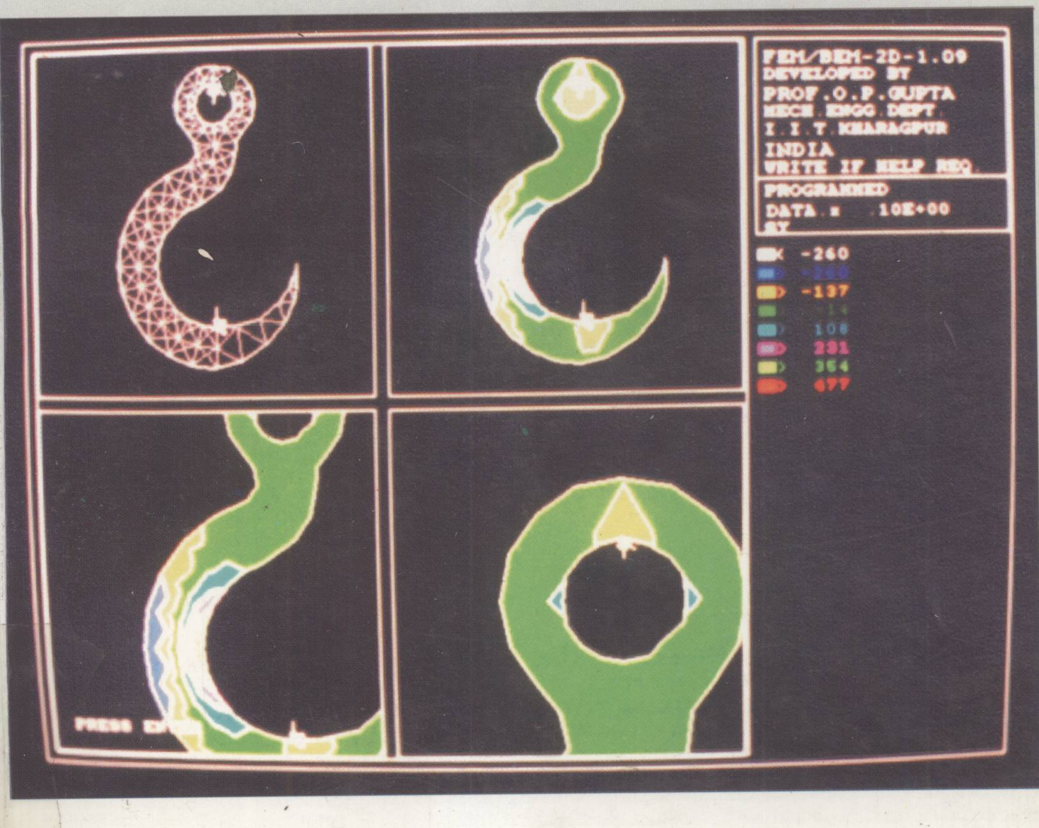
Finite and Boundary Element Methods in Engineering

O.P. Gupta



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Dedicated to
my wife, Asha and son Vikas

PREFACE

CAD, CAM, CAE are the areas which form the very basis of modern-day engineering practice and the Finite Element Method is the analytical backbone of these areas. The phenomenal growth which the last few decades have seen in the areas of numerical modelling using Finite and Boundary Element methods is due to the importance of these methods to virtually all the engineering disciplines as well as science. Software packages of all hue and sizes are available not only for Finite Element analysis but for Boundary Element analysis also. Undoubtedly, these have brought the potentials of a complex but very strong technique to the doorstep of the design engineer. Software producers sometimes claim that design engineers can use these packages with only a cursory understanding of the subject. This is true for some simple design areas of elastic stress analysis. The moment attempts are made to solve more involved problems of manufacturing or thermal or flow analysis, the user feels handicapped due to the limited information supplied in software manuals and his general lack of understanding of the basics of the subject.

Finite and Boundary Element methods now constitute important subjects in senior under-graduate and post-graduate curricula of various universities. Although a number of books deal with these topics, the author has found them either too complex or too elementary to cater to the requirements of either students or practising engineers.

At one extreme there are books which, like an encyclopaedia, cover a vast spectrum of applications of FEM with brief but very specialized treatment of each such application, requiring very advanced knowledge of the engineering field to which the application belongs. At the other extreme there are books which deal with specific applications to a particular engineering branch, such as Civil Engineering, or a particular topic, such as fluid mechanics, or which take a superficial look at the subject and deal mostly with computer codes of FEM and its implementation. These fall short of the requirements of general designers and students in several ways.

Students of engineering seeking to learn FEM find the first category of books extremely difficult to follow while the second category does not give sufficient insight into the subject if it deals with the provisions of the software package only. With engineering curricula varying widely from university to university and from one country to another and emphasis on coverage of the subject being highly dependent on the teacher, the student taking up FEM and BEM are generally not well prepared in subjects deemed prerequisites. They cannot follow the advanced research-oriented texts of the first category of books. Even practising engineers have to brush up their knowledge of the basics of such subjects as theory of elasticity, plasticity, fluid mechanics, heat transfer etc., in order to come to grips with application of numerical techniques, such as FEM and BEM in their areas. The author's experience in teaching the subject has been that students as a rule require 'freshening' in the basics of the prerequisite subjects.

This book has been written precisely for that category of readers who need a thorough and detailed coverage of essential features of basic subjects before embarking on numerical implementation in these areas. Clarity and completeness of treatment are the main objectives of the book so that a solid and wide-ranging foundation of the subject is built first. Proceeding from this stage the student will find himself well prepared to take on the latest research-oriented texts or research papers, and the practising engineer sufficiently prepared to tackle advanced applications of FEM and BEM in his field of interest using the software package available with him.

The organization of the book is such that it gradually builds up the subject starting from simple topics and application areas, leading into more involved and complex topics which represent the true practical application areas of these techniques. The classical application of FEM to elastic stress analysis is discussed first and basic concepts of the elemental stiffness matrix, its assembly and solution procedure are explained. The technique of potential energy minimization is introduced and then variational approach and weighted residue techniques are explained, leading to the formulation of problems of steady and unsteady-state heat flow (potential problems). The theory of bending of beams and plates is introduced before taking up FEM formulation for plate and shell elements. The more complex non-linear, curved and isoparametric elements are then discussed in detail and the procedure of numerical integration using the Gauss quadrature formulas is elaborated in depth. A chapter on various types of non-linearity follows. It deals with topics, such as temperature dependence of properties, plasticity and methods, such as direct iteration and the Newton-Raphson approach. The concept of tangent matrix is thoroughly explained. An important area covered in this book, generally not found in books on FEM, is the FEM formulation for problems of fluid flow leading to discussion of viscoplastic formulation for application in metal forming. The next two chapters deal with the boundary element method, a sister technique of FEM that is fast becoming a very popular tool of numerical analysis. This is another unique feature of this book.

The author feels that treatment of the boundary element method along with the finite element method not only makes discussion of techniques of numerical modelling complete, but also places the reader in an advantageous position in understanding and appreciating the research work being pursued and applications being tackled using BEM. The two chapters on this topic deal first with the fundamental concepts and then with formulation of potential problems (heat flow etc.) and problems of elastic stress analysis. The uncommon features of the mathematics associated with BEM are fully explained and wherever necessary a separate appendix has been added for clarity.

Although emphasis in the book is on fundamentals, important research topics are not left out. There is strong interest among researchers in finding ways and means of improving the reliability of finite and boundary element analyses, and in providing speedy solutions through use of automatic mesh generators and faster computer algorithms so that these methods can become truly effective tools in computer-aided engineering (CAE). A full chapter is devoted to the topics of automatic mesh generation, adaptive mesh refinement strategies for providing solutions with controlled level of error and faster matrix equation solvers, such as frontal solver, which can speedily tackle the realistic problems involving tens of thousands of nodes. These strategies can even be implemented on present-day powerful PCs.

Since the book mainly deals with fundamentals, only essential references are incorporated at the end of each chapter as additional sources relevent to the topic discussed.

The book also includes a working computer code on application of FEM basically as a tutorial program, which will help readers to understand the methodology adopted in writing such programs. It can also be used for solving simple stress analysis problems. It is hoped that the book will serve the needs of students and practising engineers who require a clear and complete understanding of the fundamentals and who also desire an exposure to the major directions of current research in this field.

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Figures 10.5, 10.6 and 10.7 are reproduced from Mattiasson, K., A. Samuelson, R.D. Wood and O.C. Zienkiewicz (eds.), NUMIFORM 86: Numerical methods in industrial forming processes—Proceedings of the 2nd international conference, Gothenburg, 25-29 August 1986. 404 pp.,Hfl.295/- US$170.00. A.A. Balkema, P.O. Box 1675, Rotterdam, Netherland.

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NOMENCLATURE

Ch. 1, 2, 3, 4, 5

$\vec{A}, \vec{B}\ldots$	Vector
$[A]^T$	Transpose of matrix $[A]$
$[A][B]$	Product of matrices $[A]$ and $[B]$
a_i, b_i, c_i, e_i	Coefficients in the expression for linear shape function for i-th node (N_i)
$[B]$	Strain-displacement matrix
c	Specific heat
$[D]$	Elasticity (stress-strain) matrix
$\{d\}$	Displacement vector at any point within an element
$\{d_i\}$	Nodal displacement vector at i-th node
$\{d^e\}$	Elemental displacement vector, consisting of all nodal displacement vectors for an element
E	Coefficient of elasticity
$\vec{e}_1, \vec{e}_2, \vec{e}_3$	Unit vectors in x_1, x_2, x_3 directions
F_R	Axial force in a rod element
F, F_1, F_2	Functionals of function T, T_x, T_y or u, v, $\partial u/\partial x$ etc
$\{f_Q^e\}, \{f_q^e\}$	Load vectors corresponding to internal heat generation, $\dot{Q}$ and surface convection
$\{f_{\varepsilon 0}^e\}, \{f_p^e\}, \{f_w^e\}, \{f_{\sigma 0}^e\}$	Load vectors due to initial strain, distributed load, body force and initial stress respectively
G	Shear modulus
$[h^e], [\overline{h^e}]$	Elemental conductance matrix and matrix corresponding to surface convection respectively
$\vec{i}, \vec{j}, \vec{k}$	Unit vectors in axial directions x, y, z
k	Thermal conductivity
$[K]$	Stiffness matrix
L	Length of rod element
l_x, l_y, l_z	Direction cosines of normal to surface
l_r, l_z	Direction cosines of normal to exterior surface for axi-symmetric case

M_{ij}	Minor of a matrix
$[N]$	Shape function matrix for an element
N_i	Individual shape function, related to node i
$N_{\tau i}$	Shape function for i-th node in time domain
p	Intensity of uniformly distributed load
$\dot{Q}$	Rate of internal heat generation per unit volume
q, α	Coefficients related to convective heat transfer from surface
$\{R\}$, $\{T\}$	Overall nodal force vector
$\{R_i\}$	External force vector at i-th node
S^e	Exterior boundary surface of an element on which distributed load or convection conditions exist
T_1, T_2, T_i	Nodal temperature
T	Temperature at a point
T_x, T_y, T_z	Gradients of temperature along Cartesian direction
$\{T^e\}$	Elemental temperature vector
T_i^k	Temperature at i-th node at k-th time step
u, v, w	Displacements in axial direction x, y, z (or axi-symmetric r, z) for a point (u is also used to represent a scalar function, Ch. 3)
u_1, u_2, u_3	Displacement referred to axial directions x_1, x_2, x_3
u_1, v_i, w_i	Nodal displacements in Cartesian as well as (u_i, v_i) in cylindrical axi-symmetric coordinates
V^e	Elemental volume
W	Body for per unit volume
w, w_1, w_2	Weighting functions
x, y, z	Cartesian coordinates of a point within an element
x_i, y_i, z_i	Cartesian coordinates of i-th node
γ_{ij} $(= 2\,\varepsilon_{ij})$	Shear strain
Δ	Area of triangular element
$\{\delta\}$	Overall (global) nodal displacement vector
ε_{ij}	Components of strain tensor
$\{\varepsilon\}$	Strain vector
$\{\varepsilon_0\}$	Initial strain vector
ε_x, ε_y, γ_{xy}	Components of 2D strain vector
ν	Poisson ratio
Π	Potential energy of whole system
Π_{se}, Π_s^e	Strain energy of e-th element
Π_w, Π_{wn}	Work done by external forces (component of potential energy for n-th node, if n is specified)
ρ	Density
$\{\sigma\}$	Stress vector

$\{\sigma_0\}$	Initial stress vector
τ, τ_1, τ_2	Time, time instants
ϕ	A scalar function
$(\ ).(\ )$	Vector dot product
$(\ )\times(\ )$	Vector cross product

Ch. 6, 7

a_1, b_1, c_1, d_1	Some coefficients exclusively defined in the context of plate bending (Sec. 7.11.2)
D	Factor defined in eq. (6.33)
$[D_b]$	Matrix used in context of bending strain energy
$\{d_i\}$	Total displacement vector for i-th node containing all displacements (and rotations) as applicable in a particular context
$\{d_b^e\}$	Elemental displacement and rotation vector for plate bending
$\{d_{bi}\}$	Transverse displacement and rotation vector for node i during plate bending alone
$\{d_p^e\}$	Elemental displacement vector for in-plane stresses containing displacements u_1, v_1, u_2, v_2....
f_1, f_2	Function f at different points
$\{f_p^e\}$	Elemental vector due to distributed load of intensity P per unit area
$\{f_{\varepsilon 0}^e\}$	Elemental load vector due to initial strain (thermal strain) in case of 2D plane stresses
G	Shear modulus (also factor for transforming area integral from Cartesian to natural coordinates)
$[J]$	Jacobian of transformation from one coordinate system to other
$[K_b^e]$	Elemental stiffness matrix for plate bending alone
$[K_g^e]$	Elemental stiffness matrix in global coordinate system
$[K_p^e]$	Elemental stiffness matrix for in-plane loading
$[K_{bij}]$	Submatrices of order 3×3 being the components of elemental plate bending stiffness matrix
$[K_{pij}]$	Submatrices of order 2×2 belonging to elemental in-plane stiffness matrix (eq. 7.81)
L	Length of a beam element
L_1, L_2, L_3, L_4	Area (or volume) coordinates

M_x, M_y	Bending moments about x and y axes
M_{xy}, M_{yx}	Twisting moment (torque) about x or y axes in the context of plate bending
$N_1, N_2 \ldots$	Shape functions
$N_1^0, N_2^0 \ldots$	Shape function used to define coordinates (sometimes)
$[R]$	Rotation matrix for global to local transformation of displacement
$[T]$	Rotational transformation matrix for transforming complete elemental displacement-cum-rotation vector from global to local coordinates
$\{T\}$	Overall nodal load and nodal moment vector in bending of beam, plates
u, v, w	Displacements along Cartesian axes
u_g, v_g, w_g	Displacements in global Cartesian coordinate system
w_1, w_2	Weightages associated with respective functions (in the context of numerical integration)
$\Delta\phi_x, \Delta\phi_y$	Incremental twist along x or y axis during plate bending
θ_s, θ_n	Rotation along the edge ij and normal to it (in-plane) for triangular plate bending element
θ_x, θ_y	Rotations about x and y axes
ξ, η, ζ	Natural (local) coordinates
$\Pi^e \; (= \Pi^e_{S+p} + \Pi^e_S)$	Total elemental potential energy due to bending, in-plane and thermal strains as well as distributed load

Ch. 8, 9, 10

A, A^e	Total and elemental area for 2D domain
C_x	Acceleration in x-direction
$[D_0]$	Material matrix (eq. 10.32)
$d\bar{\varepsilon}^P$	Equivalent plastic strain increment
$d\lambda$	Parameter associated with plastic work
$d\varepsilon_x^P, d\varepsilon_y^P, \ldots d\gamma_{xy}^P$	Components of incremental plastic strain
e	Thermal energy or other type of internal energy (excluding strain energy) per unit volume
$F(\sigma_{ij}, k), F$	Yield criterion
$f_x \cdot \rho$	Component of body force per unit volume (in x-direction)—same as $f_i\rho$ for $i = 1$
$\{f^n\}$	Load vector during n-th iteration where load (such as convective heat transfer) is dependent on function (say temperature)

G	Shear modulus (Ch. 9) and scalar potential related to gravity force (see Sec. 8.5)
g	Acceleration due to gravity
H, H_0, h	Water head (H also represents slope of stress-plastic strain curve in uniaxial tension test)
i,j	Suffixes normally indicating coordinate axes
k	Thermal conductivity (also yield stress in shear—used in yield criterion)
k_x, k_y, k_x^n, k_y^n	Thermal conductivity in x, y directions
k_x', k'	Partial differential of k_x, k_y with respect to temperature
k, l	Suffixes normally indicating the nodes in an element
$[K_T^n]$	Tangent matrix after n-th iteration
$[K_T^{ne}]$	Elemental tangent matrix
m	Coefficient related to frictional force (also total number of elements in domain)
$[m]$	Six component row vector (defined in eq. (10.28))
n	Total number of nodes
p	Pressure
$\bar{p}$	Hydrostatic pressure
$\{Q^n\}$	Vector containing residues in the concerned equation when n-th iterated value of parameter $\{T\}$ is substituted in it
$Q_1, Q_2.., Q_1^n, Q_2^n ..$	Components of vector $\{Q^n\}$ which represent residues (Fig. 9.5 or equivalent)
q_x	Temperature gradient in x-direction
q_i^{ne}	Elemental contribution equivalent of Q_i^n
r	Number of elements (at boundary) where traction is specified
R	Relaxation factor during iterations
$[R]$	Rotation matrix
S, Γ and S^e, Γ^e	Exterior surface of domain and an element respectively
T_i	Component of surface traction at boundary (force/unit area) in the direction of i-th axis
T_s, T_n	Traction along and normal to the boundary
T_ξ, T_ζ	Components of traction along boundary surface in the direction of natural coordinates ξ, ζ. (direction η is normal to boundary)

t	Time
$[T]$	Rotational transformation matrix
T_i^n	Temperature at i-th node at the end of n-th iteration
u, v, w	Fluid velocities in x, y, z directions. These are the same as v_1, v_2, v_3 when coordinate axes are taken as x_1, x_2, x_3
u_i	Components of displacement at a point
V, Ω and V^e, Ω^e	Volume of domain and that of element
v_i	Component of velocity vector anywhere in the domain
v_s	Net sliding velocity at boundary (of forging)
v_ζ, v_ξ, v_η	Components of velocity vector along directions of natural coordinate for an element
v_0	A constant (Sec. 10.2)
$(v_i)_p$	i-th component of velocity vector at node p
$v_{i,j}, \dot{\varepsilon}_{ij}$	Velocity gradient (strain rate)
$\bar{v}, \bar{v}_1, \bar{v}_2, \bar{v}_n$	Specified velocities at boundaries
x_i	Coordinate along i-th Cartesian axis
α	Number of nodes in an element
α_{ij}	Components of tensor related to kinematic hardening (eq. 9.44)
$\{\Delta d^e\}$	Incremental elemental displacement vector
$\{\Delta \varepsilon_n^e\}_{in}$	Elemental inelastic strain vector in n-th iteration
δ_{ij}	Kronecker delta
$\varepsilon, \varepsilon_i$	Error
ε_{ij}	Nine components of strain tensor
$\dot{\varepsilon}_{ij}, v_{i,j}$	Nine components of strain rate (velocity gradient) tensor
$\{\dot{\varepsilon}\}$	Strain rate vector
ε_i	Magnitude of error at i-th node
$\dot{\bar{\varepsilon}}^p$	Equivalent plastic strain rate
$\dot{\varepsilon}_v$	Volumetric strain rate
$\dot{\bar{\varepsilon}}$	Equivalent (plastic) strain rate for rigid-viscoplastic material (Sec. 10.3)
λ	Penalty function (also used as second viscosity coefficient, Secs. 8.2.5, 8.3.1)
μ	Dynamic viscosity (equivalent viscosity in metal forming)
μ^0	Coefficient of friction
ν	Kinematic viscosity

ρ	Density
$\overline{\sigma}$	Equivalent stress
$\sigma_x, \sigma_y, \sigma_z$	Normal stresses (equivalent to $\sigma_{11}, \sigma_{22}, \sigma_{33}$)
σ_{ij}	Nine components of stress tensor
$\sigma'_{ij}\,(\tau_{ij}), \sigma'_x, \sigma'_y, \ldots \tau'_{zx}$	Deviatoric stress components
$\{\sigma'\}$	Deviatoric stress vector
$\sigma_{y1}, \sigma_{y2}, \sigma_{yc}$	Yield points at different locations of stress-strain curve
$\tau_{xy}, \tau_{yz}, \tau_{zx}$	Shear stresses (equivalent to $\sigma_{12}, \sigma_{23}, \sigma_{31}$)
ϕ	A scalar function such that $v_i = \partial\phi/\partial x_i$
ψ	Stream function (Sec. 8.6.1)
ω_{ij}	Rotation about k-th axis

Ch. 11, 12, 13

$b, b\,(q)$	Body force term. In thermal problem it represents $\dot{Q}/a$, a = diffusivity
$C, C(p), C(P), C_p, C_P$	A parameter C and its values at interior source point p or source point located at boundary, P
$C(P_i), \phi(P_i)$	Values of C and ϕ for source point located at node i
$\vec{e}_r$	Unit vector in the direction of r
$\vec{e}_n$	Unit vector along outward normal to surface
$\lVert e \rVert$	Energy (error) norm
$\lVert e_t \rVert$	Total energy norm
$\lVert e_\sigma \rVert$	Stress (error) norm
$[g^e\,(p)], [\bar{h}^e\,(p)]$	Row vectors associated with source point p
$K_1(P, Q), K_2(P, Q)$	First and second kernels (functions of fundamental solution)
n	Refers to the direction of outward normal to boundary
$(p), (q)$	Indicate source and observation points respectively, located inside the domain
$(P), (Q)$	Indicate source and observation points respectively, located at boundary
(Q_e)	Observation point located within an element—'e'
$r, r\,(P, Q), r\,(p, Q)$	Distance between source and observation points
T_i	Components of traction along i-th axis
u_i	Displacement in the i-th axial direction
$\breve{u}_i, \breve{T}_i, \breve{\sigma}_i, \breve{u}_{im}, \breve{T}_{im}$	Displacement, traction and stress values due to an arbitrary source of unit intensity (force in m-th axial direction)

$\overline{u}_i$, $\overline{T}_i$	Specified displacement and traction at boundary
$(u_i)_k$, $(T_i)_k$	Component of displacement or traction in i-th direction at node k
$\Vert u \Vert$	Displacement norm
w, $w\,(P, Q)$	Fundamental solution of a potential problem (for P as source point and Q, observation point)
w_1, w_2, w_3	Weighting functions
$x_i\,(P)$, $x_k\,(P)$	Cartesian coordinates of source point P(i, k may take values 1, 2, 3)
Γ_1, Γ_2, Γ	Boundary of domain
δ	Radius of an infinitesimal circle (or sphere) around source point
$\theta = \partial\phi/\partial n$	Potential gradient normal to boundary
$\{\theta^e\}$, $\{\Theta\}$	Elemental and global gradient vector
$\overline{\theta}$, $\overline{\phi}$	Values of θ and ϕ specified at boundary
$\theta^n\,(P)$, $\theta^n\,(Q)$	Value of θ at the end of n-th time step
λ, μ	Elastic constants (Sec. 12.2.2)
ρ_i	Body force per unit volume along axial direction i
σ_{ij}	Components of stress tensor
σ	Smoothened stress—supposed to be close to the exact stress value
$\overline{\sigma}$	Approximate stress value—obtained from FE analysis
τ, $\Delta\tau$	Time and time step
ϕ	Potential (for example, temperature)
ϕ_p, $\phi\,(p)$, ϕ_P , $\phi\,(P)$	Value of potential ϕ at interior source point, p, or at source point P located at boundary
$\{\phi^e\}$, $\{\Phi\}$	Elemental and global ϕ vectors (global vector includes ϕ for all nodes)
ϕ^n, $\phi^n(P)$, $\phi^n(Q)$	Values of ϕ at the end of n-th time step
Ω	Domain of analysis (object under analysis)