CRC Standard Mathematical Tables

26th Edition

Editor of Mathematics and Statistics

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GREEK ALPHABET

Greek letter	Greek name	English equivalent	Greek letter	Greek name	English equivalent
A α	Alpha	a	N ν	Nu	n
B β	Beta	b	Ξ ξ	Xi	x
Γ γ	Gamma	g	O o	Omicron	ŏ
Δ δ	Delta	d	Π π	Pi	p
E ϵ	Epsilon	ĕ	P ρ	Rho	r
Z ζ	Zeta	z	Σ σ s	Sigma	s
H η	Eta	ē	T τ	Tau	t
Θ θ ϑ	Theta	th	Υ υ	Upsilon	u
I ι	Iota	i	Φ ϕ φ	Phi	ph
K κ	Kappa	k	X χ	Chi	ch
Λ λ	Lambda	l	Ψ ψ	Psi	ps
M μ	Mu	m	Ω ω	Omega	ō

THE NUMBER OF EACH DAY OF THE YEAR

Day of Mo.	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.	Day of Mo.
1	1	32	60	91	121	152	182	213	244	274	305	335	1
2	2	33	61	92	122	153	183	214	245	275	306	336	2
3	3	34	62	93	123	154	184	215	246	276	307	337	3
4	4	35	63	94	124	155	185	216	247	277	308	338	4
5	5	36	64	95	125	156	186	217	248	278	309	339	5
6	6	37	65	96	126	157	187	218	249	279	310	340	6
7	7	38	66	97	127	158	188	219	250	280	311	341	7
8	8	39	67	98	128	159	189	220	251	281	312	342	8
9	9	40	68	99	129	160	190	221	252	282	313	343	9
10	10	41	69	100	130	161	191	222	253	283	314	344	10
11	11	42	70	101	131	162	192	223	254	284	315	345	11
12	12	43	71	102	132	163	193	224	255	285	316	346	12
13	13	44	72	103	133	164	194	225	256	286	317	347	13
14	14	45	73	104	134	165	195	226	257	287	318	348	14
15	15	46	74	105	135	166	196	227	258	288	319	349	15
16	16	47	75	106	136	167	197	228	259	289	320	350	16
17	17	48	76	107	137	168	198	229	260	290	321	351	17
18	18	49	77	108	138	169	199	230	261	291	322	352	18
19	19	50	78	109	139	170	200	231	262	292	323	353	19
20	20	51	79	110	140	171	201	232	263	293	324	354	20
21	21	52	80	111	141	172	202	233	264	294	325	355	21
22	22	53	81	112	142	173	203	234	265	295	326	356	22
23	23	54	82	113	143	174	204	235	266	296	327	357	23
24	24	55	83	114	144	175	205	236	267	297	328	358	24
25	25	56	84	115	145	176	206	237	268	298	329	359	25
26	26	57	85	116	146	177	207	238	269	299	330	360	26
27	27	58	86	117	147	178	208	239	270	300	331	361	27
28	28	59	87	118	148	179	209	240	271	301	332	362	28
29	29	*	88	119	149	180	210	241	272	302	333	363	29
30	30		89	120	150	181	211	242	273	303	334	364	30
31	31		90		151		212	243		304		365	31

* In leap years, after February 28, add 1 to the tabulated number.

SI SYSTEM OF MEASUREMENT

SI, which is the abbreviation of the French words "Système Internationale d'Unites," is the accepted abbreviation for the International Metric System, which has seven base units, as shown below.

UNITS FOR A SYSTEM OF MEASURES AS USED INTERNATIONALLY

Quantity measured	Unit	Abbreviation
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	degree Kelvin	K
Luminous intensity	candela	cd
Amount of substance	mole	mol

RECOMMENDED DECIMAL MULTIPLES AND SUBMULTIPLES

Multiples and submultiples	Prefixes	Symbols
10^{18}	exa	E
10^{15}	pecta	P
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10	deca	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ (greek mu)
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

CONVERSION FACTORS

Conversion Factors – Metric to English

To obtain	Multiply	By
Inches	Centimeters	0.3937007874
Feet	Meters	3.280839895
Yards	Meters	1.093613298
Miles	Kilometers	0.6213711922
Ounces	Grams	$3.527396195 \times 10^{-2}$
Pounds	Kilograms	2.204622622
Gallons(U.S. Liquid)	Liters	0.2641720524
Fluid ounces	Milliliters (cc)	$3.381402270 \times 10^{-2}$
Square inches	Square centimeters	0.1550003100
Square feet	Square meters	10.76391042
Square yards	Square meters	1.195990046
Cubic inches	Milliliters (cc)	$6.102374409 \times 10^{-2}$
Cubic feet	Cubic meters	35.31466672
Cubic yards	Cubic meters	1.307950619

Conversion Factors – English to Metric*

To obtain	Multiply	By
Microns	Mils	**25.4**
Centimeters	Inches	**2.54**
Meters	Feet	**0.3048**
Meters	Yards	**0.9144**
Kilometers	Miles	**1.609344**
Grams	Ounces	28.34952313
Kilograms	Pounds	**0.45359237**
Liters	Gallons (U.S. Liquid)	**3.785411784**
Milliliters (cc)	Fluid ounces	29.57352956
Square centimeters	Square inches	**6.4516**
Square meters	Square feet	**0.09290304**
Square meters	Square yards	**0.83612736**
Milliliters (cc)	Cubic inches	**16.387064**
Cubic meters	Cubic feet	$2.831684659 \times 10^{-2}$
Cubic meters	Cubic yards	0.764554858

Conversion Factors – General*

To obtain	Multiply	By
Atmospheres	Feet of water @ 4°C	2.950×10^{-2}
Atmospheres	Inches of mercury @ 0°C	3.342×10^{-2}
Atmospheres	Pounds per square inch	6.804×10^{-2}
BTU	Food-pounds	1.285×10^{-3}
BTU	Joules	9.480×10^{-4}
Cubic feet	Cords	**128**

*Boldface numbers are exact; others are given to ten significant figures where so indicated by the multiplier factor.

Conversion Factors – General (Continued)

To obtain	Multiply	By
Degree (angle)	Radians	57.2958
Ergs	Foot-pounds	1.356×10^{7}
Feet	Miles	**5280**
Feet of water @ 4°C	Atmospheres	33.90
Foot-pounds	Horsepower-hours	1.98×10^{6}
Foot-pounds	Kilowatt-hours	2.655×10^{6}
Food-pounds per min	Horsepower	3.3×10^{4}
Horsepower	Foot-pounds per sec	1.818×10^{-3}
Inches of mercury @ 0°C	Pounds per square inch	2.036
Joules	BTU	1054.8
Joules	Foot-pounds	1.35582
Kilowatts	BTU per min	1.758×10^{-2}
Kilowatts	Foot-pounds per min	2.26×10^{-5}
Kilowatts	Horsepower	0.745712
Knots	Miles per hour	0.86897624
Miles	Feet	1.894×10^{-4}
Nautical miles	Miles	0.86897624
Radians	Degrees	1.745×10^{-2}
Square feet	Acres	**43560**
Watts	BTU per min	17.5796

Temperature Factors

$$°F = 9/5\ (°C) + 32$$

Fahrenheit temperature = 1.8 (temperature in kelvins) –459.67

$$°C = 5/9\ [(°F) - 32]$$

Celsius temperature = temperature in kelvins -273.15
Fahrenheit temperature = 1.8 (Celsius temperature) +32

DECIMAL EQUIVALENTS OF FRACTIONS OF AN INCH

		1/64 =	0.015	625		11/32	22/64 =	0.343	75			43/64 =	0.671	875
	1/32	2/64 =	.031	25			23/64 =	.359	375	11/16	22/32	44/64 =	.687	5
		3/64 =	.046	875	3/8	12/32	24/64 =	.375				45/64 =	.703	125
1/16	2/32	4/64 =	.062	5			25/64 =	.390	625		23/32	46/64 =	.718	75
		5/64 =	.078	125		13/32	26/64 =	.406	25			47/64 =	.734	375
	3/32	6/64 =	.093	75			27/64 =	.421	875	3/4	24/32	48/64 =	.75	
		7/64 =	.109	375	7/16	14/32	28/64 =	.437	5			49/64 =	.765	625
1/8	4/32	8/64 =	.125				29/64 =	.453	125		25/32	50/64 =	.781	25
		9/64 =	.140	625		15/32	30/64 =	.468	75			51/64 =	.796	875
	5/32	10/64 =	.156	25			31/64 =	.484	375	13/16	26/32	52/64 =	.812	5
		11/64 =	.171	875	1/2	16/32	32/64 =	.50				53/64 =	.828	125
3/16	6/32	12/64 =	.187	5			33/64 =	.515	625		27/32	54/64 =	.843	75
		13/64 =	.203	125		17/32	34/64 =	.531	25			55/64 =	.859	375
	7/32	14/64 =	.218	75			35/64 =	.546	875	7/8	28/32	56/64 =	.875	
		15/64 =	.234	375	9/16	18/32	36/64 =	.562	5			57/64 =	.890	625
1/4	8/32	16/64 =	.25				37/64 =	.578	125		29/32	58/64 =	.906	25
		17/64 =	.265	625		19/32	38/64 =	.593	75			59/64 =	.921	875
	9/32	18/64 =	.281	25			39/64 =	.609	375	15/16	30/32	60/64 =	.937	5
		19/64 =	.296	875	5/8	20/32	40/64 =	.625				61/64 =	.953	125
5/16	10/32	20/64 =	.312	5			41/64 =	.640	625		31/32	62/64 =	.968	75
		21/64 =	.328	125		21/32	42/64 =	.656	25			63/64 =	.984	375

PHYSICAL CONSTANTS

Equatorial radius of the earth = 6378.388 km = 3963.34 miles (statute).
Polar radius of the earth, 6356.912 km = 3949.99 miles (statute).
1 degree of latitude at 40° = 69 miles.
1 international nautical mile = 1.15078 miles (statute) = 1852 m = 6076.115 ft.
Mean density of the earth = 5.522 g/cm^3 = 344.7 lb/ft^3.
Constant of gravitation, (6.673 ± 0.003) × 10^{-8} cm^3 gm^{-1} s^{-2}.
Acceleration due to gravity at sea level, latitude 45° = 980.665cm/s^2 = 32.1740 ft/sec^2.
Length of seconds pendulum at sea level, latitude 45° = 99.3574 cm = 39.1171 in.
1 knot (international) = 101.269 ft/min = 1.6878 ft/sec = 1.1508 miles (statute)/hr.
1 micron = 10^{-4} cm.
1 angstrom = 10^{-8} cm.
Mass of hydrogen atom = (1.67339 ± 0.0031) × 10^{-24} g.
Density of mercury at 0°C = 13.5955 g/ml.
Density of water at 3.98°C = 1.000000 g/ml.
Density, maximum, of water, at 3.98°C = 0.999973 g/cm^3.
Density of dry air at 0°C, 760 mm = 1.2929 g/liter.
Velocity of sound in dry air at 0°C = 331.36 m/s - 1087.1 ft/sec.
Velocity of light in vacuum = (2.997925 ± 0.000002) × 10^{10} cm/s.
Heat of fusion of water 0°C = 79.71 cal/g.
Heat of vaporization of water 100°C = 539.55 cal/g.
Electrochemical equivalent of silver 0.001118 g/sec international amp.
Absolute wave length of red cadmium light in air at 15°C, 760 mm pressure = 6438.4696 A.
Wave length of orange-red line of krypton 86 = 6057.802 A.

π CONSTANTS

π = 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37511
$1/\pi$ = 0.31830 98861 83790 67153 77675 26745 02872 40689 19291 48091
π^2 = 9.8690 44010 89358 61883 44909 99876 15113 53136 99407 24079
$\log_e\pi$ = 1.14472 98858 49400 17414 34273 51353 05871 16472 94812 91531
$\log_{10}\pi$ = 0.49714 98726 94133 85435 12682 88290 89887 36516 78324 38044
$\log_{10}\sqrt{2\pi}$ = 0.39908 99341 79057 52478 25035 91507 69595 02099 34102 92128

CONSTANTS INVOLVING e

e = 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69996
$1/e$ = 0.36787 94411 71442 32159 55237 70161 46086 74458 11131 03177
e^2 = 7.38905 60989 30650 22723 04274 60575 00781 31803 15570 55185
$M = \log_{10}e$ = 0.43429 44819 03251 82765 11289 18916 60508 22943 97005 80367
$1/M = \log_e 10$ = 2.30258 50929 94045 68401 79914 54684 36420 76011 01488 62877
$\log_{10}M$ = 9.63778 43113 00536 78912 29674 98645 −10

π^e AND e^π CONSTANTS

π^e = 22.45915 77183 61045 47342 71522
e^π = 23.14069 26327 79269 00572 90864
$e^{-\pi}$ = 0.04321 39182 63772 24977 44177
$e^{\pi/2}$ = 4.81047 73809 65351 65547 30357
$= e^{-\pi/2}$ = 0.20787 95763 50761 90854 69556

NUMERICAL CONSTANTS

$\sqrt{2}$ = 1.41421 35623 73095 04880 16887 24209 69807 85696 71875 37695
$\sqrt[3]{2}$ = 1.25992 10498 94873 16476 72106 07278 22835 05702 51464 70151
$\log_e 2$ = 0.69314 71805 59945 30941 72321 21458 17656 80755 00134 36026
$\log_{10}2$ = 0.30102 99956 63981 19521 37388 94724 49302 67881 89881 46211
$\sqrt{3}$ = 1.73205 08075 68877 29352 74463 41505 87236 69428 05253 81039
$\sqrt[3]{3}$ = 1.44224 95703 07408 38232 16383 10780 10958 83918 69253 49935
$\log_e 3$ = 1.09861 22886 68109 69139 52452 36922 52570 46474 90557 82275
$\log_{10}3$ = 0.47712 12547 19662 43729 50279 03255 11530 92001 28864 19070

OTHER CONSTANTS

Euler's Constant γ = 0.57721 56649 01532 86061
$\log_e \gamma$ = −0.54953 93129 81644 82234
Golden Ratio ϕ = 1.61803 39887 49894 84820 45868 34365 63811 77203 09180

NUMBERS CONTAINING π

	Number	Logarithm		Number	Logarithm
π	3.1415 9265	0.4971 4987	$2\pi^2$	19.7392 0881	1.2953 2971
2π	6.2831 8531	0.7981 7991	$\pi/180$	0.0174 5329	8.2418 7741 − 10
3π	9.4247 7796	0.9742 7113	$180/\pi$	57.2957 7951	1.7581 2260
4π	12.5663 7061	1.0992 0992	$4\pi^2$	39.4784 1762	1.5963 5972
8π	25.1327 4122	1.4002 3992	$1/\pi^2$	0.1013 2118	9.0057 0032 − 10
$\pi/2$	1.5707 9633	0.1961 1990	$1/(2\pi^2)$	0.0506 6059	8.7046 7032 − 10
$\pi/3$	1.0471 9755	0.0200 2864	$1/(4\pi^2)$	0.0253 3029	8.4036 4032 − 10
$\pi/4$	0.7853 9816	9.8950 8992 − 10	$\sqrt{\pi}$	1.7724 5385	0.2485 7492
$\pi/6$	0.5235 9877	9.7189 9862 − 10	$\sqrt{\frac{\pi}{2}}$	0.8862 2692	9.9475 4492 − 10
$\pi/8$	0.3926 9908	9.5940 5992 − 10	$\sqrt{\frac{\pi}{4}}$	0.4431 1346	9.6465 1492 − 10
$2\pi/3$	2.0943 9510	0.3210 5862	$\sqrt{\frac{\pi}{2}}$	1.2533 1413	0.0980 5992
$4\pi/3$	4.1887 9021	0.6220 8862	$\sqrt{\frac{2}{\pi}}$	0.7978 8460	9.9019 4012 − 10
$1/\pi$	0.3183 0988	9.5028 5012 − 10	π^3	31.0062 7671	1.4914 4961
$2/\pi$	0.6366 1977	9.8038 8012 − 10	$\sqrt{\pi}$	1.4645 9189	0.1657 1662
$4/\pi$	1.2732 3954	0.1049 1012	$1/\sqrt{\pi}$	0.6827 8412	9.8342 8342 − 10
$1/(2\pi)$	0.1591 5494	9.2018 2012 − 10	$\sqrt[3]{\pi^2}$	2.1450 2942	0.3314 3322
$1/(4\pi)$	0.0795 7747	8.9007 9012 − 10	$1/\sqrt{\pi}$	0.5641 8958	9.7514 2512 − 10
$1/(6\pi)$	0.0530 5165	8.7246 9892 − 10	$1/\sqrt{2\pi}$	0.3989 4231	9.6009 1012 − 10
$1/(8\pi)$	0.0397 8874	8.5997 6012 − 10	$2/\sqrt{\pi}$	1.1283 7916	0.0524 5512
π^4	9.8696 0440	0.9942 9972			

MULTIPLES OF $\frac{\pi}{2}$

n	$n\frac{\pi}{2}$	n	$n\frac{\pi}{2}$	n	$n\frac{\pi}{2}$	n	$n\frac{\pi}{2}$
1	1.57079 63268	26	40.84070 44967	51	80.11061 26665	76	119.38502 08364
2	3.14159 26536	27	42.41150 08235	52	81.68140 89933	77	120.95131 71632
3	4.71238 89804	28	43.98229 71503	53	83.25220 53201	78	122.52211 34900
4	6.28318 53072	29	45.55309 34771	54	84.82300 16469	79	124.09290 98168
5	7.85398 16340	30	47.12388 98038	55	86.39379 79737	80	125.66370 61436
6	9.42477 79608	31	48.69468 61306	56	87.96459 43005	81	127.23450 24704
7	10.99557 42876	32	50.26548 24574	57	89.53539 06273	82	128.80529 87972
8	12.56637 06144	33	51.83627 87842	58	91.10618 69541	83	130.37609 51240
9	14.13716 69412	34	53.40707 51110	59	92.67698 32809	84	131.94689 14508
10	15.70796 32679	35	54.97787 14378	60	94.24777 96077	85	133.51768 77776
11	17.27875 95947	36	56.54866 77646	61	95.81857 59345	86	135.08848 41044
12	18.84955 59215	37	58.11946 40914	62	97.38937 22613	87	136.65928 04312
13	20.42035 22483	38	59.69026 04182	63	98.96016 85881	88	138.23007 67580
14	21.99114 85751	39	61.26105 67450	64	100.53096 49149	89	139.80087 30847
15	23.56194 49019	40	62.83185 30718	65	102.10176 12417	90	141.37166 94115
16	25.13274 12287	41	64.40264 93986	66	103.67255 75685	91	142.94246 57383
17	26.70353 75555	42	65.97344 57254	67	105.24335 38953	92	144.51326 20651
18	28.27433 38823	43	67.54424 20522	68	106.81415 02221	93	146.08405 83919
19	29.84513 02091	44	69.11503 83790	69	108.38494 65488	94	147.65485 47187
20	31.41592 65359	45	70.68583 47058	70	109.95574 28765	95	149.22565 10455
21	32.98672 28627	46	72.25663 10326	71	111.52653 92024	96	150.79644 73723
22	34.55751 91895	47	73.82742 73594	72	113.09733 55292	97	152.36724 36991
23	36.12831 55163	48	75.39822 36862	73	114.66813 18560	98	153.93804 00259
24	37.69911 18431	49	76.96902 00129	74	116.23892 81828	99	155.50883 63527
25	39.26990 81699	50	78.53981 63397	75	117.80972 45096	100	157.07963 26795

FACTORS AND EXPANSIONS

$(a \pm b)^2 = a^2 \pm 2ab + b^2$.

$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$.

$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$.

$a^2 - b^2 = (a - b)(a + b)$.

$a^2 + b^2 = (a + b\sqrt{-1})(a - b\sqrt{-1})$.

$a^3 + b^3 = (a - b)(a^2 + ab + b^2)$.

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$a^4 + b^4 = (a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2)$.

$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})$.

$a^n - b^n = (a + b)(a^{n-1} - a^{n-2}b + \ldots - b^{n-1})$.

for even values of n.

$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \ldots + b^{n-1})$,

for odd values of n.

$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$.

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.

$(a + b + c)^3 = a^3 + b^3 + c^3 + 3a^2(b + c) + 3b^2(a + c) + 3c^2(a + b) + 6abc$.

$(a + b + c + d + \ldots)^2 = a^2 + b^2 + c^2 + d^2 + \ldots + 2a(b + c + d + \ldots) + 2b(c + d + \ldots) + 2c(d + \ldots) + \ldots$

See also under Series.

POWERS AND ROOTS

$a^x \times a^y = a^{(x+y)}$. $\quad a^0 = 1$ [if $a \neq 0$] $\quad (ab)^x = a^x b^x$.

$\frac{a^x}{a^y} = a^{(x-y)}$. $\quad a^{-x} = \frac{1}{a^x}$. $\quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$.

$(a^x)^y = a^{xy}$. $\quad a^{\frac{1}{x}} = \sqrt[x]{a}$. $\quad \sqrt[x]{ab} = \sqrt[x]{a}\,\sqrt[x]{b}$.

$\sqrt[x]{\sqrt[y]{a}} = \sqrt[xy]{a}$. $\quad a^{\frac{x}{y}} = \sqrt[y]{a^x}$. $\quad \sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$

PROPORTION

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$,

$\frac{a-b}{b} = \frac{c-d}{d}$, $\quad \frac{a-b}{a+b} = \frac{c-d}{c+d}$.

ARITHMETIC PROGRESSION*

An arithmetic progression is a sequence of numbers such that each number differs from the previous number by a constant amount, called the *common difference*.

If a_1 is the first term; a_n the nth term; d the common difference; n the number of terms; and s_n the sum of n terms—

$$a_n = a_1 + (n - 1)d, \quad s_n = \frac{n}{2}[a_1 + a_n].$$

$$s_n = \frac{n}{2}[2a_1 + (n - 1)d].$$

The arithmetic mean between a and b is given by $\frac{a + b}{2}$.

GEOMETRIC PROGRESSION*

A geometric progression is a sequence of numbers such that each number bears a constant ratio, called the *common ratio*, to the previous number.

If a_1 is the first term; a_n the nth term; r the common ratio; n the number of terms; and s_n the sum of n terms

$$a_n = a_1 r^{n-1}; \quad s_n = a_1 \frac{1 - r^n}{1 - r}$$

$$= a_1 \frac{r^n - 1}{r - 1}, \quad r \neq 1.$$

$$= \frac{a_1 - ra_n}{1 - r}$$

$$= \frac{ra_n - a_1}{r - 1}$$

If $|r| < 1$, then the sum of an infinite geometrical progression converges to the limiting value

$$\frac{a_1}{1 - r}, \qquad \left[s_\infty = \lim_{n \to \infty} \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1}{1 - r}\right]$$

The geometric mean between a and b is given by $\sqrt{ab}$.

*It is customary to represent a_n by l in a finite progression and refer to it as the last term.

HARMONIC PROGRESSION

A sequence of numbers whose reciprocals form an arithmetic progression is called an harmonic progression. Thus

$$\frac{1}{a_1}, \quad \frac{1}{a_1 + d}, \quad \frac{1}{a_1 + 2d}, \cdots, \frac{1}{a_1 + (n - 1)d}, \cdots,$$

where

$$\frac{1}{a_n} = \frac{1}{a_1 + (n - 1)d}$$

forms an harmonic progression. The harmonic mean between a and b is given by $\frac{2ab}{a + b}$.

If A, G, H respectively represent the arithmetic mean, geometric mean, and harmonic mean between a and b, then $G^2 = AH$.

QUADRATIC EQUATIONS

Any quadratic equation may be reduced to the form,—

$$ax^2 + bx + c = 0$$

Then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If a, b, and c are real then:
If $b^2 - 4ac$ is positive, the roots are real and unequal;
If $b^2 - 4ac$ is zero, the roots are real and equal;
If $b^2 - 4ac$ is negative, the roots are imaginary and unequal.

CUBIC EQUATIONS

A cubic equation, $y^3 + py^2 + qy + r = 0$ may be reduced to the form,—

$$x^3 + ax + b = 0$$

by substituting for y the value, $x - \frac{p}{3}$. Here

$$a = \tfrac{1}{3}(3q - p^2) \text{ and } b = \tfrac{1}{27}(2p^3 - 9pq + 27r).$$

For solution let,—

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}, \qquad B = -\sqrt[3]{+\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}},$$

then the values of x will be given by,

$$x = A + B, \quad -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}, \quad -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3}.$$

If p, q, r are real, then:

If $\frac{b^2}{4} + \frac{a^3}{27} > 0$, there will be one real root and two conjugate complex roots;

If $\frac{b^2}{4} + \frac{a^3}{27} = 0$, there will be three real roots of which at least two are equal;

If $\frac{b^2}{4} + \frac{a^3}{27} < 0$, there will be three real and unequal roots.

Trignometric Solution of the Cubic Equation

The form $x^3 + ax + b = 0$ with $ab \neq 0$ can always be solved by transforming it to the trignometric identity

$$4\cos^3\theta - 3\cos\theta - \cos(3\theta) \equiv 0.$$

Let $x = m\cos\theta$, then

$$x^3 + ax + b \equiv m^3\cos^3\theta + am\cos\theta + b \equiv 4\cos^3\theta - 3\cos\theta - \cos(3\theta) \equiv 0.$$

Hence

$$\frac{4}{m^3} = -\frac{3}{am} = \frac{-\cos(3\theta)}{b},$$

from which follows that

$$m = 2\sqrt{-\frac{a}{3}}, \quad \cos(3\theta) = \frac{3b}{am}.$$

Any solution θ_1 which satisfies $\cos(3\theta) = \frac{3b}{am}$, will also have the solutions

$$\theta_1 + \frac{2\pi}{3} \quad \text{and} \quad \theta_1 + \frac{4\pi}{3}.$$

The roots of the cubic $x^3 + ax + b = 0$ are

$$2\sqrt{-\frac{a}{3}}\cos\theta_1, \quad 2\sqrt{-\frac{a}{3}}\cos\left(\theta_1 + \frac{2\pi}{3}\right), \quad 2\sqrt{-\frac{a}{3}}\cos\left(\theta_1 + \frac{4\pi}{3}\right).$$

Example where hyperbolic functions are necessary for solution with latter procedure

The roots of the equation $x^3 - x + 2 = 0$ may be found as follows:

Here

$$a = -1, \quad b = 2, \quad m = 2\sqrt{\tfrac{1}{3}} = 1.155$$

$$\cos(3\theta) = \frac{6}{-1.155} = -5.196$$

$$\cos(3\theta) = -\cos(3\theta - \pi) = -\cosh[i(3\theta - \pi)] = -5.196.$$

Using hyperbolic function tables for $\cosh [i(3\theta - \pi)] = 5.196$, it is found that

$$i(3\theta - \pi) = 2.332.$$

Thus

$$3\theta - \pi = -i(2.332).$$

$$3\theta = \pi - i(2.332)$$

$$\theta_1 = \frac{\pi}{3} - i(0.777)$$

$$\theta_1 + \frac{2\pi}{3} = \pi - i(0.777)$$

$$\theta_1 + \frac{4\pi}{3} = \frac{5\pi}{3} - i(0.777)$$

$$\cos\theta_1 = \cos\left[\frac{\pi}{3} - i(0.777)\right]$$

$$= \left(\cos\frac{\pi}{3}\right)[\cos i(0.777)] + \left(\sin\frac{\pi}{3}\right)[\sin i(0.777)]$$

$$= \left(\cos\frac{\pi}{3}\right)(\cosh 0.777) + i\left(\sin\frac{\pi}{3}\right)(\sinh 0.777)$$

$$= (0.5)(1.317) + i(0.866)(0.858) = 0.659 + i(0.743).$$

Note that

$$\cos\mu = \cosh(i\mu) \quad \text{and} \quad \sin\mu = -i\sinh(i\mu).$$

Similarly

$$\cos\left(\theta_1 + \frac{2\pi}{3}\right) = \cos[\pi - i(0.777)]$$

$$= (\cos\pi)(\cosh 0.777) + i(\sin\pi)(\sinh 0.777)$$

$$= -1.317,$$

and

$$\cos\left(\theta_1 + \frac{4\pi}{3}\right) = \cos\left[\frac{5\pi}{3} - i(0.777)\right]$$

$$= \left(\cos\frac{5\pi}{3}\right)(\cosh 0.777) + i\left(\sin\frac{5\pi}{3}\right)(\sinh 0.777)$$

$$= (0.5)(1.317) - i(0.866)(0.858) = 0.659 - i(0.743).$$

The required roots are

$$1.155[0.659 + i(0.743)] = 0.760 + i(0.858)$$

$$(1.155)(-1.317) = -1.520$$

$$(1.155)[0.659 - i(0.743)] = 0.760 - i(0.858).$$

QUARTIC OR BIQUADRATIC EQUATION

A quartic equation,

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

has the *resolvent cubic equation*

$$y^3 - by^2 + (ac - 4d)y - a^2d + 4bd - c^2 = 0.$$

Let y be any root of this equation, and

$$R = \sqrt{\frac{a^2}{4} - b + y}.$$

If $R \neq 0$, then let

$$D = \sqrt{\frac{3a^2}{4} - R^2 - 2b + \frac{4ab - 8c - a^3}{4R}}$$

and

$$E = \sqrt{\frac{3a^2}{4} - R^2 - 2b - \frac{4ab - 8c - a^3}{4R}}$$

If $R = 0$, then let

$$D = \sqrt{\frac{3a^2}{4} - 2b + 2\sqrt{y^2 - 4d}}$$

and

$$E = \sqrt{\frac{3a^2}{4} - 2b - 2\sqrt{y^2 - 4d}}.$$

Then the four roots of the original equation are given by

$$x = -\frac{a}{4} + \frac{R}{2} \pm \frac{D}{2}$$

and

$$x = -\frac{a}{4} - \frac{R}{2} \pm \frac{E}{2}$$

EQUATION $x^n = c$

Using DeMoivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta;\ i = \sqrt{-1},$$

the equation $x^n = c$ has n roots given by

$$x = \sqrt[n]{c}\left(\cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}\right) \text{ if } c > 0,$$

or

$$x = \sqrt[n]{-c}\left(\cos\frac{(2m+1)\pi}{n} + i\sin\frac{(2m+1)\pi}{n}\right) \text{ if } c < 0,$$

where m takes the n values $0, 1, 2, \ldots, (n-1)$ giving n roots.