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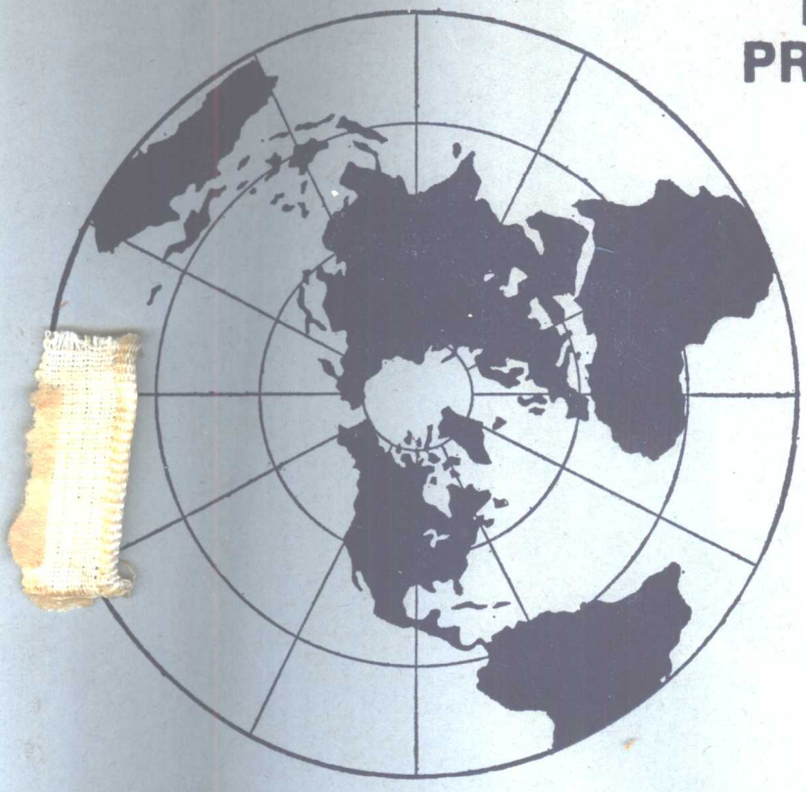

INTERNATIONAL SYMPOSIUM ON PROBABILITY AND STATISTICS IN THE ATMOSPHERIC SCIENCES

JUNE 1-4, 1971
HONOLULU, HAWAII

INTERNATIONAL SYMPOSIUM

ON

PROBABILITY AND STATISTICS

IN THE

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FOREWORD

The International Symposium on Probability and Statistics in the Atmospheric Sciences sponsored by the American Meteorological Society and cosponsored by the World Meteorological Organization was held June 1-4, 1971, in Honolulu, Hawaii. It follows three years after the First National Conference on Statistical Meteorology held in Hartford, Connecticut. During this time the number of workers in this field has grown, but not as fast as the number of problems which need to be solved. Research efforts such as the Global Atmospheric Research Program (GARP) collect large amounts of data which require new methods of analyses.

The international aspect of the meeting was made possible by the location in Honolulu, Hawaii, and by a travel grant from the National Science Foundation for non United States citizens.

We have attempted to accept papers which present new and innovative approaches to probabilistic and statistical problems in the atmospheric sciences. Three main areas of emphasis are:

(1) Stochastic Dynamic Prediction

(2) Time Series Analysis

(3) Probability Forecasts

In addition, there are a variety of papers presenting techniques for a broad range of applications.

Richard H. Jones
Program Chairman

AMS Committee on Meteorological Statistics

Richard H. Jones, Chairman
Joseph G. Bryan
Donald L. Gilman
Paul R. Julian
Allan H. Murphy
Paul T. Schickendanz

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ON NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

One dimensional nonlinear stochastic differential equations of the first order such as

$$dx(t,\omega) = f(t, x(t,\omega))dt \qquad (1.1)$$

$$dx(t,\omega) = f(t, x(t,\omega), \omega)dt \qquad (1.2)$$

have been treated by some authors.(e.g. Bernstein (1938), Ito(1951)) In these equations ω is a variable on a probability space $(\Omega, \mathcal{B}, P)$. If the initial value $x(0,\omega) = \xi$ is a numerical constant, the process $x(t,\omega)$ $(t > 0)$ defined by (1.1) reduces to a deterministic function of t. The stochastic differential equation of the form

$$dx(t,\omega) = A(t, x(t,\omega))dt + dw(t,\omega) \qquad (1.3)$$

is included in (1.2), where $w(t,\omega)$ is a Wiener process with

$$E\,dw = 0, \quad E(dw)^2 = \sigma^2 dt$$

Let $x(t) = (x_1(t),\ldots,x_n(t))'$ (' denotes the transposed) be the state of a system which is governed by a differential equation

$$\dot{x} = A(t, x) \qquad (1.4)$$

where $A(t,x) = (A_1(t,x),\ldots,A_n(t,x))'$. In the real situation, (1.4) is only an ideal relation and $x(t)$ is usually influenced by other complicated unexpected factors and may be expressed by

$$dx = A(t,x)dt + dw \qquad (1.5)$$

where $w = (w_1(t,\omega),\ldots,w_n(t,\omega))'$ is a multi-dimensional Wiener process with $E\{dw_i(s,\omega)dw_j(t,\omega)\} = \sigma_i^2\delta_{ij}\,\delta_{st}\,dt$ $(\sigma_i^2 \geq 0, i,j = 1,\ldots,n)$; that is an n-dimensional version of (1.3).

As an another example, suppose that $x = x(t,\omega)$ is a one dimensional stochastic process which has derivatives with probability one up to the $(n-1)$th order and satisfies a stochastic differential equation

$$dx^{(n-1)} = F(x, \dot{x}, \ldots, x^{(n-1)}, t)dt + dw. \qquad (1.6)$$

This equation reduces to (1.5) by setting

$$x_1 = x, \quad x_2 = \dot{x}, \ldots, \dot{x}_{n-1} = x_n \text{ (formally)},$$

$$A_1 = \ldots A_{n-1} = 0,$$

$$A_n(t,x) = F(x_1,\ldots,x_n,t) \quad (x = (x_1,\ldots,x_n)')$$

and $\sigma_1^2 = \ldots = \sigma_{n-1}^2 = 0, \ \sigma_n^2 > 0.$

In this paper we shall first consider the conditional probability law of the solution $x(t,\omega)$ of nonlinear stochastic differential equation such as (1.5) or (1.6) under a given initial condition, by means of Kolmogorov's parabolic partial differential equation for a Markov process; this idea has been suggested by Grenander(1959). Secondly, we shall try a method of successive approximation of the process $x(t,\omega)$ by means of Wiener integral.

At the present stage, these methods are rather theoretical and their practical use is left in future.

In the following sections, for the sake of simplicity, we denote stochastic processes $x(t,\omega)$ and $w(t,\omega)$ by $x(t)$ and $w(t)$ respectively in which ω's are omitted.

2. KOLMOGOROV EQUATION

Let $f(s,\xi;t,x)$ $(\xi = (\xi_1,\ldots,\xi_n), x = (x_1,\ldots,x_n), s < t)$ be the transition probability density function of a strictly continuous Markov process whose state space is the n dimensional Euclidian space R_n, that is

$$P(s,\xi;t,E) = P_r\{x(t)\in E \mid x(s) = \xi\} = \int_E f(s,\xi;t,x)dx \quad (s<t) \qquad (2.1)$$

for any Borel set $E \subset R_n$ and $\xi \in R_n$. We assume that

(1) $f(s,\xi;t,x)$ has continuous derivatives up to the third order with respect to each components of ξ and x in R_n.

(2) $\lim_{h\to 0}\int_{|x-\xi|>\delta} f(s,\xi;s+h,x)dx = 0$ for any $\delta > 0$, (2.2)

$|x - \xi|$ being the distance between ξ and x. and the following limits exist:

(3) $\lim_{h\to 0}\frac{1}{h}\int_{R_n}(x_i-\xi_i)f(s,\xi;s+h,x)dx = a_i(s,\xi)$ $(i = 1,\ldots,n)$ (2.3)

(4) $\lim_{h\to 0}\frac{1}{h}\int_{R_n}(x_i-\xi_i)(x_j-\xi_j)f(s,\xi;s+h,x)dx = 2b_{ij}(s,\xi)$ $(i,j = 1,\ldots,n)$ (2.4)

(5) $\lim_{h\to 0}\frac{1}{h}\int_{R_n}\rho^3(\xi,x)f(s,\xi;s+h,x)dx = 0,$ (2.5)

where

$$\rho^3(\xi,x) = \sum_{i,j,k}(x_i-\xi_i)(x_j-\xi_j)(x_k-\xi_k). \qquad (2.6)$$

Then f is differentiable with respect to s and t, and satisfies the Kolmogorov equations:

$$\frac{\partial f}{\partial s} = -\sum_{i=1}^{n} a_i(s,\xi)\frac{\partial f}{\partial \xi_i} - \sum_{i,j=1}^{n} b_{ij}(s,\xi)\frac{\partial^2 f}{\partial\xi_i\partial\xi_j} \qquad (2.7)$$

$$\frac{\partial f}{\partial t} = -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}[a_i(t,x)f] + \sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}[b_{ij}(t,x)f] \qquad (2.8)$$

Hereafter, we shall be concerned mainly with the second equation.

If the initial probability density function of $x(s)$ is $\varphi(s,\xi)$, it is obvious that the probability density function of $x(t)$ $(s<t)$,

$$\varphi(t,x) = \int_{R_n}\varphi(s,\xi)f(s,\xi;t,x)d\xi, \qquad (2.9)$$

satisfies the same equation (2.8).

According to Kolmogorov, if $a_i(t,x)$ and $b_{ij}(t,x)$ are continuously twice differentiable with respect to each x_i, the equation (2.8) has at most one non-negative continuous solution f. The same is true for φ.

Now, we consider a nonlinear stochastic differential equation

$$dx = A(t,x)dt + dw, \qquad (2.10)$$

where $x = (x_1(t),\ldots,x_n(t))'$, $A(t,x) = (A_1(t,x), \ldots,A_n(t,x))'$ and $w = (w_1(t),\ldots,w_n(t))'$ is a

multi-dimensional Wiener process with

$$E dw_i = 0,\quad E dw_i(s)dw_j(t) = \sigma_i^2 \delta_{ij}\, dt \quad (\sigma_i^2 \geq 0), \tag{2.11}$$

some of σ_i^2 may be zero.

We assume that the components of $A(t,x)$ has continuous derivative with respect to t and two times continuously differentiable with respect to each x_i. Then, if $x(\tau)(0 \leq \tau \leq t)$ is given, the subsequent process $x(\tau)(t < \tau)$ is stochastically determined by (2.10) and continuous in t with probability one. Therefore $x(t)$ is a Markov process with a transition probability density function $f(s,\xi;t,x)$ which satisfies the condition (2.2); we assume morover the first condition (1) stated above. With regard to the conditions (2.3),(2.4) and (2.5), by means of (2.11), we get

$$\begin{aligned} a_i(t,x) &= \lim_{h\downarrow 0} E\left\{ \frac{x_i(t+h) - x_i(t)}{h} \,\middle|\, x(t) = x \right\} \\ &= \lim_{h\downarrow 0} E\left\{ A_i(t,x) + \frac{w_i(t+h) - w_i(t)}{h} \right\} = A_i(t,x) \end{aligned} \tag{2.12}$$

$$\begin{aligned} b_{ij}(t,x) &= \lim_{h\downarrow 0} \frac{1}{2h} E\left\{ (x_i(t+h) - x_i(t))(x_j(t+h) - x_j(t)) \,\middle|\, x(t) = x \right\} \\ &= \lim_{h\downarrow 0} \frac{h}{2} E\left\{ \left(A_i(t,x) + \frac{w_i(t+h) - w_i(t)}{h}\right)\left(A_j(t,x) + \frac{w_j(t+h) - w_j(t)}{h}\right) \right\} \\ &= \begin{cases} \sigma_i^2/2 & (i = j) \\ 0 & (i \neq j) \end{cases} \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &\lim_{h\downarrow 0} \frac{1}{h} E\left\{ (x_i(t+h) - x_i(t))(x_j(t+h) - x_j(t))(x_k(t+h) - x_k(t)) \,\middle|\, x(t) = x \right\} \\ &= \lim_{h\downarrow 0} \frac{1}{h} E\left(A_i h + w_i(t+h) - w_i(t) \right)\left(A_j h + w_j(t+h) - w_j(t) \right)\left(A_k h + w_k(t+h) - w_k(t) \right) \\ &= 0 \end{aligned}$$

Therefore, the Kolmogorov second equation for our process given by (2.10) becomes

$$\frac{\partial f}{\partial t} = \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2 \frac{\partial^2 f}{\partial x_i^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[A_i(t,x) f] \tag{2.14}$$

where $f = f(s,\xi;t,x)$, $\xi = (\xi_1, \cdots, \xi_n)$, $x = (x_1, \ldots, x_n) \in R_n$ and $s < t$.

If σ_i^2 $(i = 1,\ldots,n)$ are all positibe, namely if $(\sigma_i^2 \delta_{ij})$ is positive definite, the fundamental solution of the parabolic partial differential equation (2.14) can be obtained, at least theoretically, by the Dressel's method.(Dressel(1940), (1946)).

Our problem is in the degenerate case where some of σ_i^2 are zero, which occurs for instance for the stochastic differential equation such as (1.6).

3. A METHOD OF THE SOLUTION OF DEGENERATE EQUATION

The degenerate parabolic equation corresponding to the nonlinear stochastic equation (1.6) is given by

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x_n^2} - \sum_{i=1}^{n} x_{i+1} \frac{\partial f}{\partial x_i} - A \frac{\partial f}{\partial x_n} - \frac{\partial A}{\partial x_n} f \tag{3.1}$$

where $f = f(s,\xi;t,x)$, $\sigma^2 = \sigma_n^2 > 0$ and we assume here that $A = A(x)$ is independent of t.

Let f be of the form

$$f = g(s,\xi;t,x) + \alpha(x) h(s,\xi;t,x_1,\ldots,x_{n-1}) \tag{3.2}$$

Substituting (3.2) in (3.1) we have

$$\begin{aligned} \frac{\partial g}{\partial t} + \alpha \frac{\partial h}{\partial t} &= \frac{\sigma^2}{2} \frac{\partial^2 g}{\partial x_n^2} - A \frac{\partial g}{\partial x_n} - \frac{\partial A}{\partial x_n} g \\ &\quad - \sum_{i=1}^{n-1} x_{i+1} \left(\frac{\partial g}{\partial x_i} + \alpha \frac{\partial h}{\partial x_i} + \frac{\partial \alpha}{\partial x_i} h \right) \\ &\quad + \left(\frac{\sigma^2}{2} \frac{\partial^2 \alpha}{\partial x_n^2} - A \frac{\partial \alpha}{\partial x_n} - \alpha \frac{\partial A}{\partial x_n} \right) h \end{aligned} \tag{3.3}$$

Therefore, if α and g satisfies

$$\frac{\sigma^2}{2} \frac{\partial^2 \alpha}{\partial x_n^2} - A \frac{\partial \alpha}{\partial x_n} - \alpha \frac{\partial A}{\partial x_n} = 0 \tag{3.4}$$

and

$$\frac{\partial g}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 g}{\partial x_n^2} - A \frac{\partial g}{\partial x_n} - \frac{\partial A}{\partial x_n} g \tag{3.5}$$

respectively, then h can be obtained from the first order linear partial differential equation

$$\alpha \frac{\partial h}{\partial t} + \sum_{i=1}^{n-1} \alpha x_{i+1} \frac{\partial h}{\partial x_i} + \left(\sum_{i=1}^{n-1} x_{i+1} \frac{\partial \alpha}{\partial x_i} \right) h = - \sum_{i=1}^{n-1} x_{i+1} \frac{\partial g}{\partial x_i} \tag{3.6}$$

where α is a particular solution of (3.4) given by

$$\alpha = \exp(B(x_n)) \int^{x_n} \exp(-2B(x))\, dx \tag{3.7}$$

$$B(x) = \frac{1}{\sigma^2} \int^{x} A\, dx_n \tag{3.8}$$

and g is the solution of (3.5). The equation (3.5) is transformed by

$$g = u \exp(B(x_n)) \quad (u = u(s,\xi;t,x)) \tag{3.9}$$

into the standard form

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x_n^2} - q u \quad \left(q = \frac{1}{2}\left(\frac{1}{\sigma^2} A^2 + \frac{\partial A}{\partial x_n} \right) \right) \tag{3.10}$$

which can be solved by means of the current methods.(Dressel(1946),Itô,S.(1953,1954),(1957))

If we put $\beta = \alpha h$, solving (3.6) is equivalent to solve

$$\frac{\partial \beta}{\partial t} + \sum_{i=1}^{n-1} x_{i+1} \frac{\partial \beta}{\partial x_i} = - \sum_{i=1}^{n-1} x_{i+1} \frac{\partial g}{\partial x_i} \tag{3.11}$$

Now, our initial condition is given by

$$\begin{aligned} &0 \leq f(s,\xi;t,x) < \infty \quad (s < t,\ \xi, x \in R_n) \\ &\lim_{t\downarrow s} f(s,\xi;t,\xi) = \infty \\ &\lim_{t\downarrow s} f(s,\xi;t,x) = 0 \quad (x \neq \xi) \end{aligned} \tag{3.12}$$

From (3.11) we get a characteristic curve

$$\begin{aligned} &t(\tau) = s + \tau,\quad x_i(\tau) = \xi_i + \xi_{i+1}\tau \quad (i = 1, \cdots, n-1), \\ &x_n(\tau) = \xi_n,\quad \beta(\tau) = \beta_0 + \beta_1(\tau), \end{aligned} \tag{3.13}$$

where

$$s = s(\lambda),\ \xi_i = \xi_i(\lambda)\ (i = 1, \cdots, n-1),\ \xi_n = \xi_n(\lambda),\ \beta_0 = \beta_0(\lambda) \tag{3.14}$$

is an initial curve in R_{n+2}. Corresponding to (3.12) we may assume that

$$\beta_0(\lambda) \equiv 0,\quad \beta_1(0) = 0 \tag{3.15}$$

and

$$0 \leq u(s,\xi;t,x) < \infty,\quad 0 \leq \beta(s,\xi;t,x) < \infty \tag{3.16}$$

$$\begin{aligned} &\lim_{t\downarrow s} u(s,\xi;t,\xi) = \infty \\ &\lim_{t\downarrow s} u(s,\xi;t,x) = 0 \quad (x \neq \xi) \end{aligned} \tag{3.17}$$

4. STOCHASTIC VAN DER POL EQUATION

As an example, let us take up stochastic van der Pol equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = \dot{w} \ \text{(formally)}. \tag{4.1}$$

If we put $\dot{x} = y$, then $\dot{y} = \mu(1 - x^2)y - x + \dot{w}$, that is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \mu(1 - x^2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{w} \end{bmatrix}, \tag{4.2}$$

where $E(dw)^2 = \sigma^2 dt$. Therefore we get

$$A_1(t,x,y) = y$$

$$A_2(t,x,y) = \mu(1 - x^2)y - x$$

$$B_{11}(t,x,y) = \lim_{h\downarrow 0} \frac{1}{2h} E\{(x(t+h)-x(t))^2 | x(t)=x, y(t)=y\}$$

$$= \lim_{h\downarrow 0} \frac{h y}{2} = 0$$

$$B_{12}(t,x,y) = \lim_{h\downarrow 0} \frac{1}{2h} E\{(y(t+h)-y(t))^2 | x(t)=x, y(t)=y\}$$

$$= \lim_{h\downarrow 0} \frac{h}{2} E\left\{\left(\mu(1-x^2)y - x + \frac{w(t+h)-w(t)}{h}\right)^2\right\}$$

$$= \lim_{h\downarrow 0} \frac{E(w(t+h)-w(t))^2}{2h} = \frac{\sigma^2}{2}$$

$$B_{12}(t,x,y) = \lim_{h\downarrow 0} \frac{1}{2h} E\{(x(t+h)-x(t))(y(t+h)-y(t)) | x(t)=x, y(t)=y\}$$

$$= \lim_{h\downarrow 0} \frac{h}{2} E\left\{\left(\mu(1-x^2)y - x\right)y + \left(\frac{w(t+h)-w(t)}{h}\right)y\right\}$$

$$= 0$$

Thus, the Kolmogorov equation of $f = f(s,\xi,\eta;t,x,y)$ $(s < t)$, the conditional probability density function of $(x(t),y(t))$ given $(x(s),y(s)) = (\xi,\eta)$, is given by

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial y^2} - A_1\frac{\partial f}{\partial x} - A_2\frac{\partial f}{\partial y} - Cf, \quad (4.3)$$

where $C = \partial A_2/\partial y = \mu(1 - x^2)$.

If we set

$$f = g(s,\xi,\eta;t,x,y) + \alpha(x,y)h(s,\xi,\eta;t,x), \quad (4.4)$$

from (3.7) we have

$$= \exp\{2B(y)\}\int_\eta^y \exp\{-2B(z)\}dz, \quad (4.5)$$

where

$$B(y) = \frac{1}{\sigma^2}\int^y A_2 dy = \frac{1}{2\sigma^2}(\mu(1 - x^2)y - 2xy),$$

and the characteristic curve for $\beta = \alpha h$ is given by

$$\frac{dt}{1} = \frac{dx}{y} = \frac{dy}{0} = \frac{d\beta}{-y\frac{\partial g}{\partial x}} \quad (4.6)$$

consequently

$$y = y_0, \quad x = x_0 + y_0 t$$

$$\beta = \beta_0 - y_0\int_s^t \frac{\partial}{\partial x} g(s,\xi,\eta;\tau, x_0 + y_0\tau, y_0)d\tau \quad (4.7)$$

where x_0, y_0 and β_0 are arbitrary constants. According to the initial condition (3.15), $x_0 = \xi$, $y_0 = \eta$ and $\beta_0 = 0$.

Now, the differential equation of $u = u(s,\xi,\eta;t,x,y)$ derived from

$$g = u\exp(B(y))$$

$$= u\exp\left(\frac{1}{2\sigma^2}(\mu(1 - x^2)y^2 - 2xy)\right) \quad (4.8)$$

becomes

$$L(u) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial y^2} - q(x,y)u - \frac{\partial u}{\partial t} = 0, \quad (4.9)$$

where

$$q = q(x,y) = \frac{1}{2}\left(\frac{1}{\sigma^2}A_2^2 + C\right)$$

$$= \frac{1}{2\sigma^2}(\mu(1 - x^2)y - x)^2 + \mu\sigma^2(1 - x^2)) \quad (4.10)$$

The fundamental solution of (4.9), which satisfies the initial condition (3.17), is expressed as follows.

$$u(s,\xi,\eta;t,x,y)$$

$$= Z(s,\eta;t,y) + \int_s^t d\tau\int_{-\infty}^{\infty} p(s,\eta;\tau,\zeta)Z(\tau,\zeta;t,y)d\zeta \quad (4.11)$$

where p depends also on x implicitly and

$$Z(s,\eta;t,y) = \frac{1}{\sqrt{2\pi(t-s)\sigma^2}}\exp\left\{-\frac{(y-\eta)^2}{2(t-s)\sigma^2}\right\} \quad (s<t) \quad (4.12)$$

Since $L(u) = 0$, p must satisfy the following integral equation

$$p(s,\eta;t,y) = -qZ(s,\eta;t,y) - q\int_s^t d\tau\int_{-\infty}^{\infty} p(s,\eta;\tau,\zeta)Z(\tau,\zeta;t,y)d\zeta \quad (4.13)$$

This integral equation can be solved by successive substitution.

Since $p = -qu$, p must have opesite sign to the sign of q, so that $u \geq 0$ and consequently $g \geq 0$. In the domain of (s,ξ,η,t,x,y) for which this condition is satisfied, we can get the solution f.

More general degenerate case may be treated similarly to the above example.

5. A METHOD OF SUCCESSIVE APPROXIMATION

Again, as an example, we deal with the stochastic van der Pol equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = \dot{w} \quad (5.1)$$

Let the solution x(t) be of the form

$$x(t) = x_0 + \int_0^t ((t-\tau) + a_2(t-\tau)^2 + a_3(t-\tau)^3 + \ldots)dw(\tau), \quad (5.2)$$

where $a_i\ (i \geq 2)$ are stochastic processes.

Regarding a_i as numerical constants and substituting (5.2) into (5.1), we have formally

$$\ddot{x} - \mu\dot{x} + x$$

$$= \dot{w} + (2a_2 - \mu)(w(t) - w(0)) + x_0$$

$$+ \sum_{n=1}^{\infty}\{(n+1)(n+2)a_{n+2} - \mu(n+1)a_{n+1} + a_n\}\int_0^t (t-\tau)^n dw(\tau)$$

$$\mu x^2\dot{x}$$

$$= \sum_{n=0}^{\infty}\mu(n+1)a_{n+1}\left(\sum_{i=1}^{\infty} a_i\int_0^t (t-\tau')^i dw(\tau')\right)^2\int_0^t (t-\tau)^n dw(\tau),$$

where $a_1 = 1$. Therefore if

$$a_2 = \mu/2 - x_0/(w(t) - w(0))$$

$$(n+1)(n+2)a_{n+2} - (n+1)\left\{1 - \left(\sum_{i=1}^{\infty} a_i\int_0^t (t-\tau)^i dw(\tau)\right)\right\}a_{n+1} + a_n = 0 \quad (n = 1,2,\ldots) \quad (5.3)$$

then (5.1) is satisfied by (5.2). Accordingly, we set

$$x_{k+1}(t) = \sum_{n=1}^{\infty} a_n^{(k)}\int_0^t (t-\tau)^n dw(\tau) + x_0 \quad (k = 0,1,2,\ldots) \quad (5.4)$$

where $a_n^{(k)}$ $(n = 1,2,\ldots)$ is the solution of finite difference equation

$$(n+1)(n+2)a_{n+2}^{(k)} - \mu(n+1)(1 - \chi_k^2(t))a_{n+1}^{(k)} + a_n^{(k)} = 0$$
$$(n = 1,2,\ldots;\ k = 0,1,\ldots) \tag{5.5}$$

The initial conditions are given by

$$x_0(t) \equiv 0$$
$$a_1^{(k)} \equiv 1,\ a_2^{(k)} \equiv M/2 - x_0/(w(t) - w(0)) = M/2$$
$$(k = 0,1,\ldots) \tag{5.6}$$

Therefore we get

$$a_3^{(0)} = \frac{M^2 - 1}{6}$$
$$a_4^{(0)} = \frac{M^3 - 2M}{24}$$
$$a_5^{(0)} = \frac{M^4 - 3M^2 + 1}{120} \tag{5.7}$$
$$a_6^{(0)} = \frac{M^5 - 4M^3 + 3M}{720}$$
$$\ldots\ldots\ldots\ldots$$

and generally

$$a_3^{(k)} = \frac{1}{6}(M^2(1 - x_k(t)^2) - 1)$$
$$a_4^{(k)} = \frac{1}{24}(M^3(1 - x_k(t)^2)^2 - M((1 - x_k(t)^2) + 1))$$
$$a_5^{(k)} = \frac{1}{120}(M^4(1 - x_k(t)^2)^3 - M^2((1 - x_k(t)^2)^2 + 2(1 - x_k(t)^2) + 1)$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$(k = 0,1,2,\ldots) \tag{5.8}$$

If the right hand side of (5.4) is convergent and $x_k(t)$ converges to a stochastic process $x(t)$ as $k \to \infty$ (in the quadratic mean), then the $x(t)$ is a solution which starts from $x(0) = x_0$.

6. THE MEAN VALUE FUNCTION

Let the mean value function of a stochastic process $x(t)$ be $m(t) = Ex(t)$. For the stochastic van der Pol equation we have

$$\frac{d^2m}{dt^2} - \mu(1 - m^2)\frac{dm}{dt} + m(t) + \frac{\mu}{3}\frac{dm_3}{dt} + \mu\frac{dm}{dt}\sigma^2(t) + \mu m\frac{d\sigma^2}{dt} = 0, \tag{6.1}$$

where $\sigma^2(t) = E(x(t) - m(t))^2$ and $m_3(t) = E(x(t) - m(t))^3$. Therefore, the mean value function $m(t)$ is different from the deterministic solution of van der Pol equation. This is generally true for nonlinear stochastic differential equations.

The author is intending to carry out some numerical experiments on the stochastic van der Pol equation by means of computer.

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STATISTICAL-DYNAMICAL PREDICTIONS BASED ON BOTH INITIAL STATE INFORMATION AND PRIOR PREDICTIONS*

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1. INTRODUCTION

Because of uncertainties in the initial state due to lack of perfectly complete observational data, physical predictions based on this state will necessarily contain uncertainties that can grow with time. Gleeson (1961) describes a procedure for recognizing initial uncertainties and using them to furnish probability predictions in addition to standard dynamical predictions.

It is theoretically possible that prior predictions can supply information useful for reducing uncertainties of an "initial state" which is the basis for another prediction. The present study examines two of these possibilities and outlines a method for incorporating such information, if it is theoretically valuable, into subsequent predictions of a variable and its probability.

The method is illustrated by examples based on Lagrangian and Eulerian techniques of prediction, respectively. For simplicity of presentation, kinematic rather than dynamic models will be used, although direct extension to the latter is quite possible in actual practise.

2. EXAMPLE BASED ON LAGRANGIAN METHOD

Consider the problem of predicting the future position of a unit mass of air travelling at constant speed, u, in a zonal current directed toward the east, x, at a given latitude. The equation of motion in this case is

$$\frac{du}{dt} = 0, \tag{1}$$

representing no speed variation with time, t, following the air parcel. A general solution to (1) is

$$x = x_0 + u_0 (t - t_0) , \tag{2}$$

where x_0, u_0, and t_0 are initial values of position, speed, and time, respectively, and x is position at time t.

To recognize the parcel at times t_0 and t, assume that it maintains a given constant temperature, T, which differs from that of any other air parcel on the x axis in the immediate vicinity. These conditions are summarized in the conservation equation:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0 \tag{3}$$

where partial derivatives indicate local rates of change. The prediction problem then essentially reduces to a procedure of locating the initial position, x_0, of the point of intersection of the x axis with an isotherm representing the given temperature, predicting the trajectory of that point using (2), and verifying the prediction by the observed position of the isotherm at time t.

According to the theory of errors, the error in predicted location of the parcel, is given by

$$\Delta x = \Delta x_0 + (t - t_0) \Delta u_0 , \tag{4}$$

where Δx_0 is the initial error in location, and

$$\Delta u_0 = \left[\frac{\partial u}{\partial s}\right]_0 \Delta s \tag{5}$$

is the speed error, represented in (5) by an assumed linear, constant gradient, $(\partial u/\partial s)_0$, and an error, Δs, in spatial location of an analyzed isopleth of wind speed, u_0. (Because t and t_0 are specified, no errors are indicated for them.)

The variance of x is obtained by combining (4) and (5), then squaring and averaging, to give

$$\sigma^2(x) \equiv \overline{(\Delta x)^2} = \sigma^2 (x_0) + (t-t_0)^2 \left[\frac{\partial u}{\partial s}\right]_0^2 \sigma^2(s). \tag{6}$$

The reasonable assumption is made here that Δx_0 and Δs are uncorrelated. According to a network sampling model (Gleeson, 1961), variances of x_0 and s can be evaluated from the following relations:

$$\sigma^2 (x_0) = \sigma^2(s) = 0.056a \tag{7}$$

and

$$a = R/n = \ell^2 \tag{8}$$

where 0.056 is a theoretical constant without dimensions, R is the area of the geographic region for which the synoptic analysis is made and n is the number of observation points (surface stations or instruments aloft, depending on the analysis) in the region. Thus, a can be interpreted as the average unobserved area per observation point, and ℓ (being a characteristic dimension of area a) as an average distance between observations.

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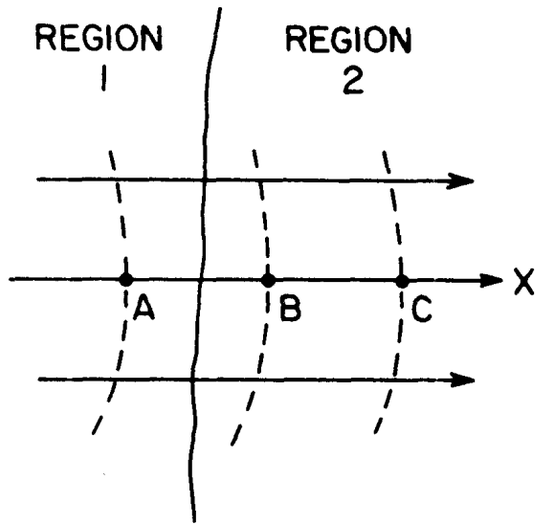

Fig. 1. Successive positions of isotherm (dashed lines) along x axis which extends from region 1 (having dense observations) to region 2 (having sparse observations). Streamlines indicated by arrows.

Substitution of (7) in (6) gives

$$\sigma^2(x) = 0.056a \left[1 + (t - t_0)^2 \left(\frac{\partial u}{\partial s}\right)_0^2\right]. \qquad (9)$$

The right side of (9) can be determined from initial data and analyses. Then, according to the model (Gleeson, 1961), the fiducial probability, P, that predictions made with (2) will lie within a specified interval, ϵ, of the true position, can be calculated from the normal distribution by use of the error function. Thus

$$P(\epsilon) = \mathrm{erf}\left(\frac{\epsilon}{\sigma}\right) \equiv \frac{1}{\sigma\sqrt{2\pi}} \int_{-\epsilon}^{\epsilon} \exp\left[-\frac{(\Delta x)^2}{2\sigma^2}\right] d(\Delta x), \qquad (10)$$

where $\sigma \equiv \sigma(x)$.

As a particular application of the foregoing equations, we now consider two successive 12-hr predictions of the location of an isotherm along the x axis, as depicted schematically in Fig. 1. At the initial time, t_0, of the first prediction, the isotherm is analyzed to be at point A in region 1 where the number of observation points is n_1. Its predicted position after 12 hr is point B in region 2. The second prediction originates at B and terminates at C, also in region 2.

Both regions 1 and 2 are assumed to have equal areas; but region 1 has four times the number of synoptic observation points that region 2 has. Thus, according to (8), if the average distance between observations is 250 miles in region 1, the corresponding distance in region 2 is 500 miles. Quite typically then, the geography represented in Figure 1 might be a land mass lying upwind from the sea.

Table 1. Values of constants and variables in initial and predicted states, for two prediction periods in first example.

	Prediction Periods 0 to 12	12 to 24
Initial State		
t_0(hr)	0	12
ℓ (mi)	250	500
x_0	A	B
u_0(mi/hr)	40	35
$(\partial u/\partial s)_0$(mi/hr/100 mi)	5	8
$\sigma^2(x_0)$(10^2 mi^2)	35.0	140.0
Predicted State		
t (hr)	12	24
x (mi)	B=A+480	C=B+420
$\sigma^2(x)$ (10^2 mi^2)	47.5	268.9
P (200 mi)	.99+	.78

Table 1 shows assumed initial-state and resultant predicted-state values of variables and values of constants in two consecutive predictions. Our main concern here is the possibility of improving the second prediction by use of information available from the first one.

In the last column of the table, note that the predicted displacement of the isotherm (and air parcel) from B to C is 420 miles, and that the predicted probability of displaced isotherms to lie within ± 200 miles of the true location is 78 per cent. According to (6), these prediction values are dependent in part on the variance of initial location of the isotherm at t_0 = 12, given by

$$\sigma^2(x_0) = 14{,}000 \text{ mi}^2, \qquad (11)$$

which is underlined in the table. By contrast, the variance of predicted location of the isotherm at t = 12, is given by

$$\sigma^2(x) = 4{,}750 \text{ mi}^2, \qquad (12)$$

which is also underlined. The relative smallness of the latter variance suggests that the earlier prediction might be useful in reducing uncertainties of the prediction in the 12- to 24-hr period.

The per cent reduction in the variance of the initially-analyzed isotherm location at t_0 = 12, accounted for by the previous prediction, is given by

$$\mathrm{PCR} = \frac{\sigma^2(x_0) - \sigma^2(x)}{\sigma^2(x_0)} = 1 - \frac{\sigma^2(x)}{\sigma^2(x_0)}. \qquad (13)$$

In this case, PCR = .66, from (11) and (12).

To see how the per cent reduction can be useful in revising a subsequent prediction, consider the extreme case in which the earlier prediction is without error. Then the predicted location of the isotherm at t = 12 hr would be as accurate as an errorless observation of temperature made from an instrument at that location at t_0 = 12 hr. One would be justified then in reanalyzing the isotherm field with all synoptic observations of temperature available for that time plus the predicted temperature plotted at

the predicted location. The revised analysis is then the basis for a new prediction of possibly greater accuracy.

In the more realistic case where $\sigma^2(x) > 0$ and $0 < PCR < 1$, the recommended procedure is to plot the temperature at its predicted location and also to plot the PCR (here, .66) nearby, for use by the analyst as a weighting factor for that temperature.

A revision of the probability prediction is also possible by analogous reasoning. The average unobserved area per observation point, is now reinterpreted as an average for two points;

$$a \equiv \frac{a}{1} + \frac{a}{2}, \qquad (14)$$

in the extreme case of perfect prediction. In general when PCR ≤ 1, we can define a smaller area, a*:

$$\frac{a}{1} + \frac{a}{1 + PCR} = a^* . \qquad (15)$$

In the present example, $a^* = (500)^2/1.66$. Also, (2) and (6) can be rewritten as

$$x^* = x_0^* + u_0 (t - t_0) \qquad (16)$$

and

$$\sigma^2(x)^* = 0.056\, a^* + 0.056\, a\, (t-t_0)^2 \left(\frac{\partial u}{\partial \varepsilon}\right)_0^2 , \qquad (17)$$

respectively,
where t_0 = 12 hr, and asterisks indicate revised values. At t = 24 hr, P(200 mi)* = .83, from (10) and (17). This is an improvement over .78 (the last value in Table 1).

Of course, the suggested modifications just described are invalid if the per cent reduction is negative. In that event, $\sigma^2(x) > \sigma^2(x_0)$ which means that no information from the earlier prediction is seen useful in improving the analysis.

3. EXAMPLE BASED ON EULERIAN METHOD

Like the previous example, the present one is based on conservation of temperature for air parcels, and also employs equation (3):

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0 , \qquad (3)$$

where the symbols have the same meanings as before.

Consider a deformation field as shown in Figures 2 and 3 where isotherms (dashed lines) are being carried toward the center by a static flow pattern whose streamlines are depicted by curved arrows. The problem now is to predict future temperatures at an arbitrary point B. (The x axis through this point is along the temperature gradient.)

The prediction equation to be used is a solution to (3) for a single time step:

$$T = T_B - u_B \left(\frac{\partial T}{\partial x}\right)_B (t - t_0), \qquad (18)$$

where T_B, u_B, and $(\partial T/\partial x)_B$ are initial values

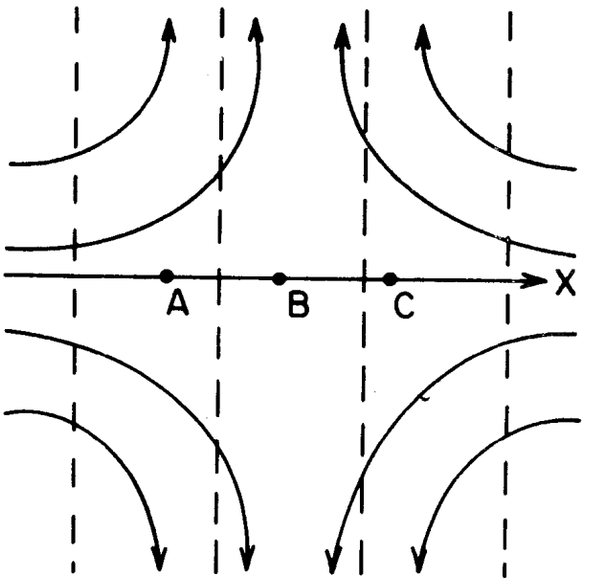

Fig. 2. Static flow pattern at t = 0. Streamlines and isotherms indicated by curved arrows and dashed lines, respectively. Temperature gradient lies along x axis.

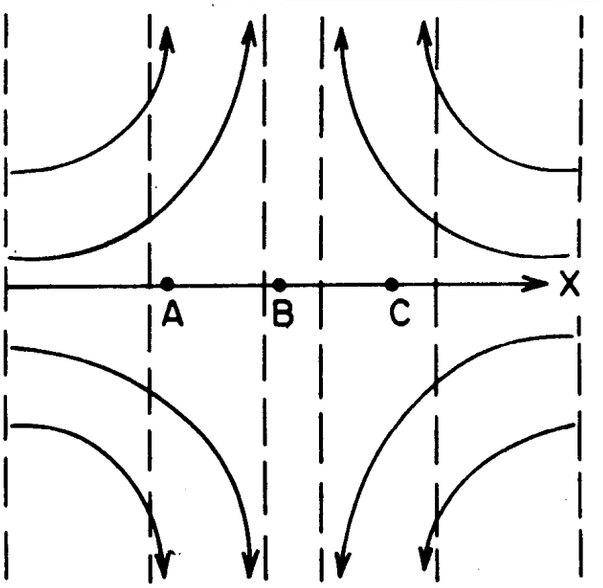

Fig. 3. Static flow pattern at t = 12 hr. See caption to Fig. 2.

obtained from a synoptic analysis at $t = t_0$. An operational version of (18) is obtained by replacing $(\partial T/\partial x)_B$ with a ratio of finite differrences. Thus,

$$T = T_B - u_B \left(\frac{T_C - T_A}{2L}\right) (t - t_0), \qquad (19)$$

where T_A and T_C are initial values of temperature interpolated at points A and C (see Fig. 2), each of which lies at the same specified distance, L, from B.

To study errors in prediction, ΔT, resulting from errors in the initial state, one can

Table 2. Values of constants and variables in initial and predicted states, for two prediction periods in second example.

	Prediction Periods	
	0 to 12	12 to 24
Initial State		
t_0 (hr)	0	12
ℓ (mi)	250	250
L (mi)	300	300
T_A (deg F)	39	30
T_B (deg F)	49	48
T_C (deg F)	59	66
u_A (mi/hr)	22	23
u_B (mi/hr)	2	3
u_C (mi/hr)	-18	-17
$(\partial T/\partial x)_A$ (deg/300 mi)	10	13
$(\partial T/\partial x)_B$ (deg/300 mi)	10	18
$(\partial T/\partial x)_C$ (deg/300 mi)	10	15
$(\partial u/\partial s)_B$ (mi/hr/300 mi)	-20	-20
σ^2 (T_B) (deg^2)	3.9	12.6
Predicted State		
t (hr)	12	24
T (deg F)	48.2	45.8
$\sigma^2(T)$ (deg^2)	6.4	20.8
P(4 deg)	.88	.62

employ the procedure described by Gleeson (1961) and demonstrated in the previous example. This leads to an equation for the variance of T, as follows:

$$\sigma^2(T) \equiv \overline{(\Delta T)^2} = 0.056\, a \left(\frac{\partial T}{\partial x}\right)_B^2 + 0.056a \frac{(t-t_0)^2}{4L^2}\left\{(u_B)^2\left[\left(\frac{\partial T}{\partial x}\right)_A^2 + \left(\frac{\partial T}{\partial x}\right)_C^2\right] + (T_C - T_A)^2 \left(\frac{\partial u}{\partial s}\right)_B^2\right\}, \quad (20)$$

where the first term on the right side represents the initial error in T_B and the other term represents initial errors of T_A, T_C and u_B; a has the same meaning as before; and $(\partial T/\partial x)_i$ and $(\partial u/\partial s)_B$ are gradients to be evaluated at points i and B, respectively, from initial analyses of temperature and the u component of the wind.

As in the previous example, we again consider two successive 12-hr predictions, but now of temperature and its probability. Table 2 displays hypothetical values depicting initial and predicted states for the two 12-hr periods.

Examination of the table shows that at t = 12 hr, the variance of the predicted temperature, 6.4 deg^2, is less than the variance of the analyzed temperature, 12.7 deg^2, at point B. The per cent reduction of the latter by use of the former is given by

$$PCR = 1 - \frac{\sigma^2(T)}{\sigma^2(T_B)} = 1 - \frac{6.4}{12.6} = .49. \quad (21)$$

Accordingly, from (15) and Table 2,

$$a^* = \frac{a}{1 + PCR} = \frac{\ell^2}{1 + PCR} = \frac{(250)^2}{1 + .49} = 41{,}900 \text{ mi}^2. \quad (22)$$

If the initial field of temperature at t_0 = 12 hr is reanalyzed in the light of the additional information about temperature provided by the 0- to 12-hr prediction (as described in the discussion of the first example), the prediction equations for the 12- to 24-hr period become

$$T^* = T_B^* - u_B \left(\frac{\partial T}{\partial x}\right)_B (t - t_0) \quad (23)$$

and

$$\sigma^2(T)^* = 0.056\, a^* \left(\frac{\partial T}{\partial x}\right)_B^2 + 0.056\, a \frac{(t-t_0)^2}{4L^2}\left\{(u_B)^2\left[\left(\frac{\partial T}{\partial x}\right)_A^2 + \left(\frac{\partial T}{\partial x}\right)_C^2\right] + (T_C - T_A)^2 \left(\frac{\partial u}{\partial s}\right)_B\right\}, \quad (24)$$

from (18) and (20), respectively, where asterisks indicate quantities whose values have been revised, as before.

By use of (22), (24), the error function, and appropriate values from Table 2, it can be determined that the revised probability for temperature predictions at t = 24 hr to fall within four degrees of the true value, is 67 per cent. This improvement over the unrevised value, 62 per cent (shown in the table), is a measure of the theoretical improvement in prediction made possible by information from the previous prediction.

4. DISCUSSION

In the two examples above, no account was taken of errors resulting from use of oversimplified prediction equations. Thus, because (2) does not recognize accelerations and (18) does not provide realistically for an increased temperature gradient with time (although this was not critically important since the wind speed was only 2 or 3 mi/hr), the prediction probabilities that were calculated must be regarded as upper limits to more realistic values that could be obtained with better physical models. But even then these more realistic probabilities would themselves be deemed upper limiting values because every model is an oversimplification to some extent.

The technique described in this study is readily extendable to complex dynamic models. Variances can be formulated in these models, and also predicted. It is then possible to make theoretical improvements in predictions of the dynamical variables and their associated probabilities. Whether such improvements are realistic must, of course, be decided ultimately in tests against actual observations.

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STOCHASTIC DYNAMIC PREDICTION: THE ENERGETICS OF UNCERTAINTY AND THE QUESTION OF CLOSURE

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1. INTRODUCTION

The numerical prediction of weather is perhaps the most unique use of the mathematical initial value problem. More than in any other physical science, an attempt is made to perform the most with the least to work with. The domain of interest is the entire earth's atmosphere. Yet, how well are the initial values known?

This paper is concerned with physical uncertainties in any initial value problem, but the weather prediction problem affords the best means to express the ideas presented.

2. THE STOCHASTIC DYNAMIC EQUATIONS

Following Lorenz (1963), the deterministic prognostic form of the hydrodynamic equations can be written in the general form

$$\dot{X}_i = \sum_{p,q} a_{ipq} X_p X_q - \sum_p b_{ip} X_p + c_i \qquad (1)$$

where: the $X_i (i=1,2,\ldots,N)$ are the N dependent variables describing the system, p and q are dummy indices, $(\dot{})$ refers to a time derivative, and the a's, b's and c's are constant coefficients describing non-linear effects and forces acting on the system. The N variables form an N-dimensional phase space whose coordinates are $X_1,\ldots,X_N$. Each point in the phase space represents a possible instantaneous state of the system. The initial state of the atmosphere is then represented by a single point and the deterministic forecast gives the trajectory of that point in phase space. In Fig. 1, the N-dimensional phase space is represented by any two dimensions. Point S is the initial position of a single point in phase space and S' is a point on the deterministic trajectory of that same point at a latter time t_1.

In view of the many limitations in the initial atmospheric dependent variables, it seems logical to express the initial conditions in terms of a probability distribution similar to the probability function in quantum mechanics as advocated by Gleeson (1968). This would mean considering an infinite ensemble of initial states in phase space with relative frequencies within the ensemble proportional to the probability densities. This approach was considered by Epstein (1969) and called the stochastic dynamic method of prediction. A stochastic model would specify the complete joint probability distribution of all the variables at each point in time, and the whole process conceived as a continuous development in time would be a stochastic method.

Consider the symmetric ensemble of points in phase space represented in Fig. 1 as a circle. With E as the initial mean of the ensemble of points, let E coincide with the initial deterministic state S. The evolution of the ensemble in time would then be computed by the stochastic dynamic equations. A hypothetical solution of the ensemble is shown at time t_1 with E' as the mean of the ensemble trajectories. Because of the non-linearity of the original deterministic equations, a time is eventually reached when E' and S' will differ as indicated in Fig. 1.

A truncated form of the formally exact stochastic equation set corresponding to (1) was given by Epstein (1969) as:

$$\dot{u}_i = \sum_{p,q} a_{ipq}(u_p u_q + \sigma_{pq}) - \sum_p b_{ip} u_p + c_i \qquad (2)$$

$$\dot{\sigma}_{ij} = \sum_{p,q} \{a_{ipq}(u_p \sigma_{jq} + u_q \sigma_{jp} + \tau_{jpq}) + a_{jpq}(u_p \sigma_{iq} + u_q \sigma_{ip} + \tau_{ipq})\} - \sum_p [b_{ip}\sigma_{jp} + b_{jp}\sigma_{ip}] \qquad (3)$$

where: u_i is the mean of X_i
σ_{ij} is the instantaneous second moment about the mean
τ_{ijk} is the instantaneous third moment about the mean.

It is evident that (2) and (3) do not form a closed set of equations - there is no prognostic equation for the τ. One can derive an equation for the third moments (cf. Fleming, 1970, hereafter referred to as F), but it involves fourth moments. The complete stochastic dynamic equations form an infinite unclosed set of coupled equations. The equations are thus unsolvable until an algorithm is devised to close the system. The algorithm or closure scheme chosen by Epstein was just to drop the third moment terms in (3). The implications of this technique and other closure methods are discussed below where the closure problem is studied more fully. It will be mentioned here, though, that even if the third