ISTITUTO NAZIONALE DI ALTA MATEMATICA

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R. CONTI

Linear differential equations and control

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PREFACE

This text is a revised version, in English, of the lecture notes of a course on Control Theory given at the « Scuola di Perfezionamento in Matematica dell'Università di Firenze » during the Spring quarter of 1975.

Both the course and the present publication were sponsored by the « Istituto Nazionale di Alta Matematica ».

Therefore I wish to express my gratitude to Directors of the « Scuola», Prof. G. Talenti, and of the « Istituto », Prof. G. Cimmino, whose joint invitation and encouragement are at the origin of this book.

While preparing it, I was assisted with valuable advices and criticisms by my former pupils, Dr. G. Anichini, Dr. A. Bacciotti and Dr. L. Pandolfi.

Prof. G. S. Goodman gave a helping hand in revising the English text.

This book is dedicated to Giovanni Sansone, whose daily presence in the life of the Istituto Matematico Ulisse Dini is a constant stimulus to all the mathematical community.

ROBERTO CONTI

Firenze, December 31, 1975

CONTENTS

INTRODUCTORY NOTES

A student with a good preparation in advanced calculus and some knowledge of the basic facts of linear algebra, general topology, functional analysis and integration theory, should find little or no difficulty in reading this book. However, for the reader's convenience, a short list of essential reference texts is indicated at the end of these notes.

A certain number of function spaces will be considered.

These functions are all either real valued, or vector valued, or matrix valued, and they all depend on a real variable t ranging in some interval Δ.

We shall always denote by:

$C(\Delta)$	the space of continuous functions,
$AC_{\mathrm{loc}}(\Delta)$	the space of locally absolutely continuous functions,
$C^{(q)}(\Delta)$	the space of functions having a continuous q-th derivative $(C^{(0)}(\Delta) = C(\Delta))$,
$C^{(\infty)}(\Delta)$	the space of functions having derivatives of arbitrary order,
$C^{(\omega)}(\Delta)$	the space of functions which admit a Taylor series expansion,
$L^p_{\mathrm{loc}}(\Delta)$	the space of measurable functions such that $\lvert f\rvert^p$ is locally Lebesgue integrable for some $p \geqslant 1$,
$L^{(\infty)}_{\mathrm{loc}}(\Delta)$	the space of measurable functions which are locally essentially bounded.

When Δ is a compact interval, we shall define norms in $C(\Delta)$, $L^p(\Delta)$, $L^{(\infty)}(\Delta)$ as follows

$$|f|_C = \sup\{|f(t)| : t \in \Delta\},$$

$$|f|_{L^p} = \left(\int_\Delta |f(t)|^p\,dt\right)^{1/p}, \qquad 1 \leqslant p,$$

$$|f|_{L^\infty} = \operatorname{ess\,sup}\{|f(t)| : t \in \Delta\}.$$

Finally, when Δ is of the type $[\tau, \omega[$, bounded or not, we shall denote by

L^p_ω the subspace of $f \in L_{\mathrm{loc}}(\Delta)$ for which

$$\int_\tau^\omega |f(t)|^p \, dt = \lim_{T \to \omega} \int_\tau^T |f(t)|^p \, dt$$

is finite, if $p \geqslant 1$, or by

L^∞_ω the subspace of measurable functions which are essentially bounded on Δ. Correspondingly,

$$|f|_{L^p_\omega} = \left(\int_\tau^\omega |f(t)|^p \, dt \right)^{1/p}, \qquad p \geqslant 1 ,$$

$$|f|_{L^\infty_\omega} = \operatorname{ess\,sup} \left\{ |f(t)| : t \in [\tau, \omega[\right\} .$$

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PART I

LINEAR DIFFERENTIAL EQUATIONS

1. Linear differential equations.

1.1 One of the simplest differential equations is represented by

$$\dot{x} - ax = 0 , \tag{1.1.1}$$

where $a \in \mathbf{R}$ is given, $x\colon t \mapsto x(t)$, a real function of $t \in \mathbf{R}$, is the unknown and $\dot{x} = dx/dt$ is its derivative.

Equation (1.1.1) is a scalar linear differential equation (LDE). A solution of (1.1.1) is any x such that

$$dx(t)/dt - ax(t) = 0 , \qquad t \in \mathbf{R} . \tag{1.1.2}$$

Let x be a solution of (1.1.1). From (1.1.2) we see that $x \in C^{(\infty)}(\mathbf{R})$. Given any compact interval $\Delta \subset \mathbf{R}$, we have $|x(t)| \leqslant \gamma_\Delta$, $t \in \Delta$, for some $\gamma_\Delta > 0$, and (1.1.2) gives

$$|d^k x(t)/dt^k| \leqslant |a|^k \gamma_\Delta , \qquad t \in \Delta , \quad (k = 1, 2, \ldots) ,$$

so that $x \in C^{(\omega)}(\Delta)$. Since Δ is arbitrary, $x \in C^{(\omega)}(\mathbf{R})$, i.e.,

$$x(t) = \sum_{k=0}^{\infty} \frac{x^{(k)}(\theta)}{k!} (t - \theta)^k , \qquad t, \theta \in \mathbf{R} .$$

From (1.1.2) again we have

$$x^{(k)}(\theta) = a^k x(\theta) , \qquad (k = 1, 2, \ldots),$$

and finally

$$x(t) = \exp\big(a(t - \theta)\big) x(\theta) , \qquad t, \theta \in \mathbf{R} . \tag{1.1.3}$$

Conversely, differentiating (1.1.3) we have (1.1.2), so that we have:

THEOREM 1.1.1: *The solutions of* (1.1.1) *are represented by* (1.1.3). □

1.2 Let us now denote by $a\colon t \mapsto a(t)$ a given real function of t defined on an open interval

$$J =]\alpha, \omega[\,, \qquad -\infty \leqslant \alpha < \omega \leqslant +\infty\,,$$

and let $a \in C^{(0)}(J)$. Then (1.1.1) is a particular case of the scalar LDE

$$\dot{x} - a(t)\,x = 0 \tag{1.2.1}$$

corresponding to $J = \mathbb{R}$, a constant. A solution of (1.2.1) is any x such that

$$dx(t)/dt - a(t)\,x(t) = 0 \qquad t \in J\,. \tag{1.2.2}$$

Therefore, if x is a solution of (1.2.1), it follows that $x \in C^{(1)}(J)$. When a is a constant we can write

$$\exp\big(a(t-\theta)\big) = \exp\Big(\int_\theta^t a\,ds\Big)$$

which suggests the following extension of Theorem 1.1.1:

THEOREM 1.2.1: *The solutions of* (1.2.1) *are represented by*

$$x(t) = \exp\Big(\int_\theta^t a(s)\,ds\Big)\,x(\theta)\,, \qquad t, \theta \in J\,. \tag{1.2.3}$$

PROOF: Let x be a solution. Then from (1.2.2)

$$d\Big(x(t)\exp\Big(-\int_\theta^t a(s)\,ds\Big)\Big)\Big/dt = 0\,, \qquad t, \theta \in J\,,$$

and (1.2.3) follows.

Conversely, (1.2.3) gives (1.2.2) by differentiation. □

1.3 The assumption $a \in C^{(0)}(J)$ is not always met in applications. A more suitable assumption is, instead, $a \in L^1_{\mathrm{loc}}(J)$. In this case x is a (Caratheodory) solution of (1.2.1) if $x \in AC_{\mathrm{loc}}(J)$ and (1.2.2) is satisfied almost everywhere in J, i.e.,

$$dx(t)/dt - a(t)\,x(t) = 0\,, \quad \text{a.e. } t \in J\,. \tag{1.3.1}$$

A solution is still represented by (1.2.3), where the integral is Lebesgue instead of Riemann.

To see the difference between the two definitions, consider

EXAMPLE 1.3.1: Let a be defined by $a(t) = 2^{-1}|t|^{-\frac{1}{2}}$, $t \neq 0$. Then, according to the first definition (Sec. 1.2), we have $J =]-\infty, 0[$, or $J =]0, +\infty[$, and the solutions, represented by

$$x(t) = \exp\left(|t|^{\frac{1}{2}}\right)c\,, \qquad c = \text{constant}\,, \tag{1.3.2}$$

are defined separately for $t < 0$ and $t > 0$. According to the new definition, we have $J = \mathbb{R}$ and (1.3.2) represents solutions defined for $t \in \mathbb{R}$. □

1.4 Frequent use will be made in the sequel of the so-called Gronwall lemma, i.e., of

THEOREM 1.4.1: *Let $a \in L^1_{\text{loc}}(J)$, $a(t) \geq 0$ and let $u \in C^{(0)}(J)$. If for some $c \in \mathbb{R}$ we have*

$$u(t) \leq \int_\theta^t a(s)u(s)\,ds + c\,, \qquad \theta \leq t\,, \tag{1.4.1}$$

then

$$u(t) \leq c \exp\left(\int_\theta^t a(s)\,ds\right), \qquad \theta \leq t\,. \tag{1.4.2}$$

If

$$u(t) \leq \int_t^\theta a(s)u(s)\,ds + c\,, \qquad t \leq \theta\,, \tag{1.4.3}$$

then

$$u(t) \leq c \exp \int_t^\theta a(s)\,ds\,, \qquad t \leq \theta\,. \tag{1.4.4}$$

PROOF: Put

$$\int_\theta^t a(s)u(s)\,ds = v(t)\,.$$