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经济 MATHEMATICS
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两大部类再生产数学模型研究*

摘 要

张小弦 林少宫

本文根据马克思再生产理论建立了两大部类再生产数学模型。此模型的特点是生产资料实物平衡方程和两大部类等价交换实现条件均能得到满足，符合马克思关于两大部类再生产过程所给出的基本假设。

本文着重对积累率与两大部类增长的关系进行了讨论。可以看到，使经济保持均衡增长的总积累率由两大部类的产出比唯一确定。通过详尽的数学分析，我们的主要结果是：

（1）第二部类资本有机构成的提高使第二部类的增长速度减慢，从而使第一部类的生产增长加快。

（2）第一部类资本有机构成的提高使得两个部类的增长率都下降。当第一部类剩余价值率大于1，并且积累率较低时，资本有机构成的提高可能使第二部类的生产增长更快。本文得到了使消费资料生产增长更快的充要条件。

（3）第一部类资本有机构成的提高使生产资料生产在整个社会生产中的比重通常具有增加的趋势，但从数学的观点看存在出现相反趋势的可能性。

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A STUDY ON THE MATHEMATICAL MO–DEL OF REPRODUCTION OF THE "TWO DEPARTMENTS"

Abstract

By
Zhang Xiaoxian
Lin Shaogong

Department of Mathematics
Huazhong University of S & T

Abstrat

In this paper a mathematical model of reproduction is built of the "two departments", based on Marx' s theory of reproduction. The model is characterized by the fact that the physical balance equations for consumption means and the realization conditions for exchange of equal values between the two departments are to be satisfied in accordance with Marx's basic assumptions concerning the process of reproduction.

Discussions are much focused on the relationship between the accumulation rate and the growth rates of the two departments. It is seen that the total accumulation rate that keeps the economy in balanced growth is uniquely determined by the output ratio between the two departments.

1. Introduction

Karl Marx's theory of the reproduction of social capital is a reflection of the general content and common laws of all the social reprodution movements characterised by socialized production and commodity production. It is very important to understand and study these laws for accelerating the socialist economic construction. since New China was founded Marx's theory has been discussed three times with respect to the theoretical problems of social reproduction and the ratio of the well known "two departments"in connection with China's practical conditions. The first time was at the start of the first Five-Year Plan; The second was in the early 1960s, a period for reajusting the national economy; and the third is in recent years, beginning in 1979, with an effort to carry out the economic reform end explore the right way of China' s socialist modernization. These discussions help much to understand the importance of achieving an over-all balance of the economy and thus improve the national economic plan.However, there are still some important problems which are not fully expounded, One reason for this is that people are used to the traditional qualitative analysis and the principle plus example method to the neglect of quantitative analysis.Thus, in this article, we try to construct a mathematical model of the two departments reproduction in accordance with Marx's reproduction theory and make some quantitative preliminary analysis wltk it.

Being different from the neoclassical general equilibrium theory, the main concepts of Marx' s reproduction theory are reproduction, analysis of value and physical constituents of social products,and realization conditions of reproduction necessary to the coordinated growth of an economic system.

Merx' s reproduction schemata employ linear production sets. Oskar Lange(3)thought that Marx reproduction theory contains the idea of input-output analysis and can be represented by the input-output tables. Thus, it seem proper that the mathematical model in this article is to be based on linear production set with difference equations describing dynamic processes of production, exchange and consumption of the two departments.

The reproduction model is a simplified vension of social reproduction. However, we still can analyse the general tendency of economic development by this model. The relation between accumulation rete and growth rates of the two departments is discussed in section 3, it can be used to provide a satisfactory explanation of the "high accumulation, low growth'' phenomenon.

One important problem in Marx's reproduction theory is the so-called''priority of the growth of means of production'' which is sometimes considered as a law of socialist develop ment.Although some authors try to prove or disprove this ''law'' in mathematical terms, the problem hasn' been t completley solved. It is easy to see, in section 4, that no general conclusion can be reached merely by giving a few examples. The problem is approached in this article by constructing a mathematical model of reproduction and making the corresponding analysis. Changes of the organic composition of capital in each department are considered both on their short term and long term effects. Theorem 4 and 4' give the necessary and condition for a faster growth rate of department I. so far, we have not known of any report with results similar to ours.

2. Model

When Marx set forth his theory of reproduction in ''Capital'', two important premises were given:

(A1) The whole of social product …is divided into two departments (I) the means of production…and(I I)the mean s o fconsumption.

The value of annual production, P, is broken down as: Constant capital C, variable capital c and surplus valus M, or $P_i = C_i + V_i + M_i$, $i = 1, 2$.

Besides this,other basic assumptuins were given as follows:

(A2) All the means of production are used up in both avlue and physical terms in the course of one cycle, say, a year:

$P_1(t) = C_1(t+1) + C_2(t+1)$.

(A3)The means of consumption are fully allocated in the ocurse of one production cycle;reserves of such goods are added to the consumption fund in the next cycle and acquired by workers in the corresponding department. $P_2(t) = V_1(t+1) + V_2(t+1) + \overline{M}_t$,where $\overline{M}_t$ denotes consumptions outside the sphere of production.

(A4) The organic composition of capital is constant in each department, $H_i = C_i/V_i = const.$

(A5) The rate of surplus value is constant overtime: $Z_i = M_i/V_i = const$.

(A6) There is no foreign trade in the system.

(A7) Exchange between the departments is conducted on an equivalent basis in terms of value.

(A8) The rate of accumulation in department I is constant.

Price or value variations are not mentioned in the above assumptions .since these variations must affect the organic composition of capital, H, and the rate of surplus value, Z, our discussion proceed, for simplicity only under assumptions(A4)and (A5).

Now, from (A2) and (A3) we have:

$$P_1(t) = C_1(t+1) + C_2(t+1), \qquad (2.1)$$

$$P_2(t) = V_1(t+1) + V_2(t+1) + M_t, \qquad (2.2)$$

or in matrix form,

$$X = A_{t+1}X_{t+1} + \begin{pmatrix} 0 \\ M \end{pmatrix}_t, \qquad (2.3)$$

where $X_t = \begin{pmatrix} P_1(t) \\ P_2(t) \end{pmatrix}$ is an output vector of the two departments for year t; $\begin{pmatrix} 0 \\ M_t \end{pmatrix}$ is a consumption vector outside the sphere of proauction for year t; $A_t = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_t$ is a technical coefficient matrix for year t, in which

$$a_{1i} = C_i/P_i,\ a_{2i} = V_i/P_i,\ i = 1,\ 2 \qquad (2.4)$$

Let $a_{3i} = M_i/P_i,\ i = 1,\ 2$ (2.5)

then $a_{1i} + a_{2i} + a_{3i} = (C_i + V_i + M_i)/P_i = 1,\ i = 1, 2$ (2.6)

Generally speaking, $a_{ij} > 0$.

From (A4) and (A5)

$$H_i = C_i/V_i = a_{1i}/a_{2i},\ Z_i = M_i/V_i = a_{3i}/a_{2i}, i = 1, 2 \qquad (2.7)$$

With the help of (2.6), we have

$$a_{1i} = H_i/(H_i + 1 + Z_i),\ a_{2i} = 1/(H_i + 1 + Z_i), a_{3i} = Z_i/(H_i + 1 + Z_i). \qquad (2.8)$$

(2.3) is the physical balance equations, it means that all of the products' means of production) of department I, in year t, is used up as constant capital required by both departments for year t+1, and that the products (means of consumption) of department II, in year t, is used up as the variable capital required by both departments for year t+1 and as a current consumption outside the sphere of production. (A6) presupposes that the economy is a closed system without foreign trade. As a result social production has to be constrained by physical compensation (supply and demand are in balance).

According to (A7), however, the movement of the total social capital (social production) also has to be constrained by value compensation, that is, the two departments must exchange their products on the equivalent basis so that the reproduction is realizable.

Obviously, if for each department there has not been any value change in trade, then an equivalent exchange of products between two departments is realized. we may write

$$P_1(t) = C_1(t+1) + V_1(t+1) + \overline{M}_1(t), \qquad (2.9)$$

$$P_2(t) = C_2(t+1) + V_2(t+1) + \overline{M}_2(t), \qquad (2.10)$$

in which $\overline{M}_i(t) \geqslant 0$ are assumed. They are consumptions outside the sphere of production of each department separatly and $\overline{M}_1(t) + \overline{M}_2(t) = M_t$. (2.9) and (2.10) are value balance equations for both departments' products. From (2.1) and (2.2),

$$V_1(t+1) + \overline{M}_1(t) = C_2(t+1),$$

this is the extended reproduction realization condion of Marx. If we put $V_1(t+1) = V_1(t)$ and $C_2(t)+1 = C_2(t)$, then $\overline{M}_1(t) = M_1(t)$ due to (A4), and

$$V_1(t) + M_1(t) = C_2(t),$$

which is the simple reproduction realization condition.

Now, according to (A1), (A2), (A3), (A6), and (A7) the reproducton model is

$$(M_1)\begin{cases} P_1(t)=C_1(t+1)+C_2(t+1), \\ P_2(t)=V_1(t+1)+V_2(t+1)+\overline{M}_1(t)+\overline{M}_2(t), \\ P_1(t)=C_1(t+1)+V_1(t+1)+\overline{M}_1(t), \\ P_2(t)=C_2(t+1)+V_2(t+1)+\overline{M}_2(t). \end{cases}$$

or

$$\begin{cases} X_t=A_{t+1}X_{t+1}+\begin{pmatrix}0\\ \overline{M}_t\end{pmatrix}, \\ X_t=\begin{pmatrix}a_{11}+a_{21} & 0\\ 0 & a_{12}+a_{22}\end{pmatrix}_{t+1}X_{t+1}+\begin{pmatrix}\overline{M}_1(t)\\ \overline{M}_2(t)\end{pmatrix}. \end{cases}$$

since the equations in(M_1)are not independent, it is better to take off one,say,the second one, and leave the other three.

$$(M_2)\begin{cases} P_1(t)=a_{11}P_1(t+1)+a_{12}P_2(t+1), & (2\text{˙}1) \\ P_1(t)=(a_{11}+a_{21})P_1(t+1)+\overline{M}_1(t), & (2.9) \\ P_2(t)=(a_{12}+a_{22})P_2(t+1)+\overline{M}_2(t)\text{˙} & (2.10) \end{cases}$$

It appears that $\overline{M}_1(t)$ and $\overline{M}_2(t)$ are exogeneous variables in (M_2),however,they are not independent of each other since (2.1) requires that the supply of means of production equals the demand for it. That is, $C_2(t+1)=V_1(t+1)+\overline{M}_1(t)$. In accordance with (A8), the accumulation rate of department I,denoted by S_1, is usually given,and from S_1 we obtain $\overline{M}_1(t)$ Then we can solve for $P_1(t+1)$ and $P_2(t+1)$ so that $\overline{M}_2(t)$ is determined by the equations.

Accumulation rates in Marx (1) is defined as:

$$S_i=(\triangle C_i+\triangle V_i)/M_i,\quad i=1,\ 2 \qquad (2.11)$$

i.e. $0\leqslant S_i\leqslant 1$, $\overline{M}_i=(1-S_i)M_i=(1-S_i)a_{3i}P_i$, $i=1,\ 2$.

The aggregate accumulation rate is then defined as

$$S=(\triangle C+\triangle V)/M=(S_1M_1+S_2M_2)/(M_1+M_2). \qquad (2.12)$$

For the purpose of comparing the growth rates of output in the two departments, it is more convenient to write(M_2) as follows:

$$
(M)\begin{cases}
(1-a_{11}(t+1)R_1(t)P_1(t)=a_{12}(t+1)R_2(t)P_2(t), & (2.13)\\
1-(a_{11}(t+1)+a_{21}(t+1))R_1(t)=(1-S_1(t))a_{31}(t), \quad 0\leqslant S_1\leqslant 1 & (2.14)\\
1-(a_{12}(t+1)+a_{22}(t+1))R_2(t)=(1-S_2(t))a_{32}(t), \quad 0\leqslant S_2\leqslant 1 & (2.15)\\
P_i(t+1)=R_i(t)P_i(t), \ i=1,\ 2;\ t=1,\ 2,\ 3,\ \cdots & (2.16)
\end{cases}
$$

If technical coefficients are unchanged(when (A4) and (A5) hold) then,

$$
(M')\begin{cases}
(1-a_{11}R_1(t))P_1(t)=a_{12}R_2(t)P_2(t), & (2.13')\\
S_1(t)=((a_{11}+a_{21})/a_{31})(R_1(t)-1), \ 0\leqslant S_1\leqslant 1 & (2.14')\\
S_2(t)=((a_{12}+a_{22})/a_{31})(R_2(t)-1), \ 0\leqslant S_2\leqslant 1 & (2.15')\\
P_i(t+1)=R_i(t)P_i(t), \ i=1,\ 2;\ t=1,\ 2,\ 3,\ \cdots & (2.16')
\end{cases}
$$

It' s easy to see that (2.13) is the basic equation of (M).

Let $g_t(R_1(t),\ R_2(t))=(a_{11}R_1(t)-1,\ a_{12}R_2(t))$, then (2.13')is equivalent to $g_t\cdot X_t=0$.

Define sets

$G^{-1}=\{(R_1,\ R_2):\ S.\ t\ (2.14')\ \text{and}\ (2.15')\}$,

$G^{-1}(X)\ \{(R_1,\ R_2)\in G^{-1};\ g(R_1,\ R_2)\cdot X=0\}$,

$\underline{X}=\{X>0;\ (R_1,\ R_2)\in G^{-1},\ g\cdot X=0\}$.

It is shown, in Figure 1, that the coordinate transformation

$$
\begin{cases}
P_1=a_{11}R_1-1,\\
P_2=a_{12}R_2,
\end{cases} \qquad (2.17)
$$

links the coordinate system of OP_1P_2 with that of $O'R_1R_2$. The set G^{-1}, rectangle ABCD, with $1=\min_i R_i\leqslant R_j\leqslant \max R_j=\tilde{R}_i$, indicates the set of all possible growth rates that conforms to the current technology. The set $G^{-1}(X_t)$, line FH, indicates the set of possible growth rates that conforms to the technology and the output X_t. The set $\underline{X}$ is the sh adow area. we can expect that the growth rates of both departments are $(R_1, R_2) \in G^{-1}(X_t)$, and, for properly choosen accumulation retes (S_1, S_2), $X_{t+1} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} X_t$.

Beginning with some t, if $R(t) = R_1(t) = R_2(t)$ and $(R_1, R_2) \in G^{-1}(X_t)$, then (R_1, R_2) is on line l and $X_{t+1} = R(t) \cdot X_t$. Growth rate $R(t)$ will be called the equilibrium growth uate and the ray on which vector X_t lies will be called an eq-rilibrium trajectory.

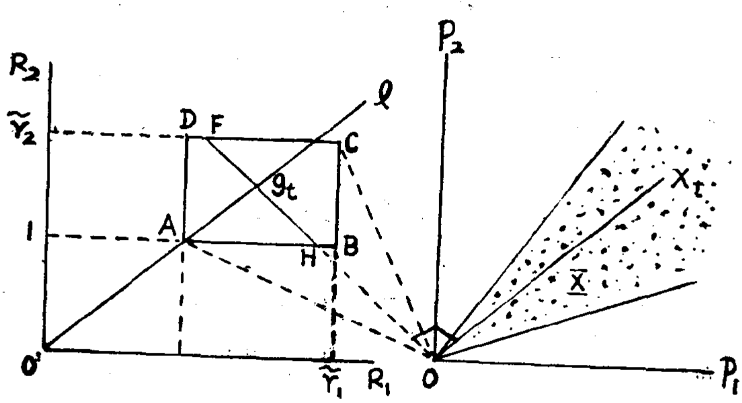

Figure 1

3. Properties

With mathematical model (M′), the properties of exten-sive extended reproduction are discussed as follows:

Property 1 Let $(R_1(t), R_2(t)), \in G^{-1}(X_t)$. If the rate of department I, $R_1(t)$ increases, then the growth rate of department Ⅱ, $R_2(t)$ has to decline, and vice versa.

Proof: From (2.13′), $(1 - a_{11}R_1(t))P_1(t) = a_{12}R_2(t)P_2(t)$

(t), the conclusion is obvious. Q. E. D.

Property 1 implies that increasing the accumulation rate of one department, thus increasing its growth rate, is at the c ost of slowing down the growth of the other department.

Property 2 $\forall (R, R) \in G^{-1}$, $\exists X(R) \in \underline{X}$ and $(R, R) \in G^{-1}(X(R))$.

Property 2 is obvious. It means that for each possible equl-ibrium growth rate of social reproduction there exists a cor-responding output vector which grows at that rate. This is not to emphasize the possible attainment of maximum growth, but the importance and possibility of coordinatating the output structure with the accumulations of both departments.

Property 3 Let $(R, R) \in G^{-1}[X(R)]$, $X(R) = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$, $\Delta(R) = P_1/P_2$, we have $\frac{d\Delta(R)}{dR} > 0$.

Proof: From (2.13′), $(1-a_{11}R)P_1 = a_{12}RP_2$,

$\Delta(R) = P_1/P_2 = a_{12}R/(1-a_{11}R)$, so $\frac{d(R)}{dR} = a_{12}/(1-a_{11}R)^2 > 0$. Q.E.D.

According to property 3, there exists only one equilibrium growth trajectory corresponding to growth rate R. The higher the growth rate, the larger the product value of department I relative to that of department II.

Now, we consider stabilities of (M′).

Property 4 Choose $(R_1(t), R_2(t)) \in G^{-1}(X_t)$. If $(R_1, R_1) \in G^{-1}$, then $(R_1(t), R_1(t)) \in G^{-1}(X_{t+1})$.

It′s easy to see, by property 4, that the social reproduciton described by model (M′)can attain the equilibrium growth trajectory with the rate $R_1(t)$ after a transition period of only one year. Thus, assumption (A8) given by Marx guarantees a steady growth in the economy. (If technical coefficients change, assumption (A10), as will be given in the section 4, is perhaps more suitable).

Property 4 is clearly embodied in Marx′s reproduction schemata, for example, in Marx (1) the extended reproduction schemata (Marx′s first example):

Base year $4000(C_1)+1000(V_1)+1000(M_1)=6000(P_1)$,

$1500(C_2)+750(V_2)+750(M_2)=3000(P_2)$.

$H_1=C_1/V_1=4$, $Z_1=M_1/V_1=1$ $H_2=C_2/V_2=2$, $Z_2=M_2/V_2=1$.

Assuming an unchanging accumlation rate for department I, $S_1=0.5$, we have

t	P_1	P_2	R_1	R_2
0	6000	3000	—	—
1	6600	3200	1.1	1.067
2	7260	3520	1.1	1.1
3	7986	3872	1.1	1.1
4	8784	4259	1.1	1.1

Obriously, this property depends on (A2). Generally speaking, if only a part of total capital is worn away and needs compesation in production, property 4 will not be true.

This property and its proof is shown in Bashmakov (4), but he fails to state the condition $(R_1, R_1) \in G^{-1}$. In effect, if $(R_1, R_1) G^{-1}$, there is no epuilibrium growth at the rate of R_1.

Property 5 Let $R_1(t) \neq R_2(t)$, $[R_1(t), R_2(t)) \in G^{-1}(X_t)$ and $[R_1(t+1), R_2(t+1)] \in G^{-1}(X_{t+1})$. If $R_2(t+1) = R_2(t)$, then

$|R_1(t+1)-R_2(t+1)|>|R_1(t)-R_2(t)|$.

There is a striking contrast between property 5 and property 4. If department Ⅱ' S accumulation rate S_2 is kept constant, the difference of the growth rates will increase in the reproduction process according to property 5. Obviously, the reproduction process cannot converge to any equilibrium growth trajectory, therefore it is in evitable that the production of the two departments will be out of proportion.

It' s more interesting, however, to analyse the whole economy in terms of the parameters associated with the two departments.

Property 6 Assume $H_1>H_2$, $(R, R)\in G^{-1}(X_t)$, and let $\bar{S}(t)$ be the aggregate accumulation rate corresponding to the growth rate R. If the accumulationrate $S(t)\gtreqless\bar{S}(t)$, then $R_1(t)\lesseqgtr R\lesseqgtr R_2(t)$.

In fact, the condition that the organic composition of capita indepartment I is greater than that in department Ⅱ usually holds, so it is interesting to see what property 6 signifies.

First of all, if an equilibrium growth is kept in the economy as described by the model (M'), then the aggregate accumulation rate is determined uniquely compatible with the output structure of the two departments. Thus, determination of a rational accumulation rate amounts to determining a rational output structure.

Secondly, an economic system will not be stable if the aggregate accumulation rate to be kept constant and the output vector are not compatible. For example, if $S(t)>\bar{S}(t)$, we have $R_1(t)<R<R_2(t)$. with the help of property 4, $(R_1(t), R_1(t))\in G^{-1}(X_{t+1})$, thus $\bar{S}(t+1)<\bar{S}(t)<S(t)=$

S(t+1). This means that the difference between the actual accumulation rate S(t+1) and the rate $\bar{S}(t+1)$ as required in an equilibrium growth will increase so that the economy will grow out of proportion sooner or later.

Finally, consider what effect the aggregate accumulation rate S(t) could have on the reproduction of the two departments. Since the basic equation(2. 13′)requires a balance of supply of and demand for means of production in order to make efficient use of the means of production, if the actual accumulation rate S(t) is greater than the rate $\bar{S}(t)$ which an equilibrium growth requires, the additional accumulation of capital must be a part of the consumption goods originally used outsideethe sphere of production. This must increase the capital accumulation in the department which has lower organic composition (department Ⅱ) and correspondingly decrease the accumulation in the department which has higher organic composition (department Ⅰ). In the process of an extensive extended reproduction this will lead to a situation in which the growth rate $R_1(t)$ decreases and the growth rate $R_2(t)$ increases.

The above theoretical analysis, however, seems to be inconsistent with the actual conditions of chinese economy. The accumulation rates in China tend to be high for most of the years while the proportions of department Ⅱ tend to be low. This may be a result of pursuing the policy of a higher growth rate of national economy and a ″faster growth of department Ⅰ″ with high accumulation and inefficient investments.

Let′s further illustrate the point in Figure 2. It′s easy to see that (3.1) is a linear equation in R_1 and R_2 on plane R, given accumulation rate S and output vector X. Property 6 implies that its slop is greater than the slop of the line, $G^{-1}(X)$. (The line given by (3.1) corresponds to a demand curve for means of production with given accumulation rate whereas the line, $G^{-1}(X)$, corresponds to a supply curve. Recall that the physically balanced equation of means of production