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Contributions (Alphabetical within each field)

Selected Works of Prof. YE Duzheng

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On Energy Dispersion in the Atmosphere

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Abstract

In this paper the energy propagation through dispersive waves in four atmospheric models is investigated. These waves are characterized by an approximate geostrophic balance. Diagrams showing the relation between group velocity, wave velocity, and wave length in the four types of atmosphere are given. It is found that:

1. In each of the four models there is always a range of wave length for which group velocity is larger than wave velocity, so that new waves can be formed ahead of initial waves.

2. Both divergence or convergence and horizontal solenoids give rise to waves with negative group velocity. But only in the presence of solenoids is there a range of wave length for which the speed of propagation of energy upstream is greater than the wave speed in the same direction. This means that only the horizontal solenoids make possible the formation of new waves upstream.

A graphical method is used to construct the distribution of phase resulting from an instantaneous point source disturbance. The phase curves are constructed for each of the four atmospheric models.

Two applications of the theory are made. The formation of a new trough over North Anerica following an intense cyclogenesis in the Gulf of Alaska is interpreted as a result of dispersion from a continuous point source of cyclonic relative vorticity into a previously straight westerly current. Computations show a pressure rise next to the region of cyclogenesis downstream and a trough farther to the east.

The blocking action observed in the west-wind belt is explained by the dispersion of an initial solitary wave. Calculations indicate that the life time of a "blocking action" is longer in high latitudes than in low latitudes; this is in agreement with observation.

I. Introduction

Margules (1905) first attempted to explain the origin of storm energy by conversion of potential energy into kinetic energy in a closed system. He obtained wind velocities of the correct order of magnitude to account for the kinetic energy of the storm from the release of potential energy of two air masses of different temperature standing side by side in a confined system. However, it is frequently observed that the kinetic energy and solenoidal energy increase at the same time in a cyclone. An excellent example has been given recently by Cressman (1948). A statistical investigation was made by Carson①, who found that an increase of the intensity of the solenoid field in the middle troposphere accompanied deepening in about 80 percent of the cases investigated. There exists, in fact, a considerable amount of evidence

① Carson, J. E., 1948: The variation of horizontal solenoidal concentration in the middle and low troposphere during cyclone formation, Master's thesis, University of Chicago.

that strongly suggests the necessity of rejecting or modifying the concept of 'internal' or 'localized' transformations of the energy of certain characteristic atmospheric circulations, and that points to the need for investigating systematically the mechanisms of energy *transfer* in the atmosphere.

One such mechanism, and the most apparent, is the advective transport of energy, with the speed of the prevailing current. But energy can be transmitted without the help of advection. This paper is devoted to a study of the nonadvective transport of energy due to the rapid adjustment between pressure field and velocity field, a process which is a consequence mainly of the earth's rotation.

Namias and Clapp (1944) has described the chain of events appearing downstream after sudden formation of an intense low in the Gulf of Alaska. The subsequent changes in circulation, which certainly cannot be explained by air-mass transport, provide an illustration of the process of rapid adjustment between pressure and velocity fields; we shall refer to the energy transmission resulting from this process as a *dispersive* transfer of energy.

The importance of dispersive processes in the atmosphere has recently been brought to the attention of meteorologists by Rossby (1945). Earlier (Rossby, 1936) he had advanced the hypothesis that the horizontal pressure gradients observed in current systems of the atmosphere and the ocean to a large extent must be interpreted as reactions to the Coriolis forces impressed upon these systems by the rotation of the earth. He then showed (Rossby, 1937, 1938) that any sudden local addition of momentum to a rotating fluid body will initiate some type of inertia oscillation of that body. A small part of the initial energy goes into inertia oscillations, but the final equilibrium configuration set up between the pressure gradient and the current system is established in only a few pendulum hours. Part of the initial energy also goes into outlying portions of the fluid through dispersion by gravitational waves, as was demonstrated later by Cahn (1945), who gave a complete mathematical analysis of the problem. Through Rossby's work it is clear that a velocity field can result in a pressure field which in turn affects the velocity field. This is the physical mechanism of the dispersion process in the atmosphere.

Generally speaking, in a dispersive process the speed of propagation of energy is different from that of the prevailing current. The energy is propagated with the group velocity which, in the one-dimensional case, can be expressed in terms of phase velocity or wave length. A simple synoptic manifestation of dispersion is the common observation that a downstream pressure rise(or fall) usually follows an upstream pressure fall(or rise).

In this paper we shall investigate the role of dispersion in certain observed phenomena in the atmosphere. Before entering into a detailed discussion the following point should be noted. Dispersion arises from unbalanced motion. Once equilibrium is reached dispersion will cease. As pointed out above, dispersion essentially arises from the *tendency* for mutual adjustments between the pressure and velocity fields, but when these two fields are in balance dispersion ceases.

II. Wave length, phase velocity, and group velocity in different atmospheric models

The close relation between energy propagation and group velocity may be explained as follows. By group velocity we mean the velocity of a group of waves as a whole. The individual waves which compose this group may advance through it or may be left behind it, while the place of the individual wave in the group is occupied in succession by other waves. If we assume that the energy of wave motion is concentrated at the crests of the wave groups, the velocity of propagation of energy will certainly be associated with the group velocity. Group velocity, phase velocity, and wave length are related by the well-known formula

$$G = C - L\mathrm{d}C/\mathrm{d}L, \tag{1}$$

where G is the group velocity, C the phase velocity, and L the wave length.

In the atmosphere and the ocean almost all forms of wave motion are dispersive, i. e., the phase velocity of individual waves depends on their wave length. The speed of energy propagation in such systems is usually different from that of the individual waves. Whether new waves will be formed ahead or in the rear of an initial wave train, or whether an increasingly long trailing wave train may develop, depends on the speed of propagation of energy. Since the energy propagation depends on group velocity which in turn is determined by wave length and phase velocity, a discussion of these three elements in different atmospheric models is desirable.

We shall consider in turn four atmospheric models:

Model A: A uniform, incompressible atmosphere of infinite depth, or of finite depth with a rigid cover.

Model B: A uniform, incompressible atmosphere with a free surface.

Model C: An incompressible atmosphere with a uniform north-south density gradient and a rigid cover.

Model D: An incompressible atmosphere with a uniform north-south density gradient and a free surface.

In each of these models there is a basic current with uniform and constant velocity U.

Model A. —The first model is the simplest and has been studied by Rossby et al. (1939) and later by Haurwitz (1940). In this atmosphere divergence and convergence are absent, and for small one-dimensional motion the vorticity equation may be written as follows:

$$\frac{\partial^2 v}{\partial x \partial t} + U\frac{\partial^2 v}{\partial x^2} + \beta v = 0, \tag{2A}$$

where v is the perturbation velocity transverse to the x-direction, and $\frac{1}{2}\beta$ represents the rate of change northward of the vertical component of the earth's angular velocity. Assuming a solution of the form $e^{i(kx-\nu t)}$ the frequency equation from (2A) is $\nu/k = U - \beta/k^2$, where ν is the frequency and k the wave number, or

$$C = U - \beta L^2, \tag{3A}$$

where $C = \nu/k$ is the phase velocity and $L = 1/k$ the wave length①. The group velocity, from (1), is then

$$G = U + \beta L^2. \tag{4A}$$

In this atmosphere the group velocity G is always positive and larger than the phase velocity C. It has a minimum value U and increases with wave length (Figure 1). Since G is always positive and larger than U, the energy is always propagated downstream with a speed larger than C. In this system new waves will be formed ahead of initial waves.

Model B. —Removing the rigid cover from the first model we obtain the second type of atmosphere. In this model divergence and convergence are possible. The vorticity equation will then take the form

$$\frac{d}{dt}\left(\frac{f+\zeta}{D}\right) = 0,$$

where D is the depth of the atmosphere, ζ the relative vorticity, and f the Coriolis parameter. Assuming the north-south motion to be geostrophic② and perturbations to be small, the vorticity equation may be written in the expanded form

$$\frac{\partial^3 D'}{\partial x^2 \partial t} + U\frac{\partial^3 D'}{\partial x^3} + \beta\frac{\partial D'}{\partial x} - \lambda^2\frac{\partial D'}{\partial t} = 0, \tag{2B}$$

where $\lambda^2 \equiv f^2/gD_0$, D_0 is the undisturbed depth, and D' the deformation of the free surface. The wave velocity is

$$C = \frac{U - \beta L^2}{1 + \lambda^2 L^2}, \tag{3B}$$

and the group velocity may be obtained by substituting (3B) in (1), the result being

$$G = \frac{U + \beta L^2 + 2\lambda^2 L^2 C}{1 + \lambda^2 L^2}. \tag{4B}$$

The graphs for C and G are shown in Figure 2. The solid curve is the wave velocity C, obtained from (3B); the dashed curve is the group velocity G, from (4B). Both C and G have the same lower limit $-\beta/\lambda^2$, corresponding to infinite wave length. Thus the phase velocity C decreases with increasing wave length asymptotically to the limiting value $-\beta/\lambda^2$, not as in the nondivergent case where it decreases without limit. At the wave length $L_s \equiv \sqrt{U/\beta}$ corresponding to zero phase velocity,

$$G = G_s \equiv \frac{2U}{1 + \lambda^2 L_s^2}.$$

The group velocity G first increases with wave length until the latter reaches a value where $d^2C/dL^2 = 0$; at this point

① For simplicity in notation we shall depart from the usual convention of defining the wave length as $2\pi/k$. Thus L, as used throughout this paper, is equal to the 'conventional' wave length divided by 2π.

② The geostrophic assumption results simply in the omission of long gravitational waves, in which we are not interested (Charney, 1947). The approximation of using the geostrophic relation after the vorticity equation has been written down does not mean that acceleration terms are neglected. The analysis of phase and group velocity in model B has been presented in Prof. Rossby's lectures and is reproduced here with his kind permission.

$$G = G_{max} \equiv \frac{1}{8}(9U + \beta\lambda^{-2}),$$

$$C = \frac{3}{16}(3U - \beta\lambda^{-2}).$$

After this wave length is reached, the group velocity decreases first rapidly and then asymptotically to the value $-\beta/\lambda^2$.

A striking difference between the nondivergent case and the present one in which convergence and divergence appear, is that in the latter the group velocity can be negative. This means that in divergent motion energy can be propagated upstream. Specific examples of this process will be given later. From Figure 2 it is seen that the group velocity is always larger than the phase velocity. Hence new waves may be formed downstream ahead of the initial disturbance, but not upstream in the rear of the initial disturbance, though energy can be propagated upstream. Since $G = C$ for $L \to \infty$, energy will be propagated with the speed of these long retrogressive waves which may therefore retrograde without changing in intensity.

The particular interest of this model lies in the second group of waves, long and retrogressive with negative group velocity. The limiting value of this negative phase or group velocity is $-\beta/\lambda^2$, as already noted. These very long waves are nondispersive waves and move with practically constant relative vorticity. The change of vorticity due to latitude is balanced by the effect of divergence, as is easily seen from (2B). For very long wave length the first two terms in (2B) may be neglected since they are inversely proportional to the square or cube of wave length. Then the remaining two terms—expressing respectively the vorticity change produced by latitude variation and by divergence—must balance each other:

$$\frac{\partial D'}{\partial t} - \beta\lambda^{-2}\frac{\partial D'}{\partial x} = 0,$$

giving a phase velocity equal to $-\beta/\lambda^2$. Taking representative values in middle latitudes we find that $-\beta/\lambda^2$ is of the order of 100 m s^{-1}. However, this unrealistically large value can never occur in the atmosphere due to the fact that the wave length is limited by the circumference of the earth. ①

Model C. ②—To investigate the effect of horizontal solenoids in purely horizontal motion we may imagine an infinitely deep atmosphere which has uniform density vertically and a uniform north-south density gradient. In this model convergence and divergence are absent,

① The reduction of the unrealistically large value of the maximum negative group velocity was suggested by J. Charney.

② It should be noted that the two following atmospherical models, C and D, are not selfconsistent. In these two models the presence of horizontal stratification implies that the basic current U must increase with height, so the assumption of constant U is not justified. However, in the present paper we are concerned only with one level. As long as vertical motion does not appear (in model C) there may not be interference between upper and lower layers and thus it may be justified to deal with one level only; but in model D, vertical motion will be present and the analysis of this model is therefore only an approximation.

and the vorticity equation is

$$\frac{\partial^2 v}{\partial x \partial t} + U \frac{\partial^2 v}{\partial x^2} + \beta v = -\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} + \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x} \equiv n,$$

α being the specific volume, p the pressure, and n the number of solenoids per unit area. For the purpose of evaluating n it is reasonable to substitute the geostrophic wind relation①; one then obtains

$$\frac{\partial^2 v}{\partial x \partial t} + U \frac{\partial^2 v}{\partial x^2} + \beta v = f\left(\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{\partial q}{\partial t}\right),$$

where $q \equiv \ln \alpha$. Since the motion is incompressible and horizontal we have

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial q}{\partial t} + U\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0.$$

Writing $s \equiv -\partial q/\partial y$ for the undisturbed value of the horizontal stratification, and assuming perturbations independent of y, the vorticity equation becomes

$$\frac{\partial^3 v}{\partial x \partial t^2} + 2U\frac{\partial^3 v}{\partial x^2 \partial t} + U^2\frac{\partial^3 v}{\partial x^3} + (\beta + fs)\frac{\partial v}{\partial t} + \beta U\frac{\partial v}{\partial x} = 0. \tag{2C}$$

Assuming a solution of the form $e^{i(kx-\nu t)}$ the wave velocity $C = \nu/k$ is found to be

$$C = U - \frac{1}{2}\sigma L^2(1 \pm l), \tag{3C}$$

where $\sigma \equiv \beta + fs$, $l \equiv (1 - L_c^2/L^2)^{\frac{1}{2}}$, and $L_c \equiv 2\sqrt{fsU}/\sigma$. This is exactly the same result as derived by Jaw (1946) using a different approach. The group velocity corresponding to (3C) is

$$G = U + \frac{1}{2}\sigma L^2(1 \pm l^{-1}). \tag{4C}$$

Equation (3C) reveals immediately one important aspect of this model, i. e., there exists a critical wave length L_c below which C becomes complex. This implies the existence of unstable waves in this type of atmosphere. The instability increases with s and U. (It is easy to see that the instability increases with s. It increases with U because the isobaric slope is proportional to U and so also is the number of horizontal solenoids per unit area.) These unstable waves are relatively short and progressive.

There are two solutions for C corresponding to the two signs in the parenthesis of (3C). For $L < L_c$, the positive sign represents a damped wave system, while the negative sign represents unstable waves. Whether we choose the positive or negative sign (or both) is dependent on the type of boundary or initial conditions imposed on the disturbance. The wave system may therefore be unstable for certain types of disturbances and damped for other types. When $L > L_c$ the waves are neutral, for either choice of sign in (3C).

Damped or unstable waves. For wave lengths smaller than the critical one the disturbance will be either damped or unstable. In either case C is a complex number. The wave velocity will be the real part of (3C)

$$C = U - \frac{1}{2}\sigma L^2, \quad \text{for} \quad L < L_c,$$

① See footnote of ① in Page 6.

and the group velocity is

$$G = U + \frac{1}{2}\sigma L^2, \qquad \text{for} \qquad L < L_c.$$

Thus G is always larger than C for this range of wave length. At the critical wave length,

$$G \to G_c \equiv U(1 + 2fs/\sigma), \qquad \text{for} \qquad L \to L_c.$$

Neutral waves. When the wave length is larger than the critical value, we may take either the positive sign or the negative sign or both in (3C), depending on the nature of the disturbance. For simplicity we will consider the first two possibilities only.

Taking the positive sign in (3C), it is readily seen that

$$C \to C_c \equiv U(1 - 2fs/\sigma), \qquad \text{for} \qquad L \to L_c;$$

and $C \to -\infty$, for $L \to \infty$. Differentiating (3C) with respect to L we have (for $L > L_c$)

$$\frac{dC}{dL} = -\frac{1}{2}\sigma L(l + 2 + l^{-1}),$$

which approaches negative infinity as $L \to L_c$, since $l \to 0$. The relation between group velocity and wave length is given by (4C) with plus sign. At the stationary wave length L_s, which can be seen to be equal to $\sqrt{U/\beta}$,

$$G \to G_s \equiv 2\beta U/(\beta - fs), \qquad \text{for} \qquad L \to L_s.$$

Also, $G \to \infty$ as $L \to L_c$ or $L \to \infty$. Hence, the group velocity will first decrease with increasing wave length until the minimum value $G_{min} \equiv U(1 + 8fs/\sigma)$ is reached, and then increases without limit. The curves for wave velocity and group velocity are shown in Figure 4, The solid curve represents wave velocity and the dashed curve group velocity; both curves have a distinct discontinuity at $L = L_c$.

Now let us consider the negative sign in the parenthesis of (3C), for the case $L > L_c$. It is readily seen that

$$C \to C_c \equiv U(1 - 2fs/\sigma), \qquad \text{for} \qquad L \to L_c,$$

$$C \to C_\infty \equiv U(1 - fs/\sigma), \qquad \text{for} \qquad L \to \infty;$$

Further, $dC/dL \to \infty$ as $L \to L_c$. Thus C increases very rapidly from C_c and then asymptotically to C_∞ as wave length further increases (Figure 3).

The expression for group velocity is given by (4C) with the minus sign, from which,

$$G \to G_\infty \equiv U(1 - fs/\sigma), \qquad \text{for} \qquad L \to \infty,$$

and $G \to -\infty$ as $L \to L_c$. From the curve for C it is seen that C increases first very rapidly and then asymptotically to G_∞. The curves for C and G are shown in Figure 3, solid line for C and dashed line for G. Both curves are discontinuous at $L = L_c$. One interesting point is that no stationary wave length exists for this case; all waves are progressive. Since $G \to C$ as L increases, long individual waves can travel downstream without changing intensity.

For the first type of disturbance-positive sign in (3C), damped waves for $L < L_c$—energy can be propagated only downstream. Since group velocity is larger than wave velocity new waves can be formed ahead of initial wave trains. For the second type of disturbance—negative sign in (3C), unstable waves for $L < L_c$—energy can be propagated in both directions. Since the speed of energy propagation in either direction can be larger in magnitude than the wave speed, new waves may be formed both downstream and upstream.

Model D — We come now to the fourth and last model, in which both horizontal solenoids and divergence or convergence are operating. The vorticity equation in this case may be written:

$$\frac{\mathrm{d}}{\mathrm{d}t}(f+\zeta)=\frac{f+\zeta}{D}\frac{\mathrm{d}D}{\mathrm{d}t}+n.$$

For small motions we have

$$\frac{\partial^2 v}{\partial x\partial t}+U\frac{\partial^2 v}{\partial x^2}+\beta v+f\frac{\partial q}{\partial t}-\frac{f}{D_0}\frac{\partial D'}{\partial t}=0.$$

In writing the above equation the geostrophic-wind relation was used in evaluating the divergence and solenoid terms of the vorticity equation. ①

The geostrophic-wind equation for the north-south velocity component is

$$v=\frac{g}{f}\frac{\partial D}{\partial x},$$

and the condition for incompressibility is

$$\frac{\partial q}{\partial t}+U\frac{\partial q}{\partial x}=sv.$$

Through elimination of D' and q we have

$$\frac{\partial^4 v}{\partial x^2\partial t^2}+2U\frac{\partial^4 v}{\partial x^3\partial t}+U^2\frac{\partial^4 v}{\partial x^4}-\lambda^2\frac{\partial^2 v}{\partial t^2}+\beta U\frac{\partial^2 v}{\partial x^2}+b\frac{\partial^2 v}{\partial x\partial t}=0, \tag{2D}$$

where $\lambda^2\equiv f^2/gD_0$ and $b\equiv\beta+fs-\lambda^2U$. Assuming a disturbance of wave form $e^{i(kx-\nu t)}$ the phase speed C is found to be

$$C=\frac{U-\frac{1}{2}L^2(b\pm la)}{1+\lambda^2L^2}, \tag{3D}$$

where $l\equiv(1-L_c^2/L^2)^{1/2}$, $a\equiv(b^2+4\beta\lambda^2U)^{1/2}$, and $L_c\equiv2\sqrt{fsU}/a$ is the critical wave length below which C becomes complex. It can be verified that (3D) reduces to Equation (3B) by putting $s=0$ and goes to (3C) by putting $\lambda^2=0$ (or $gD_0\rightarrow\infty$). The group velocity corresponding to (3D) is

$$G=\frac{U+\frac{1}{2}L^2(b\pm l^{-1}a)+2\lambda^2L^2C}{1+\lambda^2L^2}. \tag{4D}$$

Damped or unstable waves. The wave velocity and group velocity have a discontinuity at $L=L_c$ as in model C. When $L<L_c$ we may take the wave velocity to be the real part of (3D) as before:

$$C=\frac{U-\frac{1}{2}bL^2}{1+\lambda^2L^2}.$$

It decreases with increase of L. For the group velocity we have, from (3D),

$$G=\frac{U+\frac{1}{2}bL^2+2\lambda^2L^2C}{1+\lambda^2L^2}.$$

① See fotnote of ① in Page 6.

This is always positive and larger than C.

Neutral waves. For $L > L_c$ we take first the case of positive sign in (3D). As L increases, C will first decrease very rapidly and then asymptotically to its limiting negative value; thus

$$C \to C_\infty \equiv -\frac{1}{2}\lambda^{-2}(b+a), \quad \text{for} \quad L \to \infty.$$

The group velocity (plus sign in (4D)) approaches positive infinity for $L \to L_c$, and decreases very sharply to a minimum value at approximately $L = 0.9L_c$. It increases again from this wave length to a maximum value at approximately $L = 2.3L_c$ and then decreases to its asymptotic value $G_\infty = C_\infty$ for $L \to \infty$. Figure 5 shows the curves for C and G in relation to L.

Taking the negative sign in (3D), we can see that $dC/dL \to \infty$ as $L \to L_c$ and that

$$C \to C_\infty \equiv -\frac{1}{2}\lambda^{-2}(b-a), \quad \text{for} \quad L \to \infty.$$

Thus, C first increases rapidly and asymptotically to the limiting value C_∞. The group velocity (minus sign in (4D)) approaches negative infinity as $L \to L_c$ and grows very rapidly to its maximum value (which is only very slightly higher than C_∞) and then gradually falls off to a limiting value $G_\infty = C_\infty$ as $L \to \infty$ (Figure 6).

In the first case of this model—positive sign in (3D), damped waves for $L < L_c$—new waves can be formed both upstream and downstream. Since $G \to C > 0$ as L increases, long progressive waves can travel without changing intensity. In the second case—negative sign in (3D), unstable waves for $L < L_c$—new waves can be formed only downstream. In this case energy can be propagated upstream, but only for very long waves can energy travel with the individual waves, which then will not change in intensity.

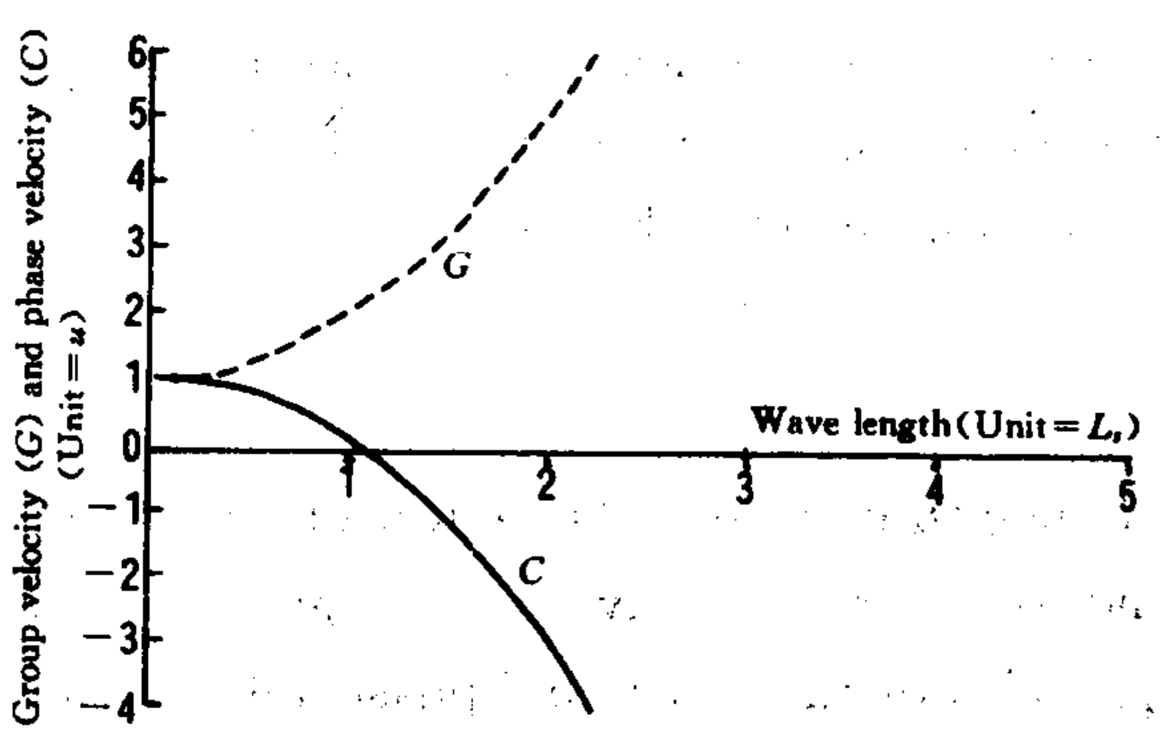

Figure 1 Group velocity (broken line) and phase velocity (solid line) as a function of wave length in model A.

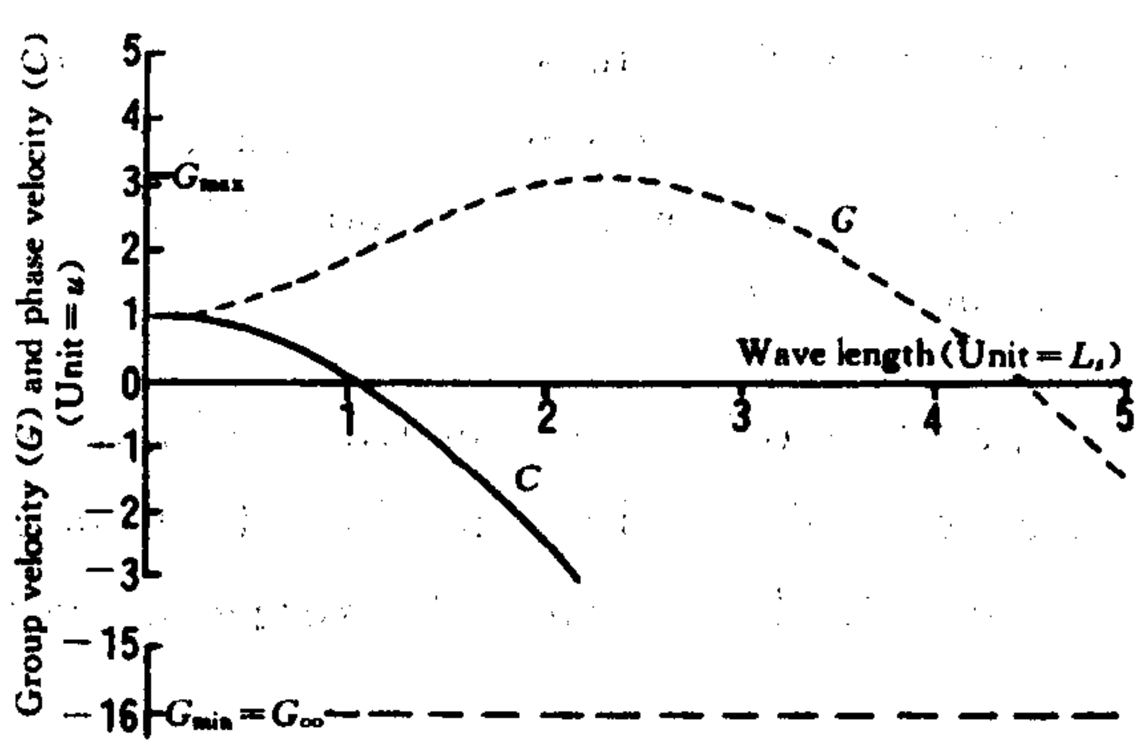

Figure 2 Group velocity and phase velocity as a function of wave length in model B. Both curves are limited by the horizontal asymptotic line $G_\infty = G_{min}$.

It is interesting to compare Figures 1, 2, 4, and 5. In Figure 1 there is only branch of waves, with only positive group velocity; the phase velocity decreases without limit while

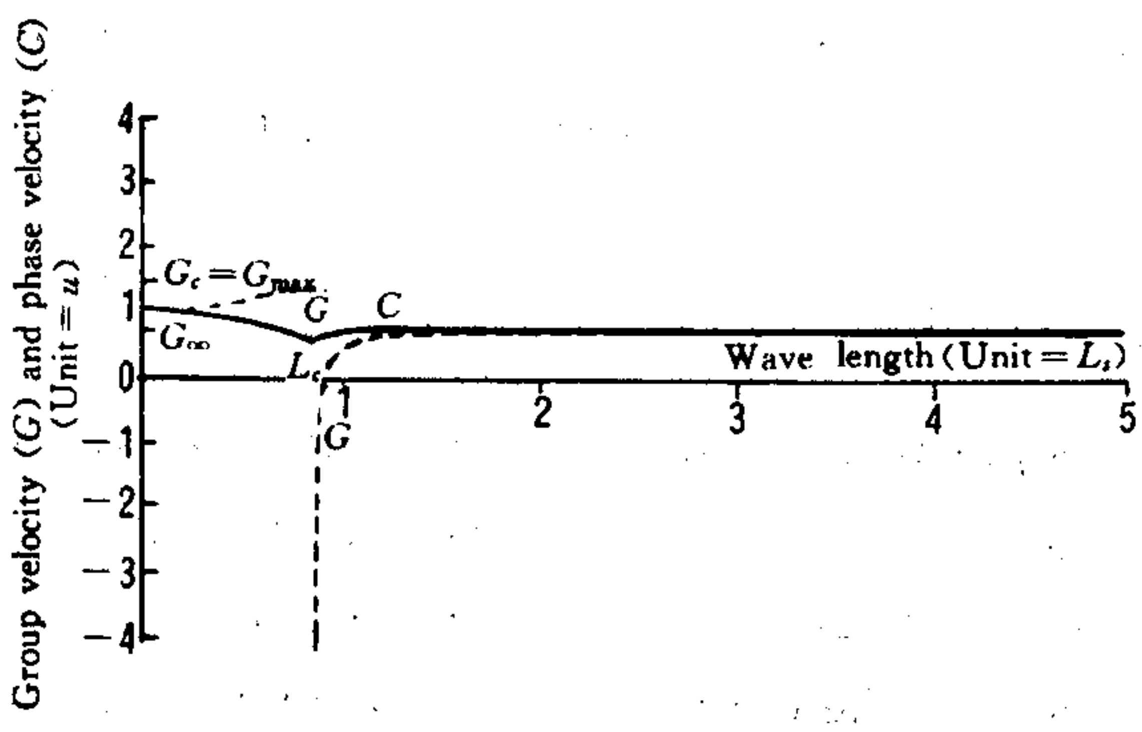

Figure 3 Group velocity and phase velocity as a function of wave length in model C when the waves with wave length below the critical value L_c are unstable (negative sign in Equation (3C)).

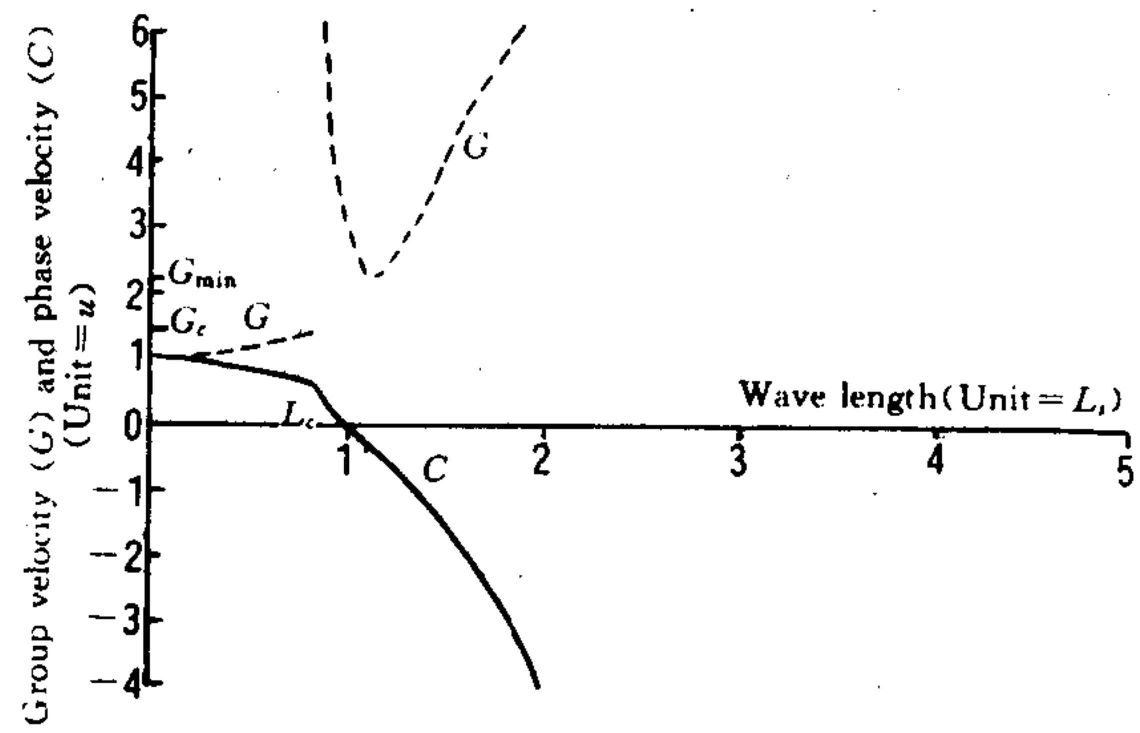

Figure 4 Group velocity and phase velocity as a function of wave length in model C when the waves with wave length below the critical value L_c are damped (possitive sign in Equation (3C)).

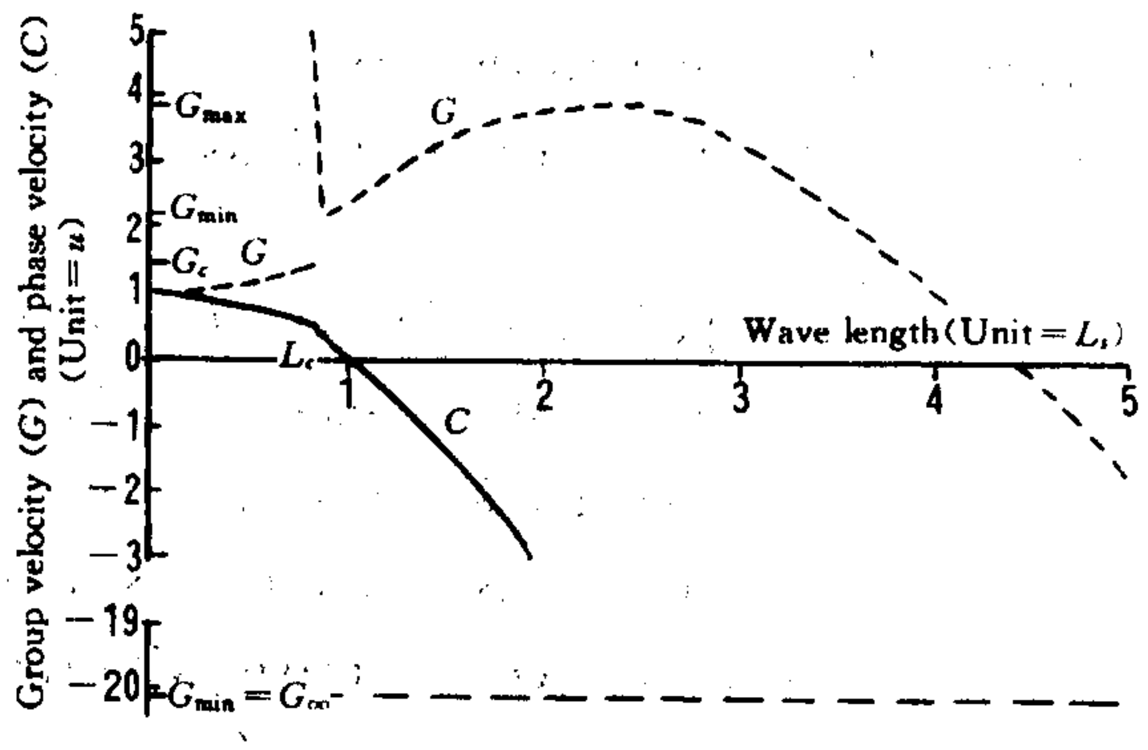

Figure 5 Group velocity and phase velocity as a function of wave length in model D when the waves with wave length below the critical value L_c are damped (positive sign in Equation (3D)). Both curves are limited by the horizontal asymptotic line $G_\infty = G_{min}$.

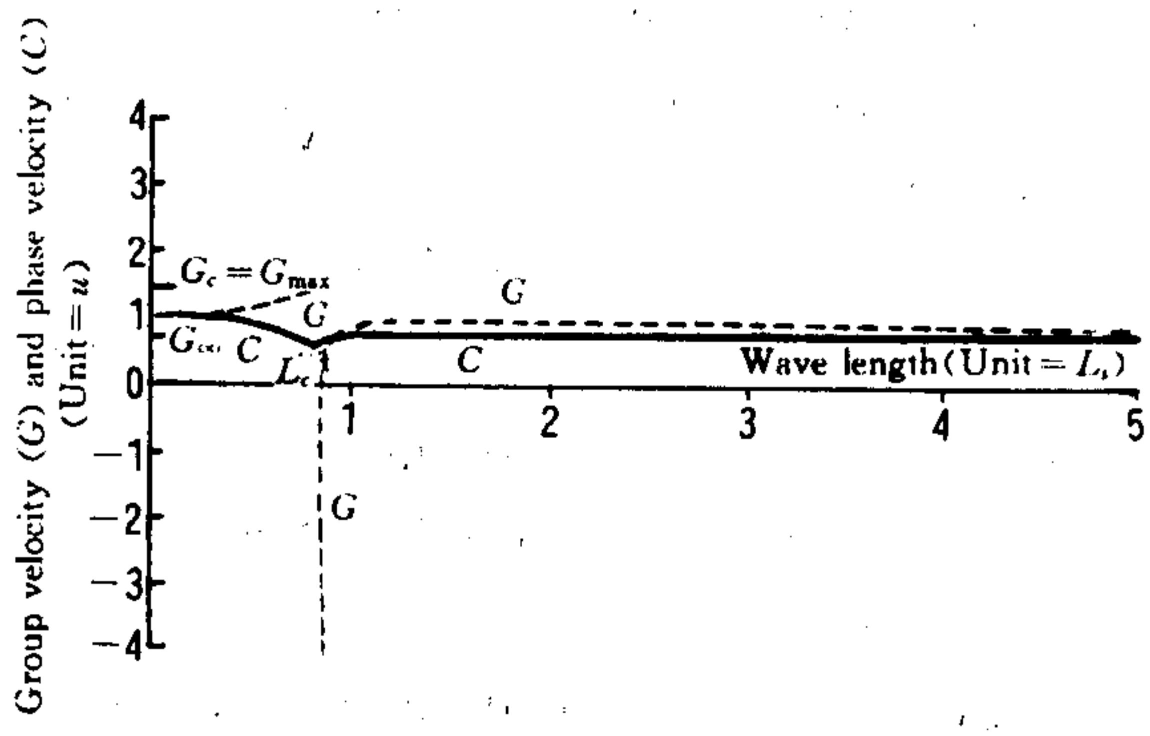

Figure 6 Group velocity and phase velocity as a function of wave length in model D when the waves with wave length below the critical value L_c are unstable (negative sign in Equation (3D)).

the group velocity increases without limit as wave length increases. The effect of divergence (Figure 2) is to add another branch of waves and to give rise to waves with negative group velocity. In the presence of divergence the magnitudes of the group and phase velocity are limited.

Comparing Figures 1 and 4 we see that the presence of solenoids also gives rise to another branch of waves, but the group velocity of this branch (from G_c to G_{min}) is also positive. The combined effect of divergence and solenoids is clearly shown in Figure 5. Both G_{max} of Figure 2 and G_{min} of Figure 4 appear in Figure 5, as well as the effect of divergence in limiting the magnitude of the group and phase velocity ($C \rightarrow G_\infty$ and $G \rightarrow G_\infty$ as $L \rightarrow \infty$).

We may call the branch of waves which appears in each of the four models the 'barotropic-nondivergent' branch, the branch of waves which appears only in the presence of divergence the 'divergent' branch, and the branch of waves which appears only in the presence of solenoids the 'solenoidal' branch. With this terminology① we can see that in Figure 1 we have only the barotropic-nondivergent branch. In Figure 2 the waves with wave length smaller than L_{max} (where $G = G_{max}$) belong to the barotropic-nondivergent branch, while for wave lengths larger than L_{max} the waves belong to the divergent branch. In Figure 4 the waves with wave length between L_c and L_{min} (where $G = G_{min}$) are of the solenoidal branch; all other wave lengths in this model may be considered to belong to the barotropic branch. In Figure 5 the solenoidal branch is between the wave length L_c and L_{min}, the divergent branch corresponds to wave length larger than L_{max} and the barotropic-nondivergent branch consists of waves of wave length smaller than L_c and between L_{min} and L_{max}.

If we consider Figures 3 and 6, we cannot separate the divergent from the solenoidal branch. In these two figures, only waves with wave length smaller than L_c are of barotropic-nondivergent character.

From the foregoing discussion we may note that the divergence affects mainly the long waves, that the solenoids affect mainly the waves of moderate wave length, and that the very short waves are not appreciably affected by either of these factors.

The foregoing analysis for models C and D was based on the assumption that the temperature gradient is not very strong so that $fs < \beta$. For a temperature gradient strong enough so that $fs > \beta$ the analysis must be modified somewhat.

Let us discuss model C first. The wave velocity for this model, when the wave length is smaller than the critical one, is

$$C = U - \frac{1}{2}\sigma L^2, \qquad \text{for} \qquad L < L_c,$$

where $\sigma \equiv \beta + fs$. The stationary wave length for this range is

$$L'_s \equiv \sqrt{2U/(\beta + fs)}.$$

This is easily verified to be larger than, equal to, or smaller than the critical wave length L_c according as fs is smaller than, equal to, or larger than β. Hence there will not be a stationary wave in the region $L < L_c$ if $fs < \beta$. In case $fs > \beta$ there will then exist two stationary waves of wave lengths $L'_s \equiv \sqrt{2U/(\beta + fs)}$ and $L_s \equiv \sqrt{U/\beta}$ respectively in the regions $L < L_c$ and $L > L_c$. If $fs > \beta$, it can be seen that both stationary waves will occur in Figure 4 while only the stationary wave of wave length L'_s will occur in Figure 3.

By differentiating the expression for L_c with respect to fs we obtain a maximum value of L_c for $fs = \beta$. This maximum value of L_c is equal to $\sqrt{U/\beta}$. Hence the value of L_c will always stay between $L'_s \equiv \sqrt{2U/(\beta + fs)}$ and $L_s \equiv \sqrt{U/\beta}$. The three wave lengths, namely, L'_s, L_c, and L_s will coincide when $fs = \beta$.

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① The terms are not rigorous because the effect of divergence or solenoids does show up in the barotropic-nondivergent branch of waves of models C and D.