CONTEMPORARY MATHEMATICS

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Lectures on Nielsen Fixed Point Theory

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American Mathematical Society

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PREFACE

These notes are based on the topics courses given at the University of California, Berkeley, in Winter 1980 and at the University of California, Los Angeles, in Winter 1981. The subject is Nielsen fixed point theory which is becoming increasingly important in geometric topology and, potentially, has applications in analysis. The approach is via covering spaces. This approach is both natural and fruitful, but no reference in the English language has been easily available. The prerequisite is minimal: the classical covering space theory and homology theory for compact polyhedra.

The Introduction explains what Nielsen theory is about. Chapter I gives the basic notions of the theory, while Chapter II is devoted to computational methods. In Chapter III we broaden the scope and introduce the Nielsen type theory for periodic points. Chapter IV provides an exposition of the latest progress in the Nielsen theory for fiber maps. Another chapter in the original courses is now sketched as §I.6 because the material is easily available in the literature. The Historical Notes and Bibliography attached are by no means complete.

The author wishes to express his gratitude to Professor T. H. Kiang who introduced him into this subject years before and whose book [Kiang (1979)] has a great influence on the presentation here. He wishes to thank Professors R. Brown and H. Schirmer for their interest in the course and their encouragement and helpful comments. He is especially indebted to Professor Brown for his help with language and in proofreading. He also wants to thank the Department of Mathematics at UCLA for hospitality during his visit and for arranging the typing of these notes. He thanks Bob Neu for his skillful typing of this manuscript.

Peking, China -- Boju Jiang

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Introduction

Let X be a space, and let $f : X \to X$ be a self-map. A fixed point of f is a solution of the equation $x = f(x)$. The set of all fixed points of f we will denote by $\mathrm{Fix}(f)$. Fixed point theory studies the nature of the fixed point set $\mathrm{Fix}(f)$ in relation to the space X and the map f, such as: existence (is $\mathrm{Fix}(f) \neq \emptyset$?); the number of fixed points $\#\mathrm{Fix}(f)$ (we will use the notation $\#S$ for the cardinality of a set S); the behavior under homotopy (how $\mathrm{Fix}(f)$ changes when f changes continuously); etc.

Fixed point theory started in the early days of topology, because of its close relationship with other branches of mathematics. Existence theorems are often proved by converting the problem into an appropriate fixed point problem. Examples are the existence of solutions for elliptic partial differential equations, and the existence of closed orbits in dynamical systems. In many problems, however, one is not satisfied with the mere existence of a solution. One wants to know the number, or at least a lower bound for the number of solutions. But the actual number of fixed points of a self-map can hardly be the subject of an interesting theory, since it can be altered by an arbitrarily small perturbation of the map. So, in topology, one proposes to determine the minimal number of fixed points in a homotopy class. This is what Nielsen fixed point theory is about. This is the theme of these notes.

Perhaps the best known fixed point theorem in topology is the Lefschetz fixed point theorem.

THEOREM (Lefschetz 1923; Hopf 1929) Let X be a compact polyhedron, and let $f : X \to X$ be a map. Define the Lefschetz number $L(f)$ of f to be

$$L(f) := \sum_q (-1)^q \operatorname{trace}(f_{q*} : H_q(X;\mathbb{Q}) \to H_q(X;\mathbb{Q})) ,$$

where $H_*(X;\mathbb{Q})$ is the rational homology of X. If $L(f) \neq 0$, then every map homotopic to f has a fixed point.

The Lefschetz number is the total algebraic count of fixed points. It is a homotopy invariant and is easily computable. But it counts the fixed points "by multiplicity", just like what one does when one says an equation of degree n has n roots. So, the Lefschetz theorem, along with its special case, the Brouwer fixed point theorem, and its generalization, the widely used Leray-Schauder theorem in functional analysis, can tell existence only.

In contrast, the (chronologically) first result of Nielsen theory has set a beautiful example of a different type of theorem.

THEOREM (Nielsen-Brouwer 1921) Let $f : T^2 \to T^2$ be a self-map of the torus. Suppose the endomorphism induced by f on the fundamental group $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ is represented by the 2×2 integral matrix A. Then the least number of fixed points in the homotopy class of f equals the absolute value of the determinant of $E - A$, where E is the identity matrix; in symbols,

$$\operatorname{Min}\{\#\operatorname{Fix}(g) \mid g \simeq f\} = |\det(E - A)| .$$

It can be shown that $\det(E - A)$ is exactly $L(f)$ on tori. This latter theorem says much more than the Lefschetz theorem specialized to the torus, since it gives a lower bound for the number of fixed points, or it confirms the existence of a homotopic map which is fixed point free. The proof was via the universal covering space $\mathbb{R}^2$ of the torus. From this instance evolved the central notions of Nielsen theory -- the fixed point classes and the Nielsen number.

Roughly speaking, Nielsen theory has two aspects. The geometric aspect concerns the comparison of the Nielsen number with the least number of fixed points in a homotopy class of self-maps. The algebraic aspect deals with the problem of computation for the Nielsen number. We choose to concentrate more on the latter aspect, partly because of the richness and difficulty of the theory, partly because of its importance to applications. As to the former aspect, we will confine ourselves to quoting the main results without proof, and recommend the books [Brown (1971)], Chapter VIII, and [Kiang (1979)], Chapter IV, for excellent expositions of earlier results, and the paper [Jiang (1980)] for the latest improvements and simplifications. We will also restrict our exposition to self-maps of compact polyhedra, since there seems to be no essential difficulty in extending further to compact ANRs or even to compact maps on noncompact ANRs by means of the method of domination (cf. [Brown (1969), (1971)] and [You]).

Nielsen theory is based on the theory of covering spaces. We will take this point of view consistently, as Nielsen himself did. An alternative way is to consider nonempty fixed point classes only, and use paths instead of covering spaces to define them. This is certainly more convenient for some geometric questions. But the covering space approach is theoretically more satisfactory, especially for computational problems, since the nonemptiness of certain fixed point classes is often the conclusion of the analysis, not the assumption.

Now let us introduce the basic idea of Nielsen theory by an elementary example.

PROPOSITION. Let $f : S^1 \to S^1$ be a self-map of the circle. Suppose the degree of f is d. Then the least number of fixed points in the homotopy class of f is $|1 - d|$.

Proof. Let S^1 be the unit circle on the complex plane, i.e. $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Let $p : \mathbb{R} \to S^1$ be the exponential map $p(\theta) = z = e^{i\theta}$. Then θ is the argument of z, which is a multi-valued function of z. For every $f : S^1 \to S^1$, one can always find "argument expressions" (or liftings) $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $f(e^{i\theta}) = e^{i\tilde{f}(\theta)}$, in fact a whole series of them, differing from each other by integral multiples of 2π. For definiteness let us write $\tilde{f}_0$ for the argument expression with $\tilde{f}_0(0)$ lying in $[0,2\pi)$, and write $\tilde{f}_k = \tilde{f}_0 + 2k\pi$. Since the degree of f is d, the functions $\tilde{f}_k$ are such that $\tilde{f}_k(\theta + 2\pi) = \tilde{f}_k(\theta) + 2d\pi$. For example, if $f(z) = -z^d$, then $\tilde{f}_k(\theta) = d\theta + (2k + 1)\pi$.

It is evident that if $z = e^{i\theta}$ is a fixed point of f, i.e. $z = f(z)$, then θ is a fixed point of some argument expression of f, i.e. $\theta = \tilde{f}_k(\theta)$ for some k. On the other hand, if θ is a fixed point of $\tilde{f}_k$, q is an integer, then $\theta + 2q\pi$ is a fixed point of $\tilde{f}_\ell$ iff $\ell - k = q(1 - d)$. This follows from the calculation $\tilde{f}_\ell(\theta + 2q\pi) = \tilde{f}_k(\theta + 2q\pi) + 2(\ell - k)\pi$ $= \tilde{f}_k(\theta) + 2qd\pi + 2(\ell - k)\pi = (\theta + 2q\pi) + 2\pi\{(\ell - k) - q(1 - d)\}$. Thus, if $\ell \not\equiv k \bmod (1 - d)$, then a fixed point of $\tilde{f}_k$ and a fixed point of $\tilde{f}_\ell$ can never correspond to the same fixed point of f, i.e. $p\,\mathrm{Fix}(\tilde{f}_k) \cap p\,\mathrm{Fix}(\tilde{f}_\ell) = \emptyset$.

So, the argument expressions fall into equivalence classes (called lifting classes) by the relation $\tilde{f}_k \sim \tilde{f}_\ell$ iff $k \equiv \ell \bmod (1 - d)$, and the fixed points of f split into $|1 - d|$ classes (called fixed point classes) of the form $p\,\mathrm{Fix}(\tilde{f}_k)$. That is, two fixed points are in the same class iff they come from fixed points of the same argument expression. Note that each fixed point class is by definition associated with a lifting class, so that the number of fixed point classes is $|1 - d|$ if $d \neq 1$, and is ∞ if $d = 1$. Also note that a fixed point class need not be nonempty.

Now, to prove that a map f of degree d has at least $|1 - d|$ fixed points, we only have to show that every fixed point class is nonempty, or equivalently, that every argument expression has a fixed point, if $d \neq 1$. In fact, for each k, by means of the equality $\tilde{f}_k(\theta + 2\pi) - \tilde{f}_k(\theta) = 2d\pi$, it is easily seen that the function $\theta - \tilde{f}_k(\theta)$ takes different signs when θ approaches $\pm\infty$, hence $\tilde{f}_k$ has at least one fixed point.

That $|1 - d|$ is indeed the least number of fixed points in the homotopy class is seen by checking the special map $f(z) = -z^d$. □

The following chapters can be considered as generalizations of this simplest example. See the table of contents and the introductory paragraph of every chapter.

CHAPTER 1

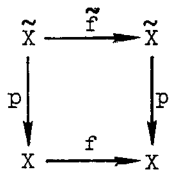
FIXED POINT CLASSES AND THE NIELSEN NUMBER

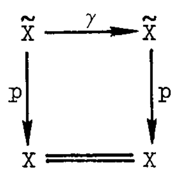
In this chapter we introduce the basic notions of Nielsen theory. The simple example of $S^1 \to S^1$ in the Introduction is generalized to self-maps of a polyhedron X, with the universal covering space $\tilde{X}$ of X playing the role of the exponential map $\mathbb{R} \to S^1$. The basic invariance theorems are in §§4-5. Section 3 is a brief introduction to the algebraic count of fixed points -- the fixed point index. We conclude this chapter by relating the Nielsen number to the least number of fixed points in a homotopy class, thus justifying the important position of the Nielsen number in the fixed point theory.

1. LIFTING CLASSES AND FIXED POINT CLASSES. We always assume X to be a connected compact polyhedron. It is well known that X has a universal covering space. (Actually the material in §§1-2 makes sense for any X with a universal covering space.) References on covering spaces: [Massey], [Spanier].

Let $p : \tilde{X} \to X$ be the universal covering of X.

1.1 DEFINITION. A lifting of a map $X \xrightarrow{f} X$ is a map $\tilde{X} \xrightarrow{\tilde{f}} \tilde{X}$ such that $p \circ \tilde{f} = f \circ p$. A covering translation is a map $\tilde{X} \xrightarrow{\gamma} \tilde{X}$ such that $p \circ \gamma = p$, i.e. a lifting of the identity map.





1.2 PROPOSITION. (i) For any $x_0 \in X$ and any $\tilde{x}_0, \tilde{x}'_0 \in p^{-1}(x_0)$, there is a unique covering translation $\gamma : \tilde{X} \to \tilde{X}$ such that $\gamma(\tilde{x}_0) = \tilde{x}'_0$. The covering translations of $\tilde{X}$ form a group $\mathcal{D} = \mathcal{D}(\tilde{X},p)$ which is isomorphic to $\pi_1(X)$.

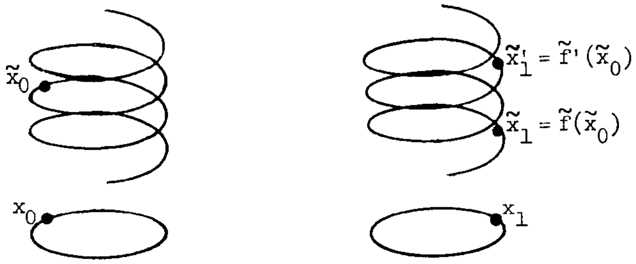
(ii) Let $f : X \to X$ be a map. For given $x_0 \in X$ and $x_1 = f(x_0)$, pick $\tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{x}_1 \in p^{-1}(x_1)$ arbitrarily. Then, there is a unique lifting

$\tilde{f}$ of f such that $\tilde{f}(\tilde{x}_0) = \tilde{x}_1$.

(iii) Suppose $\tilde{f}$ is a lifting of f, and $\alpha, \beta \in \mathcal{D}$. Then $\beta \circ \tilde{f} \circ \alpha$ is a lifting of f.

(iv) For any two liftings $\tilde{f}$ and $\tilde{f}'$ of f, there is a unique $\gamma \in \mathcal{D}$ such that $\tilde{f}' = \gamma \circ \tilde{f}$.

PROOF. (i) and (ii) are standard theorems in covering space theory. (iii) and (iv) follow from Definitions (i) and (ii). □



1.3 LEMMA. Suppose $\tilde{x} \in p^{-1}(x)$ is a fixed point of a lifting $\tilde{f}$ of f, and $\gamma \in \mathcal{D}$ is a covering translation on $\tilde{X}$. Then, a lifting $\tilde{f}'$ of f has $\gamma(\tilde{x}) \in p^{-1}(x)$ as a fixed point iff $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$.

PROOF. "If" is obvious: $\tilde{f}'(\gamma(\tilde{x})) = \gamma \circ \tilde{f} \circ \gamma^{-1}(\gamma(\tilde{x})) = \gamma \circ \tilde{f}(\tilde{x}) = \gamma(\tilde{x})$.

"Only if": Both $\tilde{f}'$ and $\gamma \circ \tilde{f} \circ \gamma^{-1}$ have $\gamma(\tilde{x})$ as a fixed point, so they agree at the point $\gamma(\tilde{x})$. By Proposition 1.2 (ii), they are the same lifting. □

1.4 DEFINITION. Two liftings $\tilde{f}$ and $\tilde{f}'$ of $f : X \to X$ are said to be <u>conjugate</u> if there exists $\gamma \in \mathcal{D}$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. <u>Lifting classes</u> := equivalence classes by conjugacy. Notation:

$$[\tilde{f}] = \{\gamma \circ \tilde{f} \circ \gamma^{-1} \mid \gamma \in \mathcal{D}\} .$$

1.5 THEOREM. (i) $\mathrm{Fix}(f) = \bigcup_{\tilde{f}} p\,\mathrm{Fix}(\tilde{f})$.

(ii) $p\,\mathrm{Fix}(\tilde{f}) = p\,\mathrm{Fix}(\tilde{f}')$ if $[\tilde{f}] = [\tilde{f}']$.

(iii) $p\,\mathrm{Fix}(\tilde{f}) \cap p\,\mathrm{Fix}(\tilde{f}') = \emptyset$ if $[\tilde{f}] \neq [\tilde{f}']$.

PROOF. (i) If $x_0 \in \mathrm{Fix}(f)$, pick $\tilde{x}_0 \in p^{-1}(x_0)$. By Proposition 1.2 (ii) there exists $\tilde{f}$ such that $\tilde{f}(\tilde{x}_0) = \tilde{x}_0$. Hence $x_0 \in p\,\mathrm{Fix}(\tilde{f})$.

(ii) If $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$, then by Lemma 1.3, $\mathrm{Fix}(\tilde{f}') = \gamma\,\mathrm{Fix}(\tilde{f})$, so that $p\,\mathrm{Fix}(\tilde{f}') = p\,\mathrm{Fix}(\tilde{f})$.

(iii) If $x_0 \in p\,\mathrm{Fix}(\tilde{f}) \cap p\,\mathrm{Fix}(\tilde{f}')$, there are $\tilde{x}_0, \tilde{x}'_0 \in p^{-1}(x_0)$ such that $\tilde{x}_0 \in \mathrm{Fix}(\tilde{f})$ and $\tilde{x}'_0 \in \mathrm{Fix}(\tilde{f}')$. Suppose $\tilde{x}'_0 = \gamma\tilde{x}_0$. By Lemma 1.3, $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$, hence $[\tilde{f}] = [\tilde{f}']$. □

1.6 DEFINITION. The subset $p\,\mathrm{Fix}(\tilde{f})$ of $\mathrm{Fix}(f)$ is called the <u>fixed point class</u> of f determined by the lifting class $[\tilde{f}]$.

1.7 THEOREM. The fixed point set $\mathrm{Fix}(f)$ splits into a disjoint union of fixed point classes. □

EXAMPLE. Lifting classes and fixed point classes of the identity map $id_X : X \to X$.

A lifting class = a conjugacy class (in the usual sense) in $\mathcal{D}$.

$p\,\mathrm{Fix}(id_{\tilde{X}}) = X$.

$p\,\mathrm{Fix}(\gamma) = \emptyset$ otherwise.

1.8 REMARK. A fixed point class is always considered to carry a label -- the lifting class determining it. Thus two empty fixed point classes are considered different if they are determined by different lifting classes.

1.9 DEFINITION. The number of lifting classes of f (and hence the number of fixed point classes, empty or not) is called the Reidemeister number of f, denoted $R(f)$. It is a positive integer or infinity.

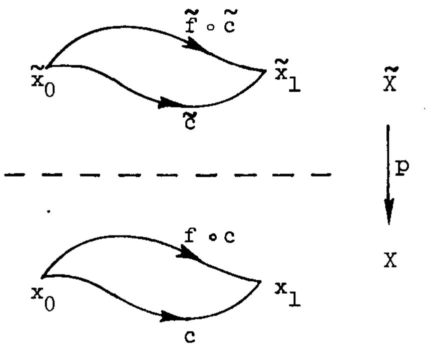
EXAMPLE. $R(f) = 1$ if X is simply-connected.

Our definition of a fixed point class is via the universal covering space. It essentially says: Two fixed points of f are in the same class iff there is a lifting $\tilde{f}$ of f having fixed points above both of them. There is another way of saying this, which does not use covering space explicitly, hence is very useful in identifying fixed point classes.

1.10 THEOREM. Two fixed points x_0 and x_1 of $f : X \to X$ belong to the same fixed point class iff there is a path c from x_0 to x_1 such that $c \simeq f \circ c$ (homotopy rel endpoints).

PROOF. "Only if". Fixed points x_0 and x_1 are in the same class, then there exists a lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ and points $\tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{x}_1 \in p^{-1}(x_1)$ such that $\tilde{f}(\tilde{x}_0) = \tilde{x}_0$ and $\tilde{f}(\tilde{x}_1) = \tilde{x}_1$.

Take a path $\tilde{c}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Since $\tilde{X}$ is simply-connected, $\tilde{c} \simeq \tilde{f} \circ \tilde{c}$. Projecting down to X, we have



$$c \simeq f \circ c$$

where $c = p \circ \tilde{c}$.

"If". Suppose $x_0 \in p\,\mathrm{Fix}(\tilde{f})$, $\tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{f}(\tilde{x}_0) = \tilde{x}_0$. We want to prove $x_1 \in p\,\mathrm{Fix}(\tilde{f})$, i.e. there exists $\tilde{x}_1 \in p^{-1}(x_1)$ such that $\tilde{f}(\tilde{x}_1) = \tilde{x}_1$.

Lift the path c from $\tilde{x}_0$ to get a path $\tilde{c}$ in $\tilde{X}$. Then $\tilde{f} \circ \tilde{c}$ projects to $f \circ c$, hence $\tilde{f} \circ \tilde{c}$ is the lift of $f \circ c$ from $\tilde{x}_0$.

Since $c \simeq f \circ c$, their lifts from the same starting point $\tilde{x}_0$ should have the same endpoint. Hence $\tilde{x}_1 = \tilde{f}(\tilde{x}_1)$, where $\tilde{x}_1$ is the other end of $\tilde{c}$. □

1.11 REMARK. Theorem 1.10 can be considered as an equivalent definition of a <u>non-empty</u> fixed point class. Its advantage: It works directly on X, hence is more convenient in geometric questions. Its disadvantage: It pays attention only to non-empty fixed point classes, hence is not satisfactory when considering the influence of homotopy on fixed point classes.

1.12 THEOREM. Every fixed point class of $f : X \to X$ is an open subset of $\mathrm{Fix}(f)$.

PROOF. Given a fixed point x_0 of f, we want to find a neighborhood U of x_0 such that any fixed point $x_1 \in U$ belongs to the same class.

Since X has a universal covering, X is locally path-connected and semilocally 1-connected. There is a neighborhood W of x_0 such that every loop in W at x_0 is trivial in X. There also is a path-connected neighborhood U of x_0 such that $U \subset W \cap f^{-1}(W)$.

Now, if $x_1 \in U \cap \mathrm{Fix}(f)$, take a path c in U from x_0 to x_1, then both c and $f \circ c$ are in W, hence $c \simeq f \circ c$. Thus x_0, x_1 are in the same class by Theorem 1.10. □

1.13 COROLLARY. Every map $f : X \to X$ has only finitely many non-empty fixed point classes, each a compact subset of X. □

1.14 COROLLARY. A continuum of fixed points lies in a single fixed point class. □

EXERCISES. 1. Let $f : S^1 \to S^1$ be of degree d. $R(f) = ?$

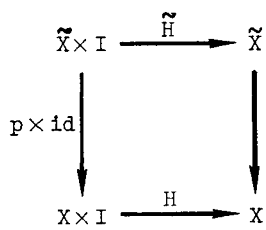
2. Let T^2 be the torus, $f : T^2 \to T^2$, the induced homomorphism on $H_1(T^2)$ given by an integral 2×2 matrix A. $R(f) = ?$

3. Discuss $R(f)$ for $f : \mathbb{R}P^2 \to \mathbb{R}P^2$.

2. THE INFLUENCE OF A HOMOTOPY. We use the following notation for a homotopy: $H = \{h_t\}_{t \in I} : f_0 \simeq f_1 : X \to X$, or $H : X \times I \to X$.

Given a homotopy $H = \{h_t\} : f_0 \simeq f_1$, we want to see its influence on fixed point classes of f_0 and f_1.

2.1 DEFINITION. A homotopy $\tilde{H} = \{\tilde{h}_t\} : \tilde{X} \to \tilde{X}$ is called a <u>lifting</u> of the homotopy $H = \{h_t\}$, if $\tilde{h}_t$ is a lifting of h_t for every $t \in I$.



2.2 BASIC OBSERVATION. Given a homotopy $H : f_0 \simeq f_1$ and a lifting $\tilde{f}_0$ of f_0, there is a unique lifting $\tilde{H}$ of H such that $\tilde{h}_0 = \tilde{f}_0$, hence they determine a lifting $\tilde{f}_1$ of f_1. (Reason: Unique lifting property of covering spaces.) Thus H gives rise to a one-one correspondence from liftings of f_0 to liftings of f_1.

$$\tilde{f}_0 \overset{H}{\rightsquigarrow} \tilde{f}_1$$

$$\tilde{f}_0 \overset{H^{-1}}{\leftsquigarrow} \tilde{f}_1$$

This correspondence preserves the conjugacy relation:

$$\{\tilde{h}_t\} : \tilde{f}_0 \simeq \tilde{f}_1 \quad \text{implies} \quad \{\gamma \circ \tilde{h}_t \circ \gamma^{-1}\} : \gamma \circ \tilde{f}_0 \circ \gamma^{-1} \simeq \gamma \circ \tilde{f}_1 \circ \gamma^{-1} .$$

2.3 DEFINITION. Let $H : f_0 \simeq f_1$ be a homotopy and $\tilde{f}_i$ be a lifting of f_i, $i = 0,1$. We say that the lifting class $[\tilde{f}_0]$ (and the fixed point class $p\,\mathrm{Fix}(\tilde{f}_0)$ of f_0) corresponds to the lifting class $[\tilde{f}_1]$ (and the fixed point class $p\,\mathrm{Fix}(\tilde{f}_1)$ of f_1) via the homotopy H, if H has a lifting $\tilde{H} : \tilde{f}_0 \simeq \tilde{f}_1$.

2.4 THEOREM. If $f_0 \simeq f_1$, then there is a one-to-one correspondence between fixed point classes of f_0 and those of f_1. Hence $R(f)$ is a homotopy invariant. □

EXAMPLE 1. A non-empty fixed point class may disappear under a homotopy.

Consider maps $S^1 \to S^1$. The universal covering is $p : \mathbb{R} \to S^1$, given by $\theta \mapsto e^{i\theta}$.

Let $H = \{h_t : z \mapsto z\, e^{it\varepsilon}\}$, where $\varepsilon > 0$ is small. Take the lifting $\tilde{H} = \{\tilde{h}_t : \theta \mapsto \theta + t\varepsilon\}$. Then $p\,\mathrm{Fix}(\tilde{h}_0) = S^1$ but $p\,\mathrm{Fix}(\tilde{h}_1) = \emptyset$.

EXAMPLE 2. The correspondence may depend on the homotopy H.

Consider maps $S^1 \to S^1$. Let $f_0 = f_1 : z \to z^{-2}$. Consider two homotopies $H' = \{h'_t : z \mapsto z^{-2}\} : f_0 \simeq f_1$ and $H = \{h_t : z \mapsto z^{-2} e^{2\pi t i}\} : f_0 \simeq f_1$.

Take $\tilde{f}_0 : \theta \mapsto -2\theta$, then H' and H lift to $\tilde{H}' = \{\tilde{h}'_t : \theta \mapsto -2\theta\}$ and $\tilde{H} = \{\tilde{h}_t : \theta \mapsto -2\theta + 2\pi t\}$ respectively. So $\tilde{f}_0$ corresponds to $\tilde{f}'_1 : \theta \mapsto -2\theta$ via H', but corresponds to $\tilde{f}_1 : \theta \mapsto -2\theta + 2\pi$ via H. Thus $p\,\mathrm{Fix}(\tilde{f}_0) = \{1\}$ corresponds to $p\,\mathrm{Fix}(\tilde{f}'_1) = \{1\}$ via H', but corresponds to $p\,\mathrm{Fix}(\tilde{f}_1) = \{e^{2\pi i/3}\}$ via H.

We now turn to another view of the above correspondence.

2.5 DEFINITION. Every homotopy $H : X \times I \to X$ gives rise to a level-preserving map $\mathbb{H} : X \times I \to X \times I$ in an obvious way:

$$\mathbb{H}(x,t) = (H(x,t),t) = (h_t(x),t) .$$

The map $\mathbb{H}$ will be called the <u>fat homotopy</u> of H, and h_t will be called the t-<u>slice</u> of $\mathbb{H}$. Similarly, for a subset $A \subset X \times I$, the subset $A_t := \{x \in X \mid (x,t) \in A\} \subset X$ will be called the t-<u>slice</u> of A.

The advantage of considering $\mathbb{H}$ is that it is a self-map of $X \times I$, so we may talk about its liftings and fixed point classes. Note that the universal covering of $X \times I$ is $p \times \mathrm{id}: \tilde{X} \times I \to X \times I$.

2.6 OBSERVATION. A lifting $\tilde{H}$ of $H \longleftrightarrow$ A lifting $\tilde{\mathbb{H}}$ of $\mathbb{H}$.

$\tilde{H}$ is a lifting of $H \Longleftrightarrow \tilde{\mathbb{H}}$ is a lifting of $\mathbb{H}$.

$\tilde{f}_0$ and $\tilde{f}_1$ correspond via $H \Longleftrightarrow \tilde{f}_0$ and $\tilde{f}_1$ are slices of an $\tilde{\mathbb{H}}$.

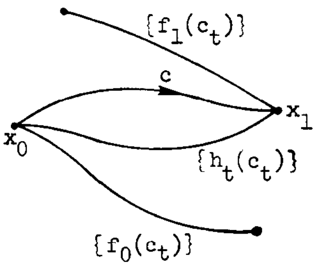
2.7 THEOREM. Let $H : f_0 \simeq f_1$ be a homotopy, $\mathbb{H}$ be its fat homotopy. Let $\tilde{f}_0$, $\tilde{f}_1$ be liftings of f_0, f_1 respectively, and let $\mathbb{F}_0 = p\,\mathrm{Fix}(\tilde{f}_0)$ and $\mathbb{F}_1 = p\,\mathrm{Fix}(\tilde{f}_1)$ be fixed point classes of f_0, f_1 respectively. Then $[\tilde{f}_0]$ corresponds to $[\tilde{f}_1]$ via H iff they are, respectively, the 0- and 1-slices of a single lifting class of $\mathbb{H}$; and $\mathbb{F}_0$ corresponds to $\mathbb{F}_1$ via H iff they are respectively the 0- and 1-slices of a single fixed point class of $\mathbb{H}$. □

This theorem is nothing but a restatement of the basic definition 2.3 in the language of fat homotopies. But it does reduce the identification of a correspondence via homotopy to the identification of a fixed point class. Thus, by 1.14 we have

2.8 COROLLARY. Let $H : f_0 \simeq f_1$ be a homotopy. Let $x_0 \in \mathrm{Fix}(f_0)$ and $x_1 \in \mathrm{Fix}(f_1)$. If $(x_0, 0)$ and $(x_1, 1)$ are connected by a continuum of fixed points of the fat homotopy $\mathbb{H}$, then the class of x_0 corresponds to the class of x_1 via H. □

Combining Theorem 2.7 with Theorem 1.10, we get

2.9 THEOREM. Let $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ be a homotopy, $x_0 \in \mathrm{Fix}(f_0)$ and $x_1 \in \mathrm{Fix}(f_1)$. Suppose x_0 belongs to a fixed point class $\mathbb{F}_0$ of f_0, and x_1 belongs to a fixed point class $\mathbb{F}_1$ of f_1. Then $\mathbb{F}_0$ corresponds to $\mathbb{F}_1$ via H iff there is a path $c = \{x_t\}_{t \in I}$ in X from x_0 to x_1 such that $\{h_t(x_t)\} \simeq \{x_t\}$ with endpoints fixed.



PROOF. $\mathbb{F}_0 \overset{H}{\rightsquigarrow} \mathbb{F}_1 \overset{2.7}{\Longleftrightarrow} (x_0, 0)$ and $(x_1, 1)$ lie in the same fixed point class of $\mathbb{H} \overset{1.10}{\Longleftrightarrow}$ there is a path $\{(x_t, s_t)\}$ in $X \times I$ from $(x_0, 0)$ to $(x_1, 1)$ such that $\{\mathbb{H}(x_t, s_t)\} = \{(h_t(x_t), s_t)\} \simeq \{(x_t, s_t)\}$, which is obviously equivalent to $\{h_t(x_t)\} \simeq \{x_t\}$. □

2.10 REMARK. This theorem can be considered as an equivalent definition of correspondence via a homotopy, for <u>non-empty</u> fixed point classes. Compare Remark 1.11.

There is still another geometric characterization of correspondence via a homotopy.

2.11 DEFINITION. A <u>deformation</u> of a homotopy $H : f_0 \simeq f_1 : X \to X$ into another $H' : f_0 \simeq f_1$ is a continuous family $\{H_u : f_0 \simeq f_1\}_{u \in I}$ with $H_0 = H$ and $H_1 = H'$.

2.12 OBSERVATION. If two homotopies $H, H' : f_0 \simeq f_1$ are deformable into each other, then they give rise to the same correspondence from the fixed point classes of f_0 to the fixed point classes of f_1. In fact, if H lifts to $\tilde{H} : \tilde{f}_0 \simeq \tilde{f}_1 : \tilde{X} \to \tilde{X}$, then the deformation $\{H_u\}_{u \in I}$ lifts to a deformation $\{\tilde{H}_u : \tilde{f}_0 \simeq \tilde{f}_1\}_{u \in I}$.

2.13 THEOREM. Let $H : f_0 \simeq f_1 : X \to X$ be a homotopy, $x_0 \in \mathrm{Fix}(f_0)$ and $x_1 \in \mathrm{Fix}(f_1)$. Then, the class of x_0 corresponds to the class of x_1 via H iff H is deformable into a homotopy $H' = \{h_t'\}_{t \in I}$ such that there is a path $c = \{x_t\}_{t \in I}$ from x_0 to x_1 with $x_t \in \mathrm{Fix}(h_t')$ for all $t \in I$.

PROOF. The "if" part follows easily from Observation 2.12 and Theorem 2.9. It remains to prove the "only if" part.

By Theorem 2.9, there is a path $c = \{x_t\}$ from x_0 to x_1 such that $\{h_t(x_t)\} \simeq \{x_t\}$, i.e. there is a $D : I \times I \to X$ with $D(0,s) = x_0$, $D(1,s) = x_1$, $D(t,0) = h_t(x_t)$, and $D(t,1) = x_t$ for all $t, s \in I$.

The polyhedron X is uniformly locally contractible (cf. [Brown (1971)], p. 39), i.e. there exists a map $\gamma : W \times I \to X$, where $W = \{(x,x') \in X \times X \mid d(x,x') < \delta\}$ for some $\delta > 0$, such that $\gamma(x,x',0) = x$, $\gamma(x,x',1) = x'$, and $\gamma(x,x,t) = x$ for all $x, x' \in X$, $t \in I$.

Let $G : \{(x,t) \mid (x,x_t) \in W\} \times I \to X$ be defined by

$$G(x,t,s) = \begin{cases} h_t(\gamma(x,x_t,2s)) & \text{if } s \le \frac{1}{2}, \\ D(t,2s-1) & \text{if } s \ge \frac{1}{2}. \end{cases}$$

It is obviously continuous. Let $\theta(t) = \delta t(1-t)$. Define a deformation $\{H_u\}_{u \in I} : X \times I \to X$ by

$$H_u(x,t) = \begin{cases} H(x,t) & \text{if } d(x,x_t) \ge \theta(t), \\ G(x,t,u - ud(x,x_t)/\theta(t)), & \text{if } d(x,x_t) \le \theta(t) > 0. \end{cases}$$

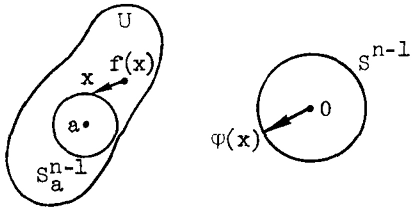
Note that $\{H_u\}$ is well-defined. The continuity is obvious except at the points with $d(x,x_t) = \theta(t) = 0$, i.e. at points with $x = x_0$, $t = 0$ or $x = x_1$, $t = 1$. The continuity at these points follows from the fact that $G(x_0,0,s) = x_0$ and $G(x_1,1,s) = x_1$ for all $s \in I$. It is easy to check that $H_u : f_0 \simeq f_1$ and $H_0 = H$, and that $H_1 = H'$ satisfies $H'(x_t,t) = x_t$, i.e. $x_t \in \mathrm{Fix}(h_t')$. □

EXERCISE. Let $H, H' : f_0 \simeq f_1 : X \to X$ be two homotopies connecting f_0 and f_1. Show that: H and H' give the same correspondence from liftings of f_0 to liftings of f_1 iff for some (hence every) point $x \in X$ the paths $\{h_t(x)\}_{t\in I}$ and $\{h'_t(x)\}_{t\in I}$ are homotopic with endpoints fixed.

3. THE FIXED POINT INDEX. The fixed point index is an indispensable tool of fixed point theory. It provides an algebraic count of fixed points in an open set. There are many different approaches to the fixed point index, all turn out to be equivalent, hence an axiomatic approach has emerged and existence and uniqueness proved. Instead of giving a self-contained treatment, we will introduce a naive, step-by-step construction of this index, and list (without proof) the most useful properties. The serious reader may consult the books [Alexandroff-Hopf], [Brown(1971)] and [Dold (1972)].

(A) THE INDEX OF AN ISOLATED FIXED POINT IN $\mathbb{R}^n$. A reasonable algebraic count of fixed points should be a generalization of the notion of multiplicity for zeros of a complex analytic function.

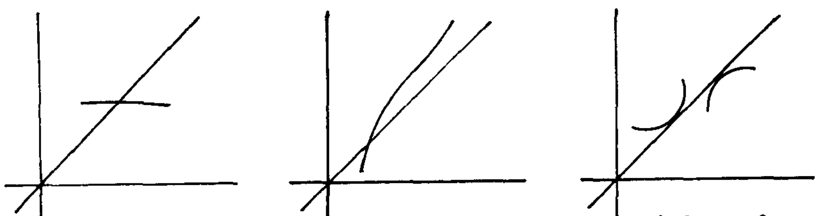
Suppose $\mathbb{R}^n \supset U \xrightarrow{f} \mathbb{R}^n$, and $a \in U$ is an isolated fixed point of f. Pick a sphere S_a^{n-1} centered at a, small enough to exclude other fixed points. On S_a^{n-1}, the vector $x - f(x) \neq 0$, so a direction field $\varphi : S_a^{n-1} \to S^{n-1}$, $\varphi(x) = \dfrac{x - f(x)}{|x - f(x)|}$, is defined.



3.1 DEFINITION. index(f,a) = degree of φ.

This definition doesn't depend on the radius of S_a^{n-1}.

EXAMPLE 1. $n = 1$. The local picture in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ of an intersection of the diagonal with the graph of f.



EXAMPLE 2. $n = 2$. Suppose f has a complex analytic expression $z \mapsto f(z)$ Then a fixed point of f is nothing but a zero of the function $z - f(z)$. Suppose z_0 is a fixed point of f. It follows from the theory of analytic functions that the multiplicity of the zero z_0 = the variance (counted by multiples of 2π) of $\arg(z - f(z))$ when z moves around z_0 once = deg φ.