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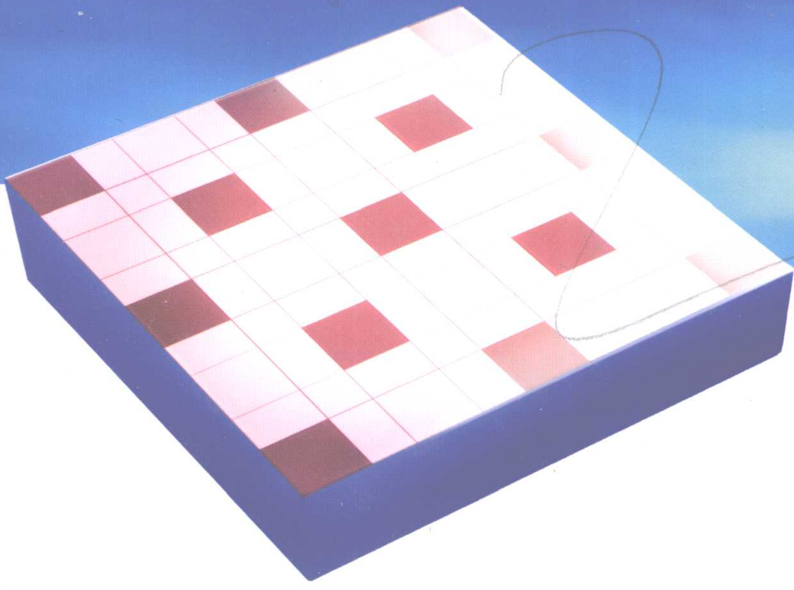
CAM 10

Industrial and Applied Mathematics in China

中国的工业与应用数学

Ta-Tsien Li
Pingwen Zhang
editors



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Preface

The China Society for Industrial and Applied Mathematics (CSIAM) had its 9th Annual Conference entitled "Industrial and Applied Mathematics in China" with 12 plenary talks from August 14 to 18, 2006 in Nanjing, China. Later on, in the 6th International Congress on Industrial and Applied Mathematics (ICIAM 2007) held from July 16 to 20, 2007 in Zurich, Switzerland, CSIAM organized an embedded meeting with the same title on July 18, 2007, which consists of two two-hour sessions with six lectures. Since all these talks concern the topic "Industrial and Applied Mathematics in China", we gather a large part of them in this volume for publication. We hope that the readers can get an impression on the present situation and trends of the industrial and applied mathematics in China from this volume and the researchers and graduate students in applied mathematics and in applied sciences can benefit from the mathematical models and methods with applications presented in this book.

We would like to take this opportunity to give our sincere thanks to all the speakers and, in particular, to those who gave their contribution to this volume for their kind help and support.

Ta-Tsien Li
July 2008

Contents

Preface

Xiaoshan Gao, Ziming Li: Mechanized Methods for Differential and Difference Equations 1

Song Jiang, Feng Xie, Jianwen Zhang: A Global Existence Result in Radiation Hydrodynamics 25

Shi Jin: Recent Computational Methods for High Frequency Waves in Heterogeneous Media 49

Ying Bao, Zhiming Ma, Yanhong Shang: Some Recent Results on Ranking Webpages and Websites 65

Lifeng Chen, Shige Peng: Report on Testing and Finding the Generating Functions g of an Option Pricing Mechanism through Market Data 79

Jun Hu, Zhongci Shi: Analysis of Nonconforming Rotated Q_1 Element for the Reissner-Mindlin Plate Problem 101

Yongji Tan: Monitoring the Corrosion of the Blast Furnace by Perturbation Method 112

Yong Xiao, Sufen Zhao, Xiaoping Wang: Numerical Study of Magnetic Properties of Nanowire Arrays 129

Zongmin Wu: Generalized B-spline 141

Xuan Zeng, Hengliang Zhu, Fan Yang, Jun Tao, Yi Wang, Jintao Xue: Mathematical Problems in System-on-Chip Design and Manufacture 166

Weiwei Qi, Ming Chen, Huitao Zhang, Peng Zhang:
A New Reconstruction Algorithm for Cone-beam CT with Unilateral Off-centered RT Multi-scan 215

Tie Zhou, Jiantao Cheng, Ming Jiang:
Bioluminescence Tomography Reconstruction by Radial Basis Function Collocation Method 229

Mechanized Methods for Differential and Difference Equations*

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Abstract

Some recent results on the mechanized methods for differential and difference equations are surveyed. The results include: the characteristic set method for differential and difference equation systems, algorithms for computing closed-form solutions of differential and difference equations, and algorithms for solving and factoring finite-dimensional linear functional systems.

1 Introduction

This paper provides a survey of some recent work on differential and difference equations by researchers at the Key Laboratory of Mathematics Mechanization and their collaborators. The work under review is greatly stimulated by Wu's method for mechanical theorem-proving in differential geometries, finding closed-form solutions of differential (difference) equations, and handling analytic and discrete mathematical objects by computers.

Differential equations describe physical laws in mechanics and geometric properties of manifolds. The characteristic set method for differential equations enables us to search for physical laws and geometric properties by computers [52]. For example, Newton's gravitational law is automatically derived form Kepler's laws [51], and "Theorema Egregium" is rediscovered by computing a characteristic set of the fundamental equations of surface theory [30].

The notion of characteristic sets for differential ideals was introduced by Ritt [42]. It plays a fundamental role in differential algebra, because

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the Hilbert Basissatz does not hold for differential ideals. The notion and algorithm of characteristic sets for polynomial and differential polynomial sets were introduced by Wu [48,50] to prove theorems in geometries and to manipulate systems of differential and algebraic equations [48,49]. Wu's work inspired a great deal of research in the communities of symbolic computation and automated reasoning. Later on, the success of Grönber bases for polynomial ideals led to methods to characterizing radical differential ideals [4]. The reader may consult [44] for more details on the recent developments of the differential characteristic set method. In this paper we briefly review Wu's scheme for differential characteristic sets and point out its recent extension to difference polynomial systems.

Integrals, special functions and combinatorial sequences are often considered as "infinite" objects. To specify them in terms of a finite amount of information on computers, one uses the differential (difference) equations annihilating these objects. For instances, automatic proofs of combinatorial identities need to find hypergeometric solutions of difference equations [37], while algorithms for symbolic integration need to compute elementary functions satisfying Risch's equation [7]. Great efforts have been made to compute closed-form solutions of linear ordinary differential (difference) equations (see, [26,39] and the references therein). There are two ways to go further: one is to look for closed-form solutions of nonlinear ordinary differential (difference) equations of some kind; the other is to develop symbolic algorithms for linear partial differential (difference) equations. We will summarize recent theoretical and algorithmic results concerning this subject.

Nonlinear differential equations arise from physics. Their analytic solutions are important for the understanding of the physical phenomena. Interesting methods to search for analytic solutions of nonlinear PDEs are given in [16,53].

Factoring polynomials helps us to solve algebraic equations. Likewise, we want to decompose differential and difference equations into those of lower orders. There have been efficient algorithms for decomposing linear ordinary differential operators [6,24,25,43]. Recent work on extending these methods to linear partial differential and difference equations [33] will be surveyed. We also mention that a decomposition algorithm for nonlinear ordinary differential equations is presented in [23].

The rest of this paper is organized as follows. In Section 2, we outline the differential characteristic method. Methods for computing rational and algebraic solutions of first-order ordinary differential and difference equations are presented in Section 3. An algebraic setting and a factorization algorithm for finite-dimensional linear functional systems are described in Sections 4 and 5, respectively.

2 The characteristic set method

The characteristic set method plays a central role in the theory and applications of mathematics mechanization. In this section, we will introduce its main features and applications in automated reasoning.

2.1 Properties of ascending chains

Let $\mathbb{K}$ be an ordinary differential field, $\mathbb{X} = \{x_1, \ldots, x_n\}$ a set of differential indeterminates, and $\mathbb{K}\{\mathbb{X}\}$ the set of *differential polynomials* in $\mathbb{X}$ with coefficients in $\mathbb{K}$. We denote $x_{i,j}$ to be the j-derivative of x_i. The *universal field* $\mathbb{E}$ over $\mathbb{K}$ is a differentially closed field containing $\mathbb{K}$ and infinitely many indeterminates. For a polynomial D and a polynomial set $\mathbb{P} \subset \mathbb{K}[\mathbb{X}]$,

$$\mathrm{Zero}(\mathbb{P}) = \{\eta \in \mathbb{E}^n |\ P(\eta) = 0, \forall P \in \mathbb{P}\}$$

is called a *variety*, and $\mathrm{Zero}(\mathbb{P}/D) = \mathrm{Zero}(\mathbb{P}) \setminus \mathrm{Zero}(D)$ is called a *quasi variety*.

A set $\mathcal{A}$ of differential polynomials is called an *ascending chain* (triangular set), or simply a chain, if after renaming the indeterminates in $\mathbb{X}$ as $\mathbb{U} = \{u_1, \ldots, u_q\}$ and $\mathbb{Y} = \{y_1, \ldots, y_p\}$, we can write $\mathcal{A}$ in the following form:

$$\begin{aligned} A_1(\mathbb{U}, y_1) &= I_1 y_{1,o_1}^{d_1} + \text{ terms of lower orders and degrees in } y_1, \\ &\cdots \\ A_p(\mathbb{U}, y_1, \ldots, y_p) &= I_p y_{p,o_p}^{d_p} + \text{ terms of lower orders and degrees in } y_p. \end{aligned} \tag{1}$$

As a matter of terminologies, o_i is called the order of A_i; I_i is called the *initial* of A_i, $S_i = \frac{\partial A_i}{\partial y_{i,o_i}}$ is called the *separant* of A_i. Write $\mathbf{I}_{\mathcal{A}} = \prod_i I_i S_i$. The *dimension* of $\mathcal{A}$ is defined to be $|\mathbb{U}| = q$, which is denoted $\dim(\mathcal{A})$. The *order* of $\mathcal{A}$ is defined to be $\mathrm{ord}(\mathcal{A}) = \sum_{i=1}^{p} o_i$. The *degree* of $\mathcal{A}$ is defined to be $\deg(\mathcal{A}) = \prod_{i=1}^{p} d_i$.

We could say that the formal solutions for a chain is basically determined. Intuitively, for a set of given values of the parameters $\mathbb{U}$, the y_i can be determined iteratively by solving univariate equations $A_i = 0$. In order to show the properties of chains, we first introduce several concepts. The *saturation ideal* of $\mathcal{A}$ is defined to be

$$\mathrm{sat}(\mathcal{A}) = \{P \in \mathbb{K}\{\mathbb{X}\} |\ \exists k \in \mathbb{N},\ \mathbf{I}_{\mathcal{A}}^k P \in (\mathcal{A})\}.$$

We may define a partial ordering among the chains in a nature way [42, 52]. It is known that any set of chains contains one with lowest order. A *characteristic set* of a differential polynomial set $\mathbb{P}$ is any chain of lowest ordering contained in $\mathbb{P}$.

A chain $\mathcal{A}$ is called *irreducible* if A_1 is an irreducible polynomial in y_{1,o_1} and A_k is an irreducible polynomial modulo $A_1, \ldots, A_{k-1}$.

Theorem 2.1. *[42, 52] Let $\mathcal{A}$ be an irreducible chain. Then* $\mathrm{sat}(\mathcal{A})$ *is a prime ideal of dimension* $\dim(\mathcal{A})$*, order* $\mathrm{ord}(\mathcal{A})$ *wrt* $\mathbb{U}$*, and degree* $\deg(\mathcal{A})$ *wrt* $\mathbb{U}$*. Conversely, a characteristic set of a prime ideal is irreducible.*

The following result shows that the dimension, order and degree of a chain are intrinsic properties.

Theorem 2.2. *[19, 22] Let $\mathcal{A}$ be a chain of form (1). If* $\mathrm{Zero}(\mathrm{sat}(\mathcal{A})) \neq \varnothing$, $\mathrm{Zero}(\mathrm{sat}(\mathcal{A}))$ *and* $\mathrm{Zero}(\mathcal{A}/\mathbf{I}_{\mathcal{A}})$ *are unmixed. More precisely, write* $\mathrm{Zero}(\mathrm{sat}(\mathcal{A}))$ *as an irredundant decomposition:* $\mathrm{Zero}(\mathrm{sat}(\mathcal{A})) = \cup_{i=1}^{r}\mathrm{Zero}\,(\mathrm{sat}(\mathcal{C}_i))$. *Then*

(1) *$\mathcal{C}_i$ is also of form (1). As a consequence,* $\dim(\mathrm{sat}(\mathcal{C}_i)) = \dim(\mathcal{A})$ *and* $\mathrm{ord}(\mathcal{C}_i) = \mathrm{ord}(\mathcal{A})$.

(2) $\deg(\mathcal{A}) \geqslant \sum_{i=1}^{r}\deg(\mathcal{C}_i)$. *Furthermore,* $\deg(\mathcal{A}) = \sum_{i=1}^{r}\deg(\mathcal{C}_i)$ *iff $\mathcal{A}$ is* saturated, *that is, the initials and seprants of $\mathcal{A}$ are invertible wrt $\mathcal{A}$.*

Another important property for chains is

Theorem 2.3. *[52] An irreducible chain admits a formal power series solution which can be computed algorithmically.*

In order to make the paper shorter, we limit to the ordinary differential case. Similar results for the partial differential case were also established, where we need to assume that the chains are either passive [49, 52] or coherent [4, 5, 27].

Similar results are also proved in the case of algebraic difference polynomials [21, 22]. However, in the difference case, we do not have algorithms to decide whether a chain is irreducible. In order to have a constructive theory, proper irreducible chains are introduced [21]. Also, Theorem 2.2 is proved only for proper irreducible chains.

2.2 Characteristic set method

The characteristic set method decomposes the zero set for a differential polynomial system in general form into the union of zero sets for chains. Since the zero set of a chain is considered to be known, this method gives a general tool to deal with differential equation systems.

Let $\mathbb{P}$ be a finite set of differential polynomials. Then we can perform the following operations:

$$\begin{array}{llllllll} \mathbb{P} & = \mathbb{P}_0 & \mathbb{P}_1 & \cdots & \mathbb{P}_i & \cdots & \mathbb{P}_m, \\ & \mathcal{B}_0 & \mathcal{B}_1 & \cdots & \mathcal{B}_i & \cdots & \mathcal{B}_m = \mathcal{C}, \\ & \mathbb{R}_0 & \mathbb{R}_1 & \cdots & \mathbb{R}_i & \cdots & \mathbb{R}_m = \varnothing, \end{array} \tag{2}$$

where $\mathcal{B}_i$ is a lowest chain in $\mathbb{P}_i$ with respect to a pre-selected partial ordering; $\mathbb{R}_i$ is the set of nonzero remainders of the polynomials in $\mathbb{P}_i$ wrt $\mathcal{B}_i$; and $\mathbb{P}_{i+1} = \mathbb{P}_0 \cup \mathcal{B}_i \cup \mathbb{R}_i$. In scheme (2), $\mathcal{B}_m = \mathcal{C}$ verifies

$$\text{prem}(\mathbb{P}, \mathcal{C}) = \{0\} \text{ and } \text{Zero}(\mathbb{P}) \subset \text{Zero}(\mathcal{C}), \tag{3}$$

where prem denotes the differential pseudo-remainder. Any chain $\mathcal{C}$ verifying the property (3) is called a *Wu characteristic set* of $\mathbb{P}$.

Theorem 2.4 (Wu's Well-ordering Principle). *[49, 52] Let $\mathcal{C}$ be a Wu characteristic set of a finite set $\mathbb{P}$ of differential polynomials. Then:*

$$\text{Zero}(\mathbb{P}) = \text{Zero}(\mathcal{C}/\mathbf{I}_{\mathcal{C}}) \bigcup \cup_i \text{Zero}(\mathbb{P} \cup \mathcal{C} \cup \{I_i\}),$$
$$\text{Zero}(\mathbb{P}) = \text{Zero}(\text{sat}(\mathcal{C})) \bigcup \cup_i \text{Zero}(\mathbb{P} \cup \mathcal{C} \cup \{I_i\}),$$

where I_i are the initials and separants of the polynomials in $\mathcal{C}$.

Using the well-ordering principle recursively, we obtain the following key result.

Theorem 2.5 (Ritt-Wu's Zero Decomposition Theorem). *[42,52] There is an algorithm which permits to determine, for a given finite set $\mathbb{P}$ of differential polynomials, a finite set of (irreducible) chains $\mathcal{A}_j$ such that*

$$\text{Zero}(\mathbb{P}) = \cup_j \text{Zero}(\mathcal{A}_j/\mathbf{I}_{\mathcal{A}_j}) = \cup_j \text{Zero}(\text{sat}(\mathcal{A}_j)).$$

Let $\mathbb{P}$ be a finite subset of $\mathbb{K}\{\mathbb{U}, \mathbb{X}\}$, and $D \in \mathbb{K}\{\mathbb{U}, \mathbb{X}\}$, where $\mathbb{U} = \{u_1, \ldots, u_m\}$ and $\mathbb{X} = \{x_1, \ldots, x_n\}$. The projection of Zero($\mathbb{P}/D$) to $\mathbb{U}$ is defined as follows:

$$\text{Proj}_{\mathbb{X}}\text{Zero}(\mathbb{P}/D) = \{e \in \mathbb{E}^m \mid \exists\, a \in \mathbb{E}^n \quad \text{s.t.} \quad (e, a) \in \text{Zero}(\mathbb{P}/D)\}.$$

Projection for quasi-varieties can be computed with the characteristic set method.

Theorem 2.6 (Projection Theorem). *[19] For a finite subset set $\mathbb{P} \subset K\{\mathbb{U}, \mathbb{X}\}$ and $D \in K\{\mathbb{U}, \mathbb{X}\}$, we can compute chains $\mathcal{A}_i$ and polynomials D_i in $\mathbb{K}[\mathbb{U}]$ such that*

$$\text{Proj}_{\mathbb{X}}\text{Zero}(\mathbb{P}/D) = \cup_{i=1}^{l} \text{Zero}(\mathcal{A}_i/D_i\mathbf{I}_{\mathcal{A}_i}).$$

The concept of characteristic sets for prime ideals was introduced by Ritt [42]. The notion of characteristic sets given above, the well-ordering principle, and the current form of zero decomposition theorems were introduced by Wu [48,49,52]. An implementation of the method can be found in [46]. In order to improve the efficiency, new characteristic set methods were proposed [4, 5, 9, 10, 18, 27, 40, 45]. The characteristic set method was used to solve certain problems for analytical functions [41].

A characteristic set method for algebraic difference equation systems was proposed in [21, 22]. It is quit surprising that there are no essential progresses for the theory and algorithms of difference characteristic set methods since the early work of Ritt and his colleagues in the 1930s. In [21], an algorithm was proposed to decompose the zero set a difference polynomial system into the union of unmixed zero sets of difference polynomial systems represented by proper irreducible chains. In [22], a new resolvent theory for difference polynomial systems was proposed.

To solve a set of equations in triangular form, we need to solve univariate equations in a cascade form. The resolvent methods were introduced to reduce the solving of equation systems into the solving of one univariate equation plus a set of linear equations [13, 22].

2.3 Wu's method of automated geometry theorem proving and discovering

A geometry theorem is called a *theorem of equality type*, if after introducing coordinates, the theorem can be expressed in the following form

$$\forall x_i[(H_1 = 0 \wedge \cdots \wedge H_s = 0 \wedge D_1 \neq 0 \wedge \cdots \wedge D_t \neq 0) \Longrightarrow (C = 0)], \quad (4)$$

where H_i, D_i, C are in $\mathbb{K}\{\mathbb{X}\}$.

For theorems of equality type, we have the following principles of mechanical theorem proving, which are consequences of Theorems 2.1 and 2.4.

Theorem 2.7. *[49] For a geometry statement of form (4), let $\mathcal{A}$ be a Wu-characteristic set of $\{H_1, \ldots, H_s\}$. If* $\mathrm{prem}(C, \mathcal{A}) = 0$*, then the statement is valid under the non-degenerate condition* $\mathbf{I}_{\mathcal{A}} \neq 0$.

Note that the non-degenerate condition $\mathbf{I}_{\mathcal{A}} \neq 0$ is generated automatically by the algorithm.

Theorem 2.8. *[52] Let $D = \prod_i D_i$. By Theorem 2.5, we have*

$$\mathrm{Zero}(\{H_1, \ldots, H_s\}/D) = \cup_{i=1}^{l} \mathrm{Zero}(\mathrm{sat}(\mathcal{A}_i)/D).$$

If $\mathrm{prem}(C, \mathcal{A}_i) = 0, i = 1, \ldots, l$*, then the statement is true. If $\mathcal{A}_i$ is irreducible and* $\mathrm{prem}(C, \mathcal{A}_i) \neq 0$*, then the statement is not valid on* $\mathrm{Zero}(\mathrm{sat}(\mathcal{A}_i)/D)$.

As an example, let us show how to prove Newton's gravitational law with Kepler's laws. The first and second Kepler laws state that each planet describes an ellipse with the sun in one focus and the radius vector drawn from the sun to a planet sweeps out equal areas in equal times. The Newton's law states that the acceleration is reversely proportional to the distance from the planet to the sun. We may use differential equations $K_1 = 0, K_2 = 0$, and $N_1 = (ar^2)' = 0$ to represent these laws:

$$\begin{aligned}
h_1 &= r^2 - x^2 - y^2 = 0,\\
h_2 &= a^2 - x''^2 - y''^2 = 0,\\
K_1 &= r - p - ex = 0 \wedge p' = 0 \wedge e' = 0,\\
K_2 &= y'x - yx' - h = 0 \wedge h' = 0,\\
d_1 &= p \neq 0 \quad \text{(The ellipse is not a line.)}.
\end{aligned}$$

Then, we need to show

$$\forall x, y, p, e, a, r[(K_1 = 0 \wedge K_2 = 0 \wedge h_1 = 0 \wedge h_2 = 0 \wedge d_1 \neq 0) \Rightarrow N_1 = 0].$$

By Theorem 2.5 ($p < e < x < y < r < a$),

$$\text{Zero}(\{K_1, p', e', h_1, h_2, n_2\}/p) = \text{Zero}(\text{sat}(\mathcal{A}_1)p),$$

where $\mathcal{A}_1$ is a chain. By computation, we have $\text{prem}(n_1, ASC_1) = 0$, which proves Newton's law.

There are two kinds of problems in differential geometry other than theorem proving. One is finding locus equations, the other is deriving geometry formulas. For a geometric configuration given by a set of polynomial equations $h_1(\mathbb{U}, x_1, \cdots, x_p) = 0, \cdots, h_r(\mathbb{U}, x_1, \cdots, x_p) = 0$, we want to find a relation between arbitrarily chosen variables $\mathbb{U}$ (parameters) and a dependent variable, say, x_1. Wu pointed out that the characteristic set method can be used to discover such unknown geometric formulas [51]. Actually, Newton's law can be deduced from Kepler's laws automatically in this way. More detailed accounts can be found in [10,11,30,45].

The characteristic set method can be used to prove a much wider class of geometry theorems. Let $\mathbb{E}$ be a differentially closed extension of $\mathbb{K}$, say, the field of meromorphic functions [42]. A *first order formula* over $\mathbb{E}$ can be defined as follows.

1. If $P \in \mathbb{K}[\mathbb{X}]$, then $P(\mathbb{X}) = 0$ is a formula.
2. If f, g are formulas, then $\neg f$, $f \wedge g$, and $f \vee g$ are formulas.
3. If f is a formula, then $\exists x_i \in \mathbb{E}(f)$ and $\forall x_i \in \mathbb{E}(f)$ are formulas.

A formula can always be written as a prefix canonical form

$$\phi = Q_1 y_1 \ldots Q_m y_m \psi(u_1, \ldots, u_d, y_1, \ldots, y_m), \tag{5}$$

where Q_k is a quantifier $\exists$ or $\forall$ and ψ a formula free of quantifiers. For a first order formula ϕ of form (5), there exists a fundamental problem:

Quantifier Elimination: Find a formula $\theta(u_1, \ldots, u_d)$ such that θ is equivalent to ϕ. If $d = 0$, we need to decide whether ϕ is valid or not.

As a consequence of Theorem 2.6, we have

Theorem 2.9. *There exists a decision procedure for the first order theory over a differentially closed field.*

3 Rational and algebraic solutions of ODEs and OΔEs

For brevity we abbreviate ordinary difference equations as OΔE.

By decomposing the zero set of a differential polynomial system into the zero sets of chains, the characteristic set method gives a complete way to describe the structure for the zero sets of equation systems. In particular, finding the solutions of differential polynomial systems can be reduced to finding those of a single differential equation or a system of equations in a single variable.

Closed-form solutions of linear ODEs and OΔEs were widely studied. On the other hand, similar results to nonlinear ODEs are very limited. In this section, we summarize some recent results on finding rational and algebraic solutions to nonlinear ODEs and OΔEs. It is interesting to see whether these results can be treated uniformity with the differential Galois theory [35].

3.1 Rational and algebraic solutions of algebraic ODEs

Let $P \in \mathbb{K}\{y\} \setminus \mathbb{K}$ be an irreducible differential polynomial in an indeterminate y and

$$\Sigma_P = \{A \in \mathbb{K}\{y\} | SA \equiv 0 \bmod \{P\}\},$$

where S is the separant of P and $\{P\}$ is the radical differential ideal generated by P. Then Σ_P is a prime ideal [42]. A generic zero of Σ_P is defined to be a *general solution* of $P = 0$. In particular, an *algebraic general solution* of $P = 0$ is a general solution $\hat{y}$ which satisfies the following equation

$$G(x, y) = \sum_{i=0}^{n} a_i(x) y^i = 0, \tag{6}$$

where a_i is a polynomial in x with degree α_i and with constant coefficients, and $G(x,y)$ is an irreducible polynomial in x,y. When $n=1$, $\hat{y}$ is called a *rational general solution* of $P=0$.

For $\alpha_0,\alpha_1,\cdots,\alpha_n\in\mathbb{Z}_{\geqslant 0}$, we define the differential polynomial

$$\mathbb{D}_{(\alpha_0;\alpha_1,\cdots,\alpha_n)} := \det(\mathcal{A}_{(h,\alpha_1;\alpha_0)}(y)|\mathcal{A}_{(h,\alpha_2;\alpha_0)}(y^2)|\cdots|\mathcal{A}_{(h,\alpha_n;\alpha_0)}(y^n)),$$

where

$$\mathcal{A}_{(h,\alpha;k)}(y) := \begin{pmatrix} \binom{k+1}{0}y_{k+1} & \binom{k+1}{1}y_k & \cdots & \binom{k+1}{\alpha}y_{k+1-\alpha} \\ \binom{k+2}{0}y_{k+2} & \binom{k+2}{1}y_{k+1} & \cdots & \binom{k+2}{\alpha}y_{k+2-\alpha} \\ \vdots & \vdots & \cdots & \vdots \\ \binom{k+h+1}{0}y_{k+h+1} & \binom{k+h+1}{1}y_{k+h} & \cdots & \binom{k+h+1}{\alpha}y_{k+h+1-\alpha} \end{pmatrix}.$$

We have

Lemma 3.1. *[1] $y(x)$ satisfies an equation of the type (6) if and only if $\mathbb{D}_{(\alpha_0;\alpha_1,\cdots,\alpha_n)}(y(x))=0$. As a consequence, we give a defining differential equation for algebraic functions.*

When $n=2,\alpha_1=\alpha_2=1$ and $\alpha_0=2$,

$$\mathbb{D}_{(2;1,1)} = \begin{vmatrix} y_3 & 3y_2 & (y^2)''' & 3(y^2)'' \\ y_4 & 4y_3 & (y^2)^{(4)} & 4(y^2)''' \\ y_5 & 5y_4 & (y^2)^{(5)} & 5(y^2)^{(4)} \\ y_6 & 6y_5 & (y^2)^{(6)} & 6(y^2)^{(5)} \end{vmatrix}.$$

We have $\mathbb{D}_{(2;1,1)}(y(x))=0$ if and only if

$$(a_{2,1}x+a_{2,0})y^2(x)+(a_{1,1}x+a_{1,0})y(x)+a_{0,2}x^2+a_{0,1}x+a_{0,0}=0$$

for constants $a_{i,j}$.

The key to find a rational and algebraic function solutions is to give a degree bound for the solution. We can give these degree bounds for first order autonomous ODEs. In what follows, let $F(y,y_1)=0$ be a first order autonomous ODE. Then we have

Theorem 3.2. *[1] If $G(x,y)=0$ defines a nontrivial algebraic solution of $F=0$, then*

(1) $\deg(G(x,y),x)=\deg(F,y_1)$,

(2) $\deg(G(x,y),y)\leqslant\deg(F,y)+\deg(F,y_1)$.

The following example shows that the bound in (2) is optimal. Let $n>m>0$ and $(n,m)=1$. Then $G=y^n-x^m$ is irreducible. $y^n-x^m=0$ is an algebraic solution of $F=y^{n-m}y_1^m-(m/n)^m=0$. Here, $\deg(G(x,y),y)=\deg(F,y)+\deg(F,y_1)$.

For rational solutions, we could give the exact degree bound [17].