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On the Formation of Quasi-geostrophic Motion in the Atmosphere

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Except in low latitudes the comparatively large scale atmospheric motions are fundamentally quasi-geostrophic. This fact has been widely applied in the field of meteorology. However, theories explaining the existence of the quasi-geostrophic motion are not many. Kibel(4) and Charney(2) introduced scale of motion in the equations of motion and found that the observed large scale motion in the atmosphere must be quasi-geostrophic. But they did not discuss the physics of the formation of quasi-geostrophic motion. The present note will provide this link.

Following Kibel(4) and Charney(2) we introduce the following transformations

$$x=L\bar{x},\quad Y=L\bar{y},\quad Z=H\bar{z},\quad u=U\bar{u},$$
$$v=U\bar{v},\quad w=W\bar{w},$$

and $t=\tau\bar{t}$ into the equations of motion. Taking the second epuation of motion as an example we have

$$\varepsilon\left(\frac{\partial\bar{v}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}+\varepsilon'\bar{w}\frac{\partial\bar{v}}{\partial\bar{z}}\right)=-\bar{u}-\frac{1}{fLU}\frac{1}{\rho}\frac{\partial p}{\partial\bar{y}} \qquad (1)$$

here

$$\varepsilon=U/fL \qquad (2)$$

$\varepsilon'=\frac{W}{U}\frac{L}{H}$; $f=2\omega\sin\varphi$, the Coriolis parameter; (x, y, z), the space coordinates; t, time; (u, v, w), the three velocity components; (L, H), the horizontal and vertical scales of motion; (U, W), the horizontal and vertical characteristic velocity components; and τ, the time scale. In obtaining (2) it is assumed that $\tau\approx U/L$. According to Kotchin(5) and Charney(2) $\varepsilon'<1$. Thus the condition for quasi-geostrophic motion is $\varepsilon<1$. Put the scale of long wave ($U=10$mps, $L=10^6$m) into the expression of ε, then $\varepsilon\approx10^{-1}$ (for middle latitude). Thus Kibel(4) and Charney(2) found that the motion of this scale must be quasi-geostrophic.

From the expression for ε (equ. (2)) it would be concluded that the larger the horizontal scale (L) of motion the more geostrophic the motion would be. However, for large scale motion, L is not independent of U they may be related approximately by Rossby's stationary wave formula:

$$L=2\pi\sqrt{\frac{U}{\beta}},$$

where β is the variation of Coriolis parameter with latitude. Put the above expression into (2) we have

$$\varepsilon=\frac{\beta L}{4\pi^2 f}.$$

The above expression states that as long as Rossby's formula is valid the smaller the horizontal scale the more geostrophic the motion would be. For middle latitudes ($\beta=1.62\times10^{-13}\,\mathrm{cm^{-1}\,sec.^{-1}}$ $f=10^{-4}\,\mathrm{sec.^{-1}}$) L would be 2.4×10^{10}cm in order to make $\varepsilon=1$. This scale is larger than the great circle of the earth. Thus the motion in the earth's atmosphere must be quasi-geostrophic, as long as Rossby's wave formula is valid.

We may also write equ. (3) in another form:

$$\varepsilon=\frac{1}{2\pi}\frac{\sqrt{\beta U}}{f}.$$

In order to get nongeostrophic motion ($\varepsilon\approx1$) in middle latitudes U must be of order of 10^6 cm/sec! This is absolutely impossible for the centrifugal acceleration is then much larger than the gravitational acceleration.

Let us discuss this problem in another way. Multiplying equ. (1) by U we then have

$$\varepsilon U\left(\frac{d\bar{v}}{d\bar{t}}\right)=-(U-U_g) \tag{3}$$

where $\frac{d\bar{v}}{d\bar{t}}=\frac{\partial\bar{v}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}+\varepsilon'\bar{w}\frac{\partial\bar{v}}{\partial\bar{z}}$ has an order 1. Thus

$$U-U_g=\pm\varepsilon U \tag{4}$$

where U_g is the geostrophic zonal wind. The ($\pm$) sign on the right side of (4) means that $(d\bar{v}/d\bar{t})$ may be positive or negative. If we take the(+)sign then ε can not be near 1. For then U_g would be much smaller than U and could be neglected in (4); *i.e.*, the pressure force could be neglected. This is not possible in the atmospheric motion (except the inertia motion). ε may be <1, but the motion is then quasi-geostrophic which is the case we are not interested in. ε may also be much larger than 1. Then $(\varepsilon-1)U\approx-U_g$. Substituting it in (2) we have

$$\varepsilon(\varepsilon-1)=-\frac{U_g}{fL}.\quad(\varepsilon>1) \tag{5}$$

If we take the ($-$) sign in equ. (4), we then obtain

$$\varepsilon(1+\varepsilon)=\frac{U_g}{fL}. \tag{6}$$

Suppose that at surface of the earth $U_g\approx0$ and substitute the zonal wind at certain level. the tropopause say, for U_g. Then from thermal wind equation and (5) and (6) we have then either

$$\varepsilon(\varepsilon-1)=+\frac{gH}{Lf^2}\frac{1}{\bar{T}}\frac{\partial\bar{T}}{\partial y}\quad(\varepsilon>1) \tag{5'}$$

or

$$\varepsilon(1+\varepsilon)=-\frac{gH}{Lf^2}\frac{1}{\bar{T}}\frac{\partial\bar{T}}{\partial y} \tag{6'}$$

Here the sign "$-$" represents mean over space.

For meteorological phenomena our atmosphere may be considered as a thin layer of air covered over the earth. For motions we are interested in (H/L) is always $\ll 1$. Thus we see that the fact that the atmosphere may be considered as a very thin layer of fluid is of fundamental importance for the occurrence of geostrophic motion. Physically this is quite easy to understand. Kotchin(5) has shown that $W/U=H/L$. Thus in the atmosphere $W\ll U$ and the motion may be considered practically horizontal. Horizontal motion is a necessary condition for geostrophy.

It may also be seen from (5′) or (6′) that the degree of non-geostrophy is proportional to $\frac{1}{f^2}\left|\frac{\partial\bar{T}}{\partial y}\right|$. Thus the nongeostrophic motion is favored by low rate of rotation of the system and high temperature contrast. This agrees with the experimental fndings of Hide(3). Indeed the criterion for nongeostrophic motion by (5′) or (6′) is, up to a constant, equivalent to the criterion derived by Kuo(6) for the axial symmetrical motion on a rotating cylinder.

Based on (5′) or (6′) we may discuss the minimum temperature contrast for the occurrence of nongeostrophic motion of different scales at different latitudes. For this purpose we may set $\varepsilon=1$. Since by difinition $\varepsilon>0$, (5′) is only possible for very large temperature contrast with high temperature to the north. This possibility is eliminated from the earth's atmosphere. We may now concentrate our attention to (6′) which may be rewritten in the following form

$$\varepsilon=\frac{gH}{2f^2L^2}\frac{\Delta\bar{T}}{\bar{T}} \tag{7}$$

for $\varepsilon\approx1$ and $-\frac{\partial\bar{T}}{\partial y}$ is put equal to $(\Delta\bar{T}/L)$. Taking $g=10^3$cm/sec^2, $H=10^6$cm and $f=10^{-4}$ sec^{-1}, we find the minmum temperature contrast for nongeostrophic motion of the scale of whole hemisphere ($L=10^9$cm) equal to

$$\Delta\bar{T}=10\bar{T}!$$

This is absolutely impossible. For the scale of half hemisphere,

$$\Delta\bar{T}=2.5\bar{T}!$$

This is still absolutely impossible. For the scale of 2.10^8 cm,

$$\bar{T}=0.4\bar{T}$$

which corresponds to a temperature gradient of 5.5° C/(deg. latitude). Such a large temperature gradient for a belt of about 20° latitude has never been observed.

Let us now calculate the minimum temperature contrast required for the occurrence of nongeostrophic motion in scale of 1000 km and 500 km at different latitudes. This is shown

Table 1.

Minimum temperature contrast ($\Delta\overline{T}$) for the occurrence of nongeostrophic motion and temperature contrast resulted from equilibrium radiation in 10° latitude belt at different latitudes ($L=10^8$cm and 5.10^7cm, $\overline{T}=250°$A).

Latitude belt	0-10°	10-20°	20-30°	30-40°	40-50°	50-60°	60-70°	70-80°	80-90°
ΔT_{c1}(°C, $L=10^8$)	1.0	8.6	22.6	42	64	86	104	118	126
ΔT_{c2}(°C, $L=5.10^7$)	0.25	2.2	5.7	10.5	16	21.5	26	29	31
ΔT_r(°C)	3	4	10	14	14	14	12	9	3

in the second and third row of Table 1.

From the table we see that this temperature contrast increases very rapidly with latitude.

In order to estimate the possibility of the nongeostrophic motion of such a scale we aslo put the temperature contrast resulted from purely radiation equilibrium in Table 1 (fourth row $\Delta\overline{T}_r$). Comparing $\Delta\overline{T}_{c1}$ and $\Delta\overline{T}_r$ we see that $\Delta\overline{T}_{c1}>\Delta\overline{T}_r$ only below 10° latitude. Thus purely under the action of radiation and earth's rotation nongeostrophic motion of scale of 10^3 km is only possible in very low latitudes. For the scale of 500 km the nongeostrophic motion is possible below 40° latitude.

From the foregoing discussison we may give the following conclusion : The fundamental reasons for the occurence of quasigeostrophic motion in the atmosphere are twofold : (1) The layer of atmosphere in which the meteorological phenomena are concerned is very thin. (2) The ratio of the temperature constrast produced by radiation to the square of the rate of earth's rotation is small.

We may now discuss the physics of quasigeostophic motion further. Though the atmospheric motion is highly geostrophic, yet for weather development the deviation from it must occur from time to time. But why this deviation does not develop so that the motion becomes highly nongeostrophic? Rossby(9)(10) first tried to answer this question. Obukhov(7), Raethjen(8) and others(1) made further investigation on this problem. They all demonstrated that the pressure (or mass) field very quickly adjusts itself to the velocity field to become quasigeostrophic balance whenever a highly nongeostrophic motion occurs. In other words, it would always be the pressure field which changes to adapt the new velocity field. Thus their studies would conclude that the observed pressure field in the atmosphere is a reaction to the Coriolis field. However, this is only half of the story. This conclusion is valid for comparatively small scale motion (not so small that earth's rotation may be neglected). For the scale of general circulation this conclusion is no longer valid.

Following Rossby(9)(10), we impart suddenly to a belt of relatively rest homogeneous atmosphere an uniform velocity U_0. Then the process of adjustment between the pressure and velocity field will start. According to Rossby the relation between the initial (U_0) and final velocity (U_f) will be

$$U_f=U_0\frac{\lambda/a}{1+\lambda/a+a/3\lambda} \qquad (9)$$

where $\lambda=\frac{1}{f}\sqrt{gD}$, D is the thickness of the atmosphere, and $2a$ is the width of the disturbed belt. Taking $D=8$ km, $f=10^{-4}$ sec.$^{-1}$, we have the following ratio of U_f/U_0 for various a :

a	500 km	3000 km	5000 km
U_f/U_0	0.85	0.41	0.25

From above table we see that for small scale ($2a$) the velocity field does not change much in the process of the adjustment, thus it must be the pressure field which changes to fit the velocity field. However, for very large scale ($2a$) it is the velocity field which changes more to give quasigeostrophic equilibrium. For intermediate scale both pressure and velocity field will change to attain mutual adjustment.

Obukhov[7] and Raethjen[8] discussed the adjustment process for circular motion. Obukhov gave the following initinal velocity field

$$\psi_0(x, y)=A\left[2+\left(\frac{R}{\lambda}\right)^2-\left(\frac{r}{R}\right)^2\right]\exp.\left(-\frac{r^2}{2R^2}\right), \quad (10)$$

$$r^2=x^2+y^2$$

Here ψ_0 is the initial stream function, λ has the same meaning as before, and R is radius of a circle within which the pressure is uniform and outside of which there is a balance between pressure and velocity field. After adjustment the final stream function becomes

$$\psi_f(x, y)=A\left[2-\left(\frac{r}{R}\right)^2\right]\exp.\left(-\frac{r^2}{2R^2}\right). \quad (11)$$

Obukhov used $R=500$ km and concluded that in the course of adjustment the velocity field has practically no change while the pressure field has a drastic change. Again this conclution is only valid for comparatively small scale nongeostrophic motion. We may calculate the ratio of initial (V_0) and final velocity (V_f) for various R. Following table is the result:

r	1000 km	2000 km	3000 km	4000 km	5000 km
V_f/V_0(R= 500 km)	0.99	0.99	0.99	0.99	0.98
V_f/V_0(R=3000 km)	0.75	0.74	0.71	0.64	0.51
V_f/V_0(R=5000 km)	0.52	0.51	0.47	0.39	0.27

From the above table we may obtain the same conclusion as before about the process of mutual adjustment between the pressure and velocity field for the case of straight flow.

Raethjen further pointed out the importent role of vertical scale in the mutual adjustment of pressure and velocity field. For a deep layer the pressure changes more and for a shallow layer the velocity field changes more to give new quasigeostrophic equilibrium if it is once disturbed.

From the foregoing discussions we may give the following statement about the production of quasigeostrophic motion: When due to some reason or other the quasigeostrophic equilibrium breaks down, then for small scale motion (not so small that the earth's rotion may be neglected) it is the pressure field to fit the new velocity field to attain new quasigeostrophic motion; for very large scale it is the velocity field which changes more to give new quasigeostrophic motion; and for intermediate scale both fields will change.

Besides the above statement we may also give further inferences: The cause of formation or destruction of comparatively small and deep weather systems, such as cold lows and warm highs, is mainly dynamic. The redistribution of mass field due to thermal processes has little effect on the formation or destruction of these systems. The redistribution of mass field is rather a result. The direct thermal process can only give small shallow system, such as warm lows and cold highs. For the motion of scale of 40-50 degrees latitude both dynamic and thermal processes are imporment. For the variation of motion of scale of hemisphere it is the thermal process which is mainly important.

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