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The Boundary Value Problems of Mathematical Physics

Translated by Jack Lohwater†

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Preface to the English Edition

In the present edition I have included "Supplements and Problems" located at the end of each chapter. This was done with the aim of illustrating the possibilities of the methods contained in the book, as well as with the desire to make good on what I have attempted to do over the course of many years for my students—to awaken their creativity, providing topics for independent work.

The source of my own initial research was the famous two-volume book *Methods of Mathematical Physics* by D. Hilbert and R. Courant, and a series of original articles and surveys on partial differential equations and their applications to problems in theoretical mechanics and physics. The works of K. O. Friedrichs, which were in keeping with my own perception of the subject, had an especially strong influence on me.

I was guided by the desire to prove, as simply as possible, that, like systems of n linear algebraic equations in n unknowns, the solvability of basic boundary value (and initial-boundary value) problems for partial differential equations is a consequence of the uniqueness theorems in a "sufficiently large" function space. This desire was successfully realized thanks to the introduction of various classes of general solutions and to an elaboration of the methods of proof for the corresponding uniqueness theorems. This was accomplished on the basis of comparatively simple integral inequalities for arbitrary functions and of *a priori* estimates of the solutions of the problems without enlisting any special representations of those solutions.

In this present edition I included some explanations of the basic text, and corrected misprints and inaccuracies that I noticed.

In conclusion, I want to express my deep gratitude to Professor A. J. Lohwater, who, regardless of the demands of his own scientific and pedagogical work, expressed the desire to acquaint himself with my book in detail and translate it into English. He translated all six chapters, but was not able to edit the book. The tragic, untimely death of Professor Lohwater cut short work on the book. The translation of "Supplements and Problems" I completed myself, and the translation of the "Introduction" and of this preface was done under the supervision of Springer-Verlag.

I thank all who have worked on this edition, especially the editorial and production staff of Springer-Verlag.

July 1984 O. A. LADYZHENSKAYA

Contents

CHAPTER I

Preliminary Considerations

This chapter is of an introductory character and we shall present a series of concepts and theorems from functional analysis which will be used in the sequel for studying boundary value problems for differential equations. These facts will be stated without proof.

Moreover, we introduce a number of concrete functional spaces and describe properties of these spaces of interest to us. We shall either give complete proofs of some of these theorems, or else describe the fundamental steps by which the reader can reconstruct complete proofs.

In the course of the entire book we shall use Lebesgue measure and the Lebesgue integral. The reader of this book should also be familiar with the fundamentals of real variables and functional analysis (see [AK 1], [LTS 1], [SM 1; 2], [SO 5], [RN 1].

§1. Normed Spaces and Hilbert Spaces

A set E of abstract elements is called a *real (complex) linear normed space* if:

(1) E is a linear vector space with multiplication by real (complex) numbers;
(2) To every element u of E there is a real number (called the norm of the element and denoted by $\|u\|$) satisfying the following axioms:

(a) $\|u\| \geq 0$, where $\|u\| = 0$ only for the zero element;
(b) $\|u + v\| \leq \|u\| + \|v\|$, the triangle inequality;
(c) $\|\lambda u\| = |\lambda| \cdot \|u\|$.

A natural metric can be introduced into such a space: the distance $\rho(u, v)$ between two elements u and v is defined by $\rho(u, v) = \|u - v\|$. The convergence of a sequence $\{u_n\}$ of elements of E to $u \in E$ in the norm of E (in other words, strong convergence in E) is defined by $\|u_n - u\| \to 0$ as $n \to \infty$, and in abbreviated notation by $u_n \to u$.

A collection of elements $E' \subset E$ is said to be *everywhere dense* in E if any element of E is the limit, in the norm of E, of elements of E'.

If E contains a countable, everywhere dense set of elements, then E is called *separable*. The sequence $\{u_n\}_{n=1}^{\infty}$ is called convergent (or Cauchy sequence, or fundamental) if $\|u_p - u_q\| \to 0$ when $p, q \to \infty$.

If, for every Cauchy sequence $\{u_n\}_{n=1}^{\infty}$, there is a limiting element u in E, then E is called *complete* (in this case $\|u_n - u\| \to 0$ when $n \to \infty$). A complete, linear, normed space is usually called a *space of type B* or a *Banach space*. All spaces considered below will be complete and separable.

We shall be dealing basically with a particular case of the Banach spaces, namely, the Hilbert spaces. In a real Hilbert space H we define a scalar product (u, v) for an arbitrary pair of elements u and v. It is a real number satisfying the following axioms:

(a) $(u, v) = (v, u)$;
(b) $(u_1 + u_2, v) = (u_1, v) + (u_2, v)$;
(c) $(\lambda u, v) = \lambda(u, v)$;
(d) $(u, u) \geq 0$, where $(u, u) = 0$ only for the zero element $u = 0$.

In a complex Hilbert space the scalar product (u, v) is a complex number satisfying axioms (b)–(d), together with the axiom (a') $(u, v) = \overline{(v, u)}$ instead of axiom (a).

As the norm of an element u we take the number $\|u\| = \sqrt{(u, u)}$. In the definition of a Hilbert space we include the requirement that it be complete and separable.

For any two elements u and v in H we have the inequality of Cauchy, Bunyakovski, and Schwarz:

$$|(u, v)| \leq \|u\| \cdot \|v\|,$$

which we shall simply call Cauchy's inequality in the sequel.

In addition to convergence in norm (strong convergence) in the space H, we shall also consider weak convergence. A sequence $\{u_n\}$ is said to converge weakly in H to the element u if $(u_n - u, v) \to 0$ as $n \to \infty$ for all $v \in H$. For brevity, this will be denoted by $u_n \rightharpoonup u$. It is not difficult to understand that if the norms of the $\{u_n\}$ are uniformly bounded, then to prove the weak convergence of $\{u_n\}$ to u, it is enough to verify that $(u_n - u, v) \to 0$ as $n \to \infty$ only for some set V which is everywhere dense in H. A sequence $\{u_n\}$ cannot be weakly (much less strongly) convergent to two different elements of H. If $\{u_n\}$ converges to u in the norm of H, then it converges weakly to u. The converse is false. However, if, in addition to the weak convergence of $\{u_n\}$

to u, it is known that $\|u_n\| \to \|u\|$, then $\{u_n\}$ converges strongly to u. In the sequel we shall make frequent use of the following proposition:

Theorem 1.1. *If the sequence $\{u_n\}$ converges weakly to u in H, then*

$$\|u\| \leq \varliminf_{n \to \infty} \|u_n\| \leq \varlimsup_{n \to \infty} \|u_n\|,$$

where the right-hand side of this inequality is finite.

A Hilbert space (and we emphasize that, by definition, such spaces are complete), as well as any closed subspace of it, is complete with respect to weak convergence.†

A set M in a Banach space B is called *precompact* (or *precompact in B*) if every infinite sequence of elements of M contains a convergent subsequence. If the limits of all such subsequences belong to M, then M is called *compact* (or *compact in itself*). In a similar way we introduce in a Hilbert space H the notions of weak precompactness and weak compactness. We have the following criterion of weak compactness in H:

Theorem 1.2. *A set M of H is weakly precompact if and only if it is bounded.*

We mention two examples of real spaces B and H. The totality of all real-valued measurable functions $u(x)$, defined on a domain Ω of Euclidean space R^n with a finite integral

$$\|u\|_{p,\Omega} = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p} \tag{1.1}$$

with arbitrary fixed $p \geq 1$, forms a (complete) separable Banach space if its norm is defined by (1.1). This space is usually denoted by $L_p(\Omega)$. Strictly speaking, it must be understood that an element of $L_p(\Omega)$ is not any function $u(x)$ with the properties indicated, but rather the class of functions which are equivalent to it on Ω (that is, those functions which coincide with it almost everywhere on Ω). Nevertheless, for the sake of brevity we shall speak of the elements of $L_p(\Omega)$ as functions defined on Ω.

As examples of everywhere dense sets in $L_p(\Omega)$ we can take:

(a) all infinitely differentiable functions, or all polynomials, or even only polynomials with rational coefficients;
(b) the set $\dot{C}^\infty(\Omega)$ of all infinitely differentiable functions with compact supports belonging to Ω.

† This is also true for non-linear convex sets. but it is not true for all closed sets. For example, the set $S_R = \{u: \|u\| = R\}$ is closed but not weakly closed.

The space $L_2(\Omega)$ becomes a real Hilbert space if we introduce a scalar product by means of the equality

$$(u, v) = \int_\Omega u(x)v(x)\,dx.$$

Throughout most of the book we shall deal with the real spaces B and H. The exception consists of §§3, 4, 5, and 7 of Chapter II and of §2 of the present chapter in which we use complex Hilbert spaces, including the complex space $L_2(\Omega)$. The elements of this last space are the complex-valued functions $u(x) = u_1(x) + iu_2(x)$ with the scalar product defined as

$$(u, v) = \int_\Omega u(x)\overline{v(x)}\,dx.$$

We mention a number of algebraic and functional inequalities which we shall use frequently in the course of the entire book.

Cauchy's inequality:

$$\left|\sum_{i,j=1}^n a_{ij}\xi_i\eta_j\right| \le \sqrt{\sum_{i,j=1}^n a_{ij}\xi_i\xi_j}\sqrt{\sum_{i,j=1}^n a_{ij}\eta_i\eta_j}, \tag{1.2}$$

which is valid for any non-negative quadratic form $a_{ij}\xi_i\xi_j$ with $a_{ij} = a_{ji}$ and for arbitrary real $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$.

"Cauchy's inequality with ε":

$$|ab| \le \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2, \tag{1.3}$$

which holds for all $\varepsilon > 0$ and for arbitrary a and b and its generalization—Young's inequality:

$$|ab| \le \frac{1}{p}|\varepsilon a|^p + \frac{p-1}{p}\left|\frac{b}{\varepsilon}\right|^{p/(p-1)} \quad \text{for all} \quad p > 1. \tag{1.3'}$$

From the functional inequalities we need inequalities which are concrete versions of the triangle inequality and Cauchy's inequality.

For the space $L_2(\Omega)$ these take the form

$$\left(\int_\Omega (u+v)^2\,dx\right)^{1/2} \le \left(\int_\Omega u^2\,dx\right)^{1/2} + \left(\int_\Omega v^2\,dx\right)^{1/2} \tag{1.4$_1$}$$

and

$$\left|\int_\Omega uv\,dx\right| \le \left(\int_\Omega u^2\,dx\right)^{1/2}\left(\int_\Omega v^2\,dx\right)^{1/2} \tag{1.4$_2$}$$

For the space $\mathbf{L}_2(\Omega)$ consisting of the vector functions $\mathbf{u} = (u_1, \ldots, u_N)$ with $u_i \in L_2(\Omega)$, Cauchy's inequality takes the form

$$\left| \int_\Omega \sum_{i=1}^N u_i v_i \, dx \right| \leq \left(\int_\Omega \sum_{i=1}^N u_i^2 \, dx \right)^{1/2} \left(\int_\Omega \sum_{i=1}^N v_i^2 \, dx \right)^{1/2}. \tag{1.5}$$

The left-hand side of (1.5) is the modulus of the scalar product of **u** and **v**, while the right-hand side is the product of the norms of **u** and **v**. As a generalization of (1.4_2) we have Hölder's inequality,

$$\left| \int_\Omega uv \, dx \right| \leq \left(\int_\Omega |u|^p \, dx \right)^{1/p} \left(\int_\Omega |v|^{p'} \, dx \right)^{1/p'}, \tag{1.6}$$

which holds for any $u \in L_p(\Omega)$, $v \in L_{p'}(\Omega)$ and for all $p \geq 1$ (p' will always denote the exponent conjugate to p, i.e., $p' = p/(p-1)$). For $p = 1$ we have $p' = \infty$, and by $\|v\|_{p',\Omega}$ it is necessary to take $\operatorname{ess\,sup}_\Omega |v|$. The inequality

$$\left| \sum_{i=1}^N a_i b_i \right| \leq \left(\sum_{i=1}^N |a_i|^p \right)^{1/p} \left(\sum_{i=1}^N |b_i|^{p'} \right)^{1/p'}. \tag{1.7}$$

is the discrete analogue of (1.6). A generalization of (1.4_1) gives the triangle inequality for elements of $L_p(\Omega)$:

$$\|u + v\|_{p,\Omega} \leq \|u\|_{p,\Omega} + \|v\|_{p,\Omega} \qquad (p \geq 1). \tag{1.8}$$

It is also true that

$$\left| \int_\Omega \sum_{i=1}^N u_i v_i \, dx \right| \leq \left(\int_\Omega \sum_{i=1}^N |u_i|^p \, dx \right)^{1/p} \left(\int_\Omega \sum_{i=1}^N |v_i|^{p'} \, dx \right)^{1/p'} \qquad (p \geq 1). \tag{1.9}$$

§2. Some Properties of Linear Functionals and Bounded Linear Operators in Hilbert Space

A linear functional l on H (complex or real) is a linear, continuous, numerical function $l(u)$ which is defined for all $u \in H$. Linearity of l (or distributivity) means that, for arbitrary elements u_1 and u_2 of H and for arbitrary numbers λ and μ,

$$l(\lambda u_1 + \mu u_2) = \lambda l(u_1) + \mu l(u_2). \tag{2.1_1}$$

Continuity of $l(u)$ means that $l(u_n) \to l(u)$ whenever $u_n \to u$. It has been shown that if $l(u)$ satisfies (2.1_1), then continuity is equivalent to the boundedness of $l(u)$ on the surface of the unit sphere $S_1 \equiv \{u: \|u\| = 1\}$, or, similarly,

$$|l(u)| \leq c\|u\| \tag{2.1_2}$$

for all $u \in H$.

The theorem of F. Riesz asserts that a linear functional l on H may be written in the form of a scalar product

$$l(u) = (u, v),$$

where the element v is uniquely defined by $l(u)$. The quantity $\|v\|$ is called the *norm* $\|l\|$ of the linear functional l. It is clear that $\|l\| = \sup_{u\in H}(|l(u)|/\|u\|)$ is the smallest of all possible constants c for which (2.1_2) holds. Let us pass now to the linear operator on H. An operator A, defined on some set $\mathscr{D}(A)$ of H, assigns to each element $u \in \mathscr{D}(A)$ a certain element $v \in H$; this is usually written $v = Au$ or $v = A(u)$. If the equality

$$A(\lambda u_1 + \mu u_2) = \lambda A(u_1) + \mu A(u_2)$$

holds on $\mathscr{D}(A)$, then we say that A is linear (where it is assumed that $\mathscr{D}(A)$ is a linear set). If, in addition, there exists a constant c such that, for all $u \in \mathscr{D}(A)$,

$$\|Au\| \leq c\|u\|, \tag{2.2}$$

then A is called a bounded operator on $\mathscr{D}(A)$. Such an operator may be extended in a continuous way to the closure $\overline{\mathscr{D}(A)}$ in H (which will be a closed subspace of H), in which case (2.2) will hold for all $u \in \overline{\mathscr{D}(A)}$. Such an operator can be extended (in different ways if $\overline{\mathscr{D}(A)} \neq H$) to all H and still have (2.2) hold. We shall encounter various bounded operators defined on all H. The smallest c for which (2.2) holds for all $u \in H$ is called the norm of the operator A, so that

$$\|A\| = \sup_{u\in H} \frac{\|Au\|}{\|u\|}.$$

We shall be interested in two classes of bounded linear operators. One of them is the class of *self-adjoint* operators: an operator A is called *self-adjoint* if, for all $u, v \in H$,

$$(Au, v) = (u, Av). \tag{2.3}$$

The spectrum of such an operator A is real and lies in the interval $[-\|A\|, \|A\|]$. The other class is that of completely continuous operators. An operator A is called *completely continuous* if it takes any bounded set into a precompact set. The spectrum of such an operator consists of the point zero, together with an at most countable set of eigenvalues whose only possible point of accumulation is the point zero. Each of these eigenvalues, except perhaps the point zero, is of finite multiplicity. In view of this, the eigenvalues may be enumerated in the order of decreasing modulus, $|\lambda_1| \geq |\lambda_2| \geq \ldots$, with only a finite number of the λ_m on any circle of the type $|\lambda| = |\lambda_k|$. The point zero can be an eigenvalue of infinite multiplicity.

If the operator A is self-adjoint and completely continuous, then its spectrum is real and discrete with a single possible point of accumulation at zero. All eigenvalues, except perhaps zero, are of finite multiplicity, and it is possible to arrange them in order of decreasing modulus: $|\lambda_1| \geq |\lambda_2| \geq \cdots$

where $\lambda_k \to 0$ if $k \to \infty$ (here each eigenvalue, except zero, is repeated according to multiplicity). The corresponding eigenelements $\{u_k\}$ (i.e., the solutions of the equations $Au_k = \lambda_k u_k$) can be chosen in such a way that they are mutually orthogonal and normalized. The closed subspace $\mathscr{L}$ spanned by them coincides with H if $\lambda = 0$ is not a point of the discrete spectrum (i.e., if the equation $Au = 0$ has only the trivial solution $u = 0$ in H). Otherwise the orthogonal complement of $\mathscr{L}$ in H will be the eigensubspace of H corresponding to $\lambda = 0$. Let us denote the set $H \ominus \mathscr{L} = N$, and let $\{v_k\}$ be an orthogonal basis of N. Then any element $u \in H$ may be represented in the form of the sum of two series

$$u = \sum_{k=1} (u, u_k)u_k + \sum_{k=1} (u, v_k)v_k,$$

each of which may contain a finite or infinite number of terms. We have

$$\|u\|^2 = \sum_{k=1} (u, u_k)^2 + \sum_{k=1} (u, v_k)^2,$$

$$Au = \sum_{k=1} \lambda_k(u, u_k)u_k \quad \text{and} \quad \|Au\|^2 = \sum_{k=1} \lambda_k^2(u, u_k)^2.$$

Let us return now to the general completely continuous operator A and formulate some well-known results related to solving equations of the form

$$u - \lambda Au = v, \tag{2.4}$$

where v is a given element of a complex Hilbert space H and λ is a complex parameter.

For these equations we have the Fredholm solvability property, that is, the three Fredholm theorems hold for equations (2.4):

(1) If the homogeneous equation (2.4), i.e., the equation

$$u - \lambda Au = 0, \tag{2.5}$$

has only the trivial solution, then (2.4) may be solved uniquely for arbitrary $v \in H$. (In other words, this theorem asserts that the existence theorem follows from the uniqueness theorem.)

(2) The homogeneous equation (2.5) can have non-trivial (i.e., non-zero) solutions for not more than a countable number of values $\{\lambda_k\}$, each of which is of finite multiplicity. The set $\{\lambda_k\}$ cannot have a point of condensation λ_0 with $|\lambda_0| < \infty$. These exceptional values $\{\lambda_k\}$ are called characteristic numbers for A. For the adjoint operator A^* the characteristic numbers are $\{\bar{\lambda}_k\}$, that is, the equation

$$u - \lambda A^*u = 0 \tag{2.6}$$

has a non-trivial solution only for $\lambda = \bar{\lambda}_k$, and the multiplicity of $\bar{\lambda}_k$ for A^* is the same as the multiplicity of λ_k for A.

(3) Equation (2.4) with λ equal to any one of the characteristic values λ_k may be solved for those v, and only for those v, which are orthogonal to all solutions of (2.6) corresponding to $\lambda = \bar{\lambda}_k$.

If these orthogonality conditions are fulfilled, then (2.4) has infinitely many solutions. All of them can be written in the form $u = u_0 + \sum_{j=m}^{m+p} c_j u_j$, where u_0 is any solution of (2.4) with $\lambda = \lambda_k$, the c_j are arbitrary constants, and the u_j, $j = m, \ldots, m + p$, are all linearly independent solutions of (2.5) for $\lambda = \lambda_k$.

Such solvability takes place, for example, for linear algebraic systems in which u and v are vectors with n components and A is a square matrix with n^2 entries. The same thing is true of integral operators having a kernel which does not misbehave too badly.

In Chapter II we shall show that such solvability also takes place for the basic boundary value problems for elliptic equations with bounded coefficients in a bounded domain. This is not obvious, for in these problems one has to deal with unbounded operators, but they can be reduced to equations of the type (2.4) with completely continuous operators A.

We also recall a well-known fact related to the solvability of (2.4) with an arbitrary bounded operator A, namely, that such an equation may be solved uniquely for all $v \in H$ with λ such that $|\lambda| < 1/\|A\|$, and that its solution may be represented in the form of a series $u = v + \lambda A v + \lambda^2 A^2 v + \ldots$ which converges in H and also $\|u\| \leq \|v\|/(1 - |\lambda|\,\|A\|)$.

In this section we have formulated various theorems about bounded operators in a complex Hilbert space H, which are also valid in real spaces H. However, in studying the spectra of non-symmetric operators A acting on a real space H we encounter in a natural way a larger complex space, for the spectrum of such operators A may be complex.

§3. Unbounded Operators

Let us recall certain facts about unbounded linear operators A on H. Such operators are not defined for all elements u of H. The set on which A is defined is called the domain of definition of A and is denoted by $\mathscr{D}(A)$. This set is linear and, for all $u, v \in \mathscr{D}(A)$ and for all complex numbers λ, μ, satisfies

$$A(\lambda u + \mu v) = \lambda A u + \mu A v.$$

In contrast to the case of the bounded operators, a constant c does not exist in the case of an unbounded operator A for which (2.2) holds for all u in $\mathscr{D}(A)$. We shall consider only the case that $\mathscr{D}(A)$ is dense in H. The set of values of A, i.e. the range of A, will be denoted by $\mathscr{R}(A)$, so that $A(\mathscr{D}(A)) = \mathscr{R}(A) \subset H$.

We shall be interested in unbounded operators arising from differential expressions. To every such expression correspond various operators defined by indicating their domain of definition. As an example, we consider the differential expression $\mathscr{L}u(x) = d^2u(x)/dx^2$ on the interval $x \in [0, 1]$, taking as H the real functional space $L_2(0, 1)$. We can associate with $\mathscr{L}$ the operator A in this space which is defined on all infinitely differentiable functions with support in (0, 1). For $u(x)$ in $\mathscr{D}(A)$ the operator A is calculated by $Au = \mathscr{L}u = d^2u(x)/dx^2$. It is easy to see that A is unbounded on $\mathscr{D}(A)$. With the expression $\mathscr{L}u$ we can associate another unbounded operator $\tilde{A}$ whose domain of definition is the set of all infinitely differentiable functions on [0, 1]. On these functions, $\tilde{A}$ is calculated in the same way as A on $\mathscr{D}(A)$, namely, $\tilde{A}u = d^2u(x)/dx^2$. There is a natural ordering between A and $\tilde{A}$: $\mathscr{D}(A) \subset \mathscr{D}(\tilde{A})$ and $Au = \tilde{A}u$ for $u \in \mathscr{D}(A)$. In a situation like this we say that the operator $\tilde{A}$ is an *extension* of the operator A. It is clear that the domain of definition for $\mathscr{L}$ can be chosen in an infinite number of ways, and each time we are led, generally speaking, to another unbounded operator with other properties. The operators A and $\tilde{A}$ described above have fundamentally different properties. For example, A satisfies the relationship

$$(Au, v) = \int_0^1 \frac{d^2u}{dx^2} v \, dx = (u, Av), \tag{3.1}$$

which is easily verified by integration by parts, where $u(x)$ and $v(x)$ in (3.1) are arbitrary elements of $\mathscr{D}(A)$. For the operator $\tilde{A}$ this property does not hold, for

$$(Au, v) = (u, Av) + \frac{du}{dx} v \bigg|_{x=0}^{x=1} - u \frac{dv}{dx} \bigg|_{x=0}^{x=1}, \tag{3.2}$$

and the sum of the last two terms on the right-hand side of (3.2) is not zero for all $u(x)$ and $v(x)$ in $\mathscr{D}(\tilde{A})$. The property (3.1) guarantees the symmetry of A, but the operator $\tilde{A}$ is not symmetric. As is well known in general operator theory, symmetric operators possess a large number of nice properties. The theory of symmetric operators has been well developed and can be used for studying specific classes of differential operators. One of the most important concepts is that of a self-adjoint operator.

An operator A is called *self-adjoint* if it is symmetric, that is, if

$$(Au, v) = (u, Av) \quad \text{for all} \quad u, v \in \mathscr{D}(A), \tag{3.3}$$

and if the identity

$$(Au, v) = (u, w), \tag{3.4}$$

where v and w are fixed and u is an arbitrary element of $\mathscr{D}(A)$, implies that $v \in \mathscr{D}(A)$ and $w = Av$. In other words, A is self-adjoint if its adjoint operator A^* has the same domain of definition $\mathscr{D}(A)$ and $A = A^*$ on $\mathscr{D}(A)$. The identity (3.4) defines the domain of definition of A^* and its values on the domain, namely, those v for which there exists w satisfying (3.4) for all $u \in \mathscr{D}(A)$ constitute $\mathscr{D}(A^*)$ with A^*v set equal to w. In the majority of cases of operators A arising from differential expressions it is easy to verify the validity (or non-validity) of (3.3) by means of integration by parts. It is considerably more difficult to describe the domain $\mathscr{D}(A^*)$ for these operators and to determine whether it coincides with $\mathscr{D}(A)$ or not. For differential expressions containing partial derivatives, this is usually carried out in some "roundabout" way, most often with the use of inverse operators which turn out to be bounded. In point of fact, these "roundabout" methods are used for studying elliptic differential equations under one set of boundary conditions or another. For bounded operators A defined on all H, the self-adjointness is a consequence of their symmetry.

Let us return to the example of the differential operator $\mathscr{L}u = d^2u/dx^2$. We have already determined above that the corresponding operator A is symmetric. It is not difficult to see, however, that it is not self-adjoint. Actually, (3.3) holds for A not only for u and v in $\mathscr{D}(A)$, but also, for example, for $u \in \mathscr{D}(A)$ and $v \in \mathscr{D}(\tilde{A})$. This shows that $\mathscr{D}(A^*)$ is larger than $\mathscr{D}(A)$. There arises the natural question: Can A be extended so as to be self-adjoint? It turns out that it can be done, and, indeed, in an infinite number of ways. The general theory of operators gives the first step of such an extension; this is the procedure of the closure of an operator. The procedure consists of the following. Let A (where A may be non-symmetric) be defined on a dense set $\mathscr{D}(A)$ of the space H. We adjoin to $\mathscr{D}(A)$ all elements u which are limits of those sequences $\{u_n\}$ of $\mathscr{D}(A)$ for which $\{Au_n\}$ converges to some element v. We define $v = \bar{A}u$, where the symbol $\bar{A}$ denotes the closure of A. The set $\mathscr{D}(A)$, supplemented by all such elements u, constitutes $\mathscr{D}(\bar{A})$, the domain of definition of $\bar{A}$. However, this procedure does not always lead to a linear operator $\bar{A}$, for the definition requires that it be defined in a single-valued way on $\mathscr{D}(\bar{A})$, which is equivalent to the requirement that it have the value 0 on the zero element. Actually, in the procedure described above, we can encounter the case where u_n converges to $u = 0$ and Au_n converges to $v \neq 0$. If this happens, then, by what was said above, we should set $\bar{A}0$ equal to $v \neq 0$, and, by the same token, obtain an operator $\bar{A}$ which is not linear. Examples show that the situation described is possible for certain unbounded operators, and consequently an arbitrary unbounded operator does not have a closure.

It is not difficult to prove that a necessary and sufficient condition that an operator A have a closure is that the domain of definition of the conjugate operator A^* be dense in H. This criterion is easy to verify for operators defined by differential expressions with coefficients that are "not too bad," so that such operators have a closure.

In particular, the operator A that we defined above by $\mathscr{L}u = d^2u/dx^2$ has a closure. But then the question arises: How do we find the closure $\bar{A}$ for A, that is, which functions are to be adjoined to $\mathscr{D}(A)$ and how do we calculate the operator $\bar{A}$ for them? It turns out that it is not a simple thing to answer this question, especially in the case of differential expressions with partial derivatives. The answer requires the extension of the notion of derivative and the introduction of what are called "generalized derivatives." We shall go into this notion in more detail in the next section in view of its cardinal importance for all problems which will be investigated in this book.