Z'_iZ_i)^-Z'_iX_i)$, $i=1, 2$, A^- is a generalized inverse of A, and $A\geqslant B$ means $A-B\geqslant 0$.

There are two situations we shall consider.

(a) Let

$$\mu(X'_{i1})\cap\mu(X'_{i2})=\{0\},\quad i=1,\ 2, \tag{2.4}$$

$$\mu(Z'_{i1})\cap\mu(Z'_{i1})=\{0\},\quad i=1,\ 2, \tag{2.5}$$

$$\mu(X'_{11}Z^{\perp}_{11})=\mu(X'_{21}Z^{\perp}_{21})\triangleq\mu(A). \tag{2.6}$$

Theorem 1. *For two linear models* (2.1) *and* (2.2), *under the conditions* (2.4)—(2.6), $l_1\geqslant l_2$, $(\beta, \mu(A))$ *iff* $H_1\geqslant H_2$, *where*

$$H_i=[X'_{i1}X_{i1}-X'_{i1}Z_{i1}(Z'_{i1}Z_{i1})^-Z'_{i1}X_{i1}]/\sigma_i^2,\quad i=1,2. \tag{2.7}$$

Proof. We first prove that

$$X_{21}(X'_2X_2)^-X'_{21}=X_{21}(X'_{21}X_{21})^-X'_{21}; \tag{2.8}$$

$$X_{22}(X'_2X_2)^-X'_{22}=0. \tag{2.9}$$

In fact, from (2.4) there exists a nonsingular matrix $P'=(P'_1P'_2P'_3)$, such that $\mu(X'_{21})=\mu(P)$ and $\mu(X'_{22})=\mu(P'_2)$. Thus,

$$X_{21}=(Q_1\ 0\ 0)P,\quad X_{22}=(0\ Q_2\ 0)P,$$

where Q_1 and Q_2 are of full rank of columns. Therefore

$$X_{21}(X_2'X_2)^-X_{21}' = (Q_1\ 0\ 0)\begin{pmatrix} Q_1'Q_1 & 0 & 0 \\ 0 & Q_2'Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}^- \begin{pmatrix} Q_1' \\ 0 \\ 0 \end{pmatrix}$$
$$= (Q_1\ 0\ 0)\left[\begin{pmatrix} Q_1' \\ 0 \\ 0 \end{pmatrix}(Q_1\ 0\ 0)\right]^- \begin{pmatrix} Q_1' \\ 0 \\ 0 \end{pmatrix}$$
$$= X_{21}(X_{21}'X_{21})^-X_{21}',$$

this proves (2.8). And (2.9) can be proved in a similar manner. Denote by $P_{\mu(B)}$ the projector matrix onto $\mu(B)$. In view of Theorem 2.18 of Pringle and Rayner[2] and (2.8) and (2.9), we obtain

$$X_{21}M_2^-X_{21}' = \sigma_2^2\{X_{21}(X_2'X_2)^-X_{21}' - X_{21}(X_2'X_2)^-X_2'P_{\mu(Z_2)}(I + P_{\mu(Z_2)}P_{\mu(X_2)}P_{\mu(Z_2)})^{-1} \cdot P_{\mu(Z_2)}X_2(X_2'X_2)^-X_{21}'\}$$

$$= \sigma^2 X_{21}\left[(X_{21}'X_{21}) - (X_{21}(X_{21}'X_{21})^-X_{21}'\ 0)\begin{pmatrix} P_{\mu(Z_{21})} & 0 \\ 0 & P_{\mu(Z_{22})} \end{pmatrix}\right.$$
$$\begin{pmatrix} I + P_{\mu(Z_{21})}P_{\mu(X_{21})}P_{\mu(Z_{21})} & 0 \\ 0 & I + P_{\mu(Z_{22})}P_{\mu(X_{22})}P_{\mu(Z_{22})} \end{pmatrix}^{-1}$$
$$\left.\begin{pmatrix} P_{\mu(Z_{21})} & 0 \\ 0 & P_{\mu(Z_{22})} \end{pmatrix}\begin{pmatrix} X_{21}(X_{21}'X_{21})^-X_{21}' \\ 0 \end{pmatrix}\right]X_{21}'$$
$$= \sigma_2^2 X_{21}[(X_{21}'X_{21})^- - X_{21}(X_{21}'X_{21})^-X_{21}'P_{\mu(Z_{21})}(I + P_{\mu(Z_{21})}P_{\mu(X_{21})}P_{\mu(Z_{21})})^{-1} \cdot P_{\mu(Z_{21})}X_{21}(X_{21}'X_{21})^-X_{21}']X_{21}'$$
$$= X_{21}H_2^-X_{21}'.$$

By using (2.8), (2.9) and (2.6), it is easy to prove that

$$X_{21}(X_1'X_1)^-X_{21}' = X_{21}(X_{11}'X_{11})X_{21}', \quad X_{21}(X_1'X_1)^-X_{11}' = X_{21}(X_{11}'X_{11})^-X_{11}',$$
$$X_{22}(X_1'X_1)^-X_{11}' = 0.$$

According to these facts and in a similar fashion, we can prove $X_{21}M_1^-X_{21}' = X_{21}H_1^-X_{21}'$. Hence, $l_1 \geqslant l_2$, $(\beta, \mu(A))$ iff

$$(Z_{21}^\perp)'X_{21}H_2^-X_{21}'Z_{21}^\perp \geqslant (Z_{21}^\perp)'X_{21}H_1^-X_{21}'Z_{21}^\perp. \tag{2.10}$$

Denote by $\mu(A)^\perp$ the orthogonal complement of $\mu(A)$. Note that $H_i = X_{i1}'P_{\mu(Z_{i1})^\perp}X_{i1}'$, $i = 1, 2$. Therefore

$$\mu(H_i) = \mu(X_{i1}'P_{\mu(Z_{i1})^\perp}) = \{u:\ u = X_{i1}'t,\ Z_{i1}'t = 0 \text{ for some } t\}$$
$$= \mu(X_{i1}'Z_{i1}^\perp).$$

From this fact, it is easy to see that (2.10) is equivalent to

$$H_2 \geqslant H_2H_1^-H_2. \tag{2.11}$$

To complete the proof of Theorem 1, it is sufficient to show that (2.11) is equivalent to $H_1 \geqslant H_2$. As $H_i \geqslant 0$, $i = 1, 2$ with the same rank, there exists a nonsingular matrix T, such that $H_i = T\ D_iT'$, $i = 1, 2$, where D_i, $i = 1, 2$, are diagonal matrices with the same number of nonzero elements. Therefore $H_1 \geqslant H_2 \Longleftrightarrow D_1 \geqslant D_2 \Longleftrightarrow D_2 \geqslant D_2 D_1^+ D_2 \Longleftrightarrow H_2 \geqslant H_2 H_1^- H_2$, where D_1^+ is the Moore-Penrose inverse of D_1. This completes the proof of Theorem 1.

Theorem 2. *Under the conditions of Theorem 1, $l_1 > l_2$, $(\beta, \mu(A))$ iff $H_1 \geqslant H_2$, $H_2 \neq H_2 H_1^- H_2$,*

Proof. From Definition 1 and (2.10), $l_1 > l_2$, $(\beta, \mu(A))$ iff $H_1 \geqslant H_2$ and

$$\alpha_0'(Z_{21}^{\perp})' X_{21} H_2^- X_{21}' Z_{21}^{\perp} \alpha_0 > \alpha_0'(Z_{21}^{\perp})' X_{21} H_1^- X_{21} Z_{21}^{\perp} \alpha_0 \text{ for some } \alpha_0. \tag{2.12}$$

Since $\mu(H_2) = \mu(X_{21}' Z_{21}^{\perp})$, (2.12) is equivalent to $t' H_2 t > t' H_2 H_1^- H_2 t$ for some t. Note the fact $H_2 \geqslant H_1 \Longleftrightarrow H_2 \geqslant H_2 H_1^- H_2$ proved in Theorem 1, this completes the proof of Theorem 2.

Corollary 1. *Under the conditions of Theorem 1. $l_1 \cong l_2, (\beta, \mu(A))$ iff $H_1 = H_2$.*

(b) Let

$$\mu(X_{21}') \cap \mu(X_{22}') = \{0\}, \quad \mu(Z_{21}') \cap \mu(Z_{22}') = \{0\}, \tag{2.13}$$

$$\mu(X_1' Z_1^{\perp}) = \mu(X_{21}' Z_{21}^{\perp}) \triangleq \mu(B). \tag{2.14}$$

Theorem 3. *Suppose that (2.13) and (2.14) hold, then $l_1 \geqslant l_2$, $(\beta, \mu(B))$ iff $M_1 \geqslant H_2$.*

Theorem 4. *Under the conditions of Theorem 3, $l_1 > l_2$, $(\beta, \mu(B))$ iff $M_1 \geqslant H_2$, $H_2 \neq H_2 M_1^- H_2$.*

The proof of Theorems 3 and 4 can be carried out in the same way as that of Theorems 1 and 2, with suitable modifications.

Corollary 2. *Under the conditions of Theorem 3, $l_1 \cong l_2$, $(\beta, \mu(B))$ iff $M_1 = H_2$.*

Next, we consider the comparison of the linear models

$$l_i = L(X_i\beta, \ \sigma_i^2 I_{n_i}): \ y_i = X_i\beta + e_i, \ E(e_i) = 0,$$
$$\mathrm{Cov}(e_i) = \sigma_i^2 I_{n_i}, \ i = 1, 2. \tag{2.15}$$

It is easy to see that (2.15) is a special case of (2.1) and (2.2) with $Z_i = 0$, $i=1, 2$. Hence, the results below can be obtained directly from the preceding theorems and corollaries for the models (2.1) and (2.2).

Some obvious modifications of Definitions 1 and 2 for the model (2.15) is needed. If we consider the comparison of $l_i = L(X_i\beta, \ \sigma_i^2 I_{n_i})$, $i = 1, 2$, on an estimable subspace $\mu(C)$, then for three ordering relationships, the notations $l_1 \geqslant l_2$, $(\mu(C))$, $l_1 > l_2$, $(\mu(C))$ and $l_1 \cong l_2$, $(\mu(C))$ will be employed.

Let

$$X_i = \begin{pmatrix} X_{i1} \\ X_{i2} \end{pmatrix}, \quad i = 1, 2,$$

and $\mu(X_{11}') = \mu(X_{21}')$ and denote $\mu(C) = \mu(X_{11}')$. Then $l_1 \geqslant l_2$, $(\mu(C))$ iff

$$\sigma_2^2 X_{21}(X_2' X_2)^- X_{21}' \geqslant \sigma_1^2 X_{21}(X_1' X_1)^- X_{21}'.$$

Corollary 3. *Suppose* $\mu(X'_{i1}) \cap \mu(X'_{i2}) = \{0\}$, $i = 1, 2$, $\mu(X'_{11}) = \mu(X'_{21})$, *and denote* $\mu(C)$. *Then*

(i) $l_1 \geqslant l_2$, $(\mu(C)) \Longleftrightarrow X'_{11}X_{11}/\sigma_1^2 \geqslant X'_{21}X_{21}/\sigma_2^2$;

(ii) $l_1 > l_2$, $(\mu(C)) \Longleftrightarrow \begin{cases} X'_{11}X_{11}/\sigma_1^2 \geqslant X'_{21}X_{21}/\sigma_2^2; \\ \sigma_1^2 X'_{21}X_{21} \neq \sigma_2^2 X'_{21}X_{11}(X'_{11}X_{11})^- X'_{11}X_{21}; \end{cases}$

(iii) $l_1 \cong l_2$, $(\mu(C)) \Longleftrightarrow X'_{11}X_{11}/\sigma_1^2 = X'_{21}X_{21}/\sigma_2^2$.

Corollary 4. *Suppose* $\mu(X'_{21}) \cap \mu(X'_{22}) = \{0\}$, $\mu(X'_1) = \mu(X'_{21})$, *and denote* $\mu(D)$. *Then*

(i) $l_1 \geqslant l_2$, $(\mu(D)) \Longleftrightarrow X'_1X_1/\sigma_1^2 \geqslant X'_{21}X_{21}/\sigma_2^2$;

(ii) $l_1 > l_2$, $(\mu(D)) \Longleftrightarrow \begin{cases} X'_1X_1/\sigma_1^2 \geqslant X'_{21}X_{21}/\sigma_2^2; \\ \sigma_1^2 X'_{21}X_{21} \neq \sigma_2^2 X'_{21}X_1(X'_1X_1)^- X'_1X_{21}; \end{cases}$

(iii) $l_1 \cong l_2$, $(\mu(D)) \Longleftrightarrow X'_1X_1/\sigma_1^2 = X'_{21}X_{21}/\sigma_2^2$.

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THE BANDWIDTH OF THE PRODUCT OF TWO Γ-TYPE CONDENSED GRAPHS

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In this note, the bandwidth of the product of two Γ-type condensed graphs is obtained, and the relevant conclusions in Refs. [2—6] are generalized since the path, the cycle and the complete graph are the three kinds of the Γ-type condensed graphs.

I. Definitions and Examples of Γ-Type Condensed Graphs

In this note, the common concepts and symbols in ordinary works on graph theory are employed. The senses of the symbols $\lfloor r \rfloor$, $\lceil r \rceil$, $|S|$, ∂S (or $\partial_G S$) can be found in Ref. [1], and the definition of the product of graphs in Ref. [7]. Besides, J denotes the set of all integers. For integers $n \geqslant m \geqslant 1$, write $J[m, n] = \{m, m+1, \cdots, n\}$, $J_n = J(n) = J[1, n]$, and write $J_0 = \phi$.

Definition 1.1. Let G be a graph, and let $|V(G)| = n \geqslant 2$. If there is $k \in J_{n-1}$ and a numbering f: $V(G) \to J_n$ such that $|\partial W| \geqslant \min\{|W|, k\}$ for any proper subset W of $V(G)$ and $f(\partial(f^{-1}(J_i))) = J[i-k+1, i]$ for any $i \in J[k, n-1]$, then G is called a Γ-type condensed graph (shortly called Γ graph below) of kth order, and f is called a condensed numbering on G (or $V(G)$).

Example 1.1. The path P_n (only $n \geqslant 2$ is considered), the cycle C_n (only $n \geqslant 3$) and the complete graph K_n (only $n \geqslant 2$) are Γ graphs of the 1st, 2nd and $(n-1)$th orders, respectively.

Example 1. 2. Suppose that graph $G_0 \subset G_1 \subset G$, $V(G_0) = V(G)$. It is not difficult to prove that if both G and G_0 are all Γ graphs of the kth order, then so is G_1, and each condensed numbering on G is the same on G_0.

Definition 1. 2. For any $n > k \geqslant 1$, construct a graph G_{nk}, $V(G_{nk}) = \{v_1, \cdots, v_n\}$, $E(G_{nk}) = \{v_i v_j: 1 \leqslant i < j \leqslant \min\{i+k, n\}\}$, then G_{nk} is a Γ graph of kth order. Every graph isomorphis to G_{nk} is called a maximum Γ graph. It is easy to verify that every Γ graph is a subgraph of a maximum Γ graph having the same vertexes and the same order.

Example 1. 3. Let G_{nk} be the same as in Definition 1.2, and let $G \subset G_{nk}$, $V(G) = V(G_{nk})$. It is not difficult to show that if $v_i v_j \in E(G)$ for $1 \leqslant i < j \leqslant k$ or $n-k+1 \leqslant i < j \leqslant n$, and $v_j v_{i+k} \in E(G)$ when $v_i v_j \notin E(G)$ for $i \in J_{n-k}$, $j \in J[i, i+k]$, then G is a Γ graph of the kth order.

Example 1. 4. From Example 1.3 we can derive that the product $P_m \times K_k$ of the path P_m and the complete graph K_k is a Γ graph of the kth order.

II. Some Properties of the Product of Γ Graphs

Definition 2. 1. Let g and h be condensed numberings on Γ graphs G and H respectively, f and f_1 be numberings on $V(G \times H)$. If $(f(u, v) - f(u', v))(g(u) - g(u')) \geqslant 0$ and $f(V(G) \times \{v\}) = f_1(V(G) \times \{v\})$ for any $\{u, u'\} \subset V(G)$ and $v \in V(H)$, then f is called a numbering on $G \times H$ harmonious with g, and the rearranging of f_1 in the direction of g. Similarly, we also have the concepts about the numbering on $G \times H$ harmonious with h and the rearranging of f_1 in the direction of h. Now we prove

Theorem 2. 1. *Let G, H and g, h be the same as in Definition* 2.1. (i) *If f is the rearranging of f_1 in the direction of g (or h), then $B(f) \leqslant B(f_1)$*; (ii) $B(G \times H) = \min\{B(f): f$ *is a numbering on* $G \times H$ *harmonious with* g *and* $h\}$.

Theorem 2. 2. *If Γ graphs G_i and H_i have the same number of vertexes and the same order, $(i = 1, 2)$, then $B(G_1 \times G_2) = B(H_1 \times H_2)$.*

III. The Bandwidth of the Product of Γ Graphs

Let m, n, k and l be all positive integers, $k < m, l < n$. In this section it is always assumed that $\lambda = \left[\frac{m}{k}\right]$, $\mu = \left[\frac{n}{l}\right]$, $a = m - \lambda k$, $b = n - \mu l$. For any non-negative integers i and j, put $\theta(i, j) = |\min\{(-1)^j \cdot i, (-1)^i \cdot j\}|$. Let

$$B_1(\lambda, k, l, a, b) = \lambda kl + \frac{a}{2}(l + b) + \frac{1}{2}\theta(a, l - b),$$

$$B_{01}(\lambda, k, l, a, b) = \lambda kl + \frac{ab}{2} + \frac{1}{2}\theta(a, b),$$

$$B_{02}(\lambda, k, l, a, b) = (\lambda - 1)kl + \frac{1}{2}(k + a)(l + b) + \frac{1}{2}\theta(k - a, l - b),$$

$$B_0(\lambda, k, l, a, b) = \min\{B_{0i}(\lambda, k, l, a, b): i = 1, 2\},$$

$$B(\lambda, \mu, k, l, a, b) = \begin{cases} (\lambda k + a)l = ml, & \text{if } \mu \geqslant \lambda + 2; \\ B_i(\lambda, k, l, a, b), & \text{if } \mu = \lambda + i, i = 0 \text{ or } 1; \\ B(\mu, \lambda, l, k, b, a), & \text{if } \mu \leqslant \lambda. \end{cases}$$

Theorem 3. 1. *Let G_{pq} denote the Γ graph of the qth order having p vertexes, then*

$$B(G_{mk} \times G_{nl}) = B(\lambda, \mu, k, l, a, b).$$

The essentials of the proof of Theorem 3.1 is to be presented. For convenience, let us assume that the p vertexes of G_{pq} are all on the real number axis: $V(G_{pq}) = J_p$, and on G_{pq} there is a chosen condensed numbering g_{pq}, $g_{pq}(i) = i (\forall i \in J_p)$. According to Theorem 2.2, we might assume that G_{pq} is a maximum Γ graph as well. Let $G = G_{mk} \times G_{nl}$, $\Gamma = V(G) = J_m \times J_n$, and $\partial = \partial_G$. Obviously, under