REPRESENTATIONS OF GROUPS

WITH SPECIAL CONSIDERATION FOR THE NEEDS OF MODERN PHYSICS

BY

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1963

NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM

PREFACE TO THE ENGLISH TRANSLATION

This translation, whose publication has unfortunately been somewhat delayed, follows literally the text of the German edition (published in 1955) apart from a few minor alterations one of which, in Chapter VIII § 5, is concerned with the "explicit spin representation" of the rotation group, following H. FREUDENTHAL [1956]; this has given, for the first time, a basis for the numerical calculation of these important representations. Other changes have been made in the bibliography. Some titles published since 1955 have been included, particularly books on the applications of representation theory in physics. In addition the quoted German textbooks on algebra and group theory have been replaced by works in the English language.

I must cordially thank Messrs. P. G. Murphy, J. Mayer-Kalkschmidt and P. Carr, who have all taken great pains with the translation. I must also thank the North-Holland Publishing Company, at whose request this translation is published, who have co-operated energetically in overcoming the difficulties which have occurred.

Giessen, October 1962 H. BOERNER

PREFACE TO THE GERMAN EDITION

The theory of group representations is one of the best examples of interaction between physics and pure mathematics. A few years before the turn of the century the algebraist G. FROBENIUS introduced group characters and the concept of representation. Through a decade nearly every volume of the Berliner Sitzungsberichte contained one or other of the beautiful papers of FROBENIUS and I. SCHUR on this subject. Meanwhile, in the new century and in the same Berlin, the quantum theory of radiation appeared. But no one guessed that the two theories were to become so intimately related a quarter of a century later. This happened in Göttingen, where the BORN-HEISENBERG quantum mechanics originated, in the immediate spiritual and physical neighbourhood of EMMY NOETHER's circle of algebraists. The special, I must say aesthetic, beauty of this relationship lies in the fact that it concerns the basic symmetries of atomic mechanics. This, with the help of FROBENIUS's concepts, made it possible to understand many of the atom's secrets with surprising ease.

Until now, special books on representation theory have appeared only in English. In Germany, apart from some chapters in textbooks of algebra and groups theory, there are only those beautiful books on representation theory and physics together written about 1930 by great mathematicians as H. WEYL and VAN DER WAERDEN.

The contents of this book are purely mathematical; the subject-matter and method of presentation, however, are designed to satisfy the requirements of physicists. Since the book by VAN DER WAERDEN [1932], which has come out in the same "yellow collection" of the SPRINGER Verlag, deals with applications to physics, discussion of applications was omitted completely; the mathematical contents could thus be made more comprehensive and set out in more detail, in order to make it more accessible to the non-specialist. At the same time an attempt was made to arrange the subject-matter in such a way that a reader interested in any special topic need not read any more than is necessary. In particular, the lines preceding the first paragraph in each chapter are intended to serve this end. Beyond the usual material of elementary lecture courses, no previous knowledge is assumed. In fact, in the first two chapters the whole of the theories of matrices and of groups are summarized; where proofs are incomplete reference is made to suitable textbooks.

The principal concern of the book is to give the representations and characters of a number of the most important groups. Only that part of the general theory which is needed here is developed. It will be based on the field of complex numbers or an algebraically closed field. This is in conformity with the nature of the applications. It also has the result that the set of statements which one obtains constitute a beautifully simple and well-rounded theory.

Thus, there has been left out of consideration the theory of modular representations—basic field of prime characteristic—and also the question about the behaviour of the representations for an extension of the field as well as the connection with invariant theory and the applications to pure group theory.

In part the individual chapters differ widely from one another in their methods; after all, it is desirable for the reader to become acquainted with many methods for obtaining the representations†. The theory of the *group ring* is worked out completely for finite groups, obtaining the system of representations; thus the method first pointed out by E. NOETHER is used. If one has already proved complete reducibility, this method is so simple that nothing excessive is expected of the physicist reader. Other than the concept of an *algebra*—which has already a place for itself in physics—only that of an *ideal* is used. No knowledge of either is assumed.

As a concrete example of the theory of the group ring the representation theory of the *symmetric group* will be presented.

The theory of *characters* will be developed according to SCHUR—a method which can be carried over directly to compact continuous groups. Here the integral over the group is used. Differential calculations—i.e. the theory of the infinitesimal ring—are also discussed. With assumptions on differentiability, which are satisfied in all the cases considered here, the two can be dealt with conveniently and briefly. The omission of matter of greater generality, such as the treatment of more general continuous groups, seemed advisable, since much on the subject can be found in the book on almost periodic functions by MAAK [1950].

† Other methods are given in LITTLEWOOD [1950], MURNAGHAN [1938c], H. WEYL [1946], BURNSIDE [1901].

The theory of the *rational integral* representations of the *full linear group* and their connection with the representations of the symmetric group were discovered by SCHUR and developed exhaustively by H. WEYL. As the "group-theoretical foundation of tensor calculus" (WEYL) it has more fundamental than practical importance for physics. Here also with our restriction to complex numbers everything becomes clear and simple; consequently the tensor concept is used. Having constructed the rational integral representations it causes only a little additional labour to obtain all the continuous representations not only of the full linear group but also of the *real*, the *unimodular* and the *unitary* groups. The relationship between the full linear and symmetric groups has practical importance; the formulae which express the connection between the characters of the two groups give the best method for the numerical calculation of the characters of the symmetric group. From this there also results a summary of the representations of the *alternating* group, and their characters are calculated.

As for the continuous groups already mentioned, the representations of the rotation group for any dimensions are obtained. The usual rotation group d_3 is the most important in physics (its representations can be obtained from VIII, § 6 without studying the general theory). The LORENTZ group is a modification of the rotation group d_4; d_5 and d_6 have also been important in physical papers on theories of elementary particles. A summary of the characters of the rotation group (and thus also of the representations) is obtained by the method of E. CARTAN, but with the "global" turn given by STIEFEL—thus without reference to infinitesimals. For a more general treatment one would need deeper topological theorems; but since we only consider one concrete example everything is elementary, and the characters are obtained in a certain sense by an exact treatment of the group itself. Also the general theorem of PETER and WEYL is not necessary. In order to show that actual representations belong to the calculated characters it is only necessary to give the "fundamental representations", with whose help one is led on to all others. Of these the single-valued ones are the tensor representations already mentioned. The two-valued ones, the so-called *spin representations*, are obtained by two methods. The first is infinitesimal: because of the great importance in physics, it seemed appropriate to me to treat in detail the infinitesimal ring of the rotation group and the CLIFFORD algebra closely associated with it—the latter being just the general theory of PAULI matrices. The second is the direct global method of BRAUER and WEYL, in which also the CLIFFORD algebra is used.

Among the LORENTZ *groups*, the representations of the *"ordinary"*, LORENTZ group of special relativity theory can be obtained almost as easily as those of the ordinary rotation group. For a general number of dimensions, there is treated only as much as can be carried over from the corresponding rotation groups directly.

In conclusion I must express deep gratitude to Mr. H. WIELANDT, who has undertaken the labour of reading all the proofs, making many critical comments. I also thank Herren Th. BIEGLER, G. KRAFFT, R. KRIEGER and W. VELTE for assistance in the preparation of the manuscript and diagrams and for numerous suggestions of small improvements. Not least I owe my thanks to the publishers and printers, who have willingly accepted my numerous requests for changes in the composition of formulae.

Giessen, February 1955 H. BOERNER

ORIGINAL TITLE: DARSTELLUNGEN VON GRUPPEN

TRANSLATED FROM THE GERMAN

BY

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IN COOPERATION WITH

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PUBLISHERS:

NORTH-HOLLAND PUBLISHING CO. – AMSTERDAM

SOLE DISTRIBUTORS FOR U.S.A.:

INTERSCIENCE PUBLISHERS, a division of

JOHN WILEY & SONS, INC. – NEW YORK

PRINTED IN THE NETHERLANDS

BY DRUKKERIJ MEIJER, WORMERVEER AND AMSTERDAM

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CHAPTER I

MATRICES

In this chapter we shall summarize the properties of vectors and linear mappings which will be needed later. So far as matters which are dealt with in elementary courses and textbooks are concerned, the proofs are either omitted or only indicated†.

§§ 1–3 on linear mapping of vector spaces, on matrices and on algebras are basic material for the whole book, while the material of §§ 4–5 (special kinds of matrices, eigenvalues, diagonal form) is needed mainly in chapters VII and VIII. Of the contents of § 6 the operation ($\dot{+}$) occurs in the whole book; the Kronecker product is not used until chapter V. § 7 contains the concepts fundamental to representation theory—"equivalent" and "reducible"—and Schur's lemma. § 8 is needed almost solely in chapter V, the theorem of § 9 only in VIII § 6.

§ 1. Vectors

We select a set of scalars, denoted by K, with the following properties. K is a field—i.e., the four operations of arithmetic are applicable in the usual way. K contains the domain of rational numbers ("it is of characteristic zero"). Finally, we require that K also contains the roots of all algebraic equations whose coefficients lie in K ("it is algebraically closed"). This condition, though restrictive, is permissible for those parts of representation theory which are considered in this book and it is very convenient. The domain of all complex numbers satisfies it, and one may always think of this domain, especially in connection with physical applications.

A system of mathematical objects $a, b, \ldots$ is called an *n-dimensional vector space* R_n *over* K, the objects themselves *vectors*, if two operations are defined in the system: multiplication of a vector by a scalar from K and addition of two vectors (whenever a and b belong to the system, so does $\lambda a + \mu b$)††. These operations must satisfy the following con-

† See BIRKHOFF and MAC LANE [1953]; VAN DER WAERDEN [1949/50].

†† Numbers will usually be denoted by Greek letters, algebraic quantities like vectors, group elements and matrices by italic Latin letters, sets of numbers or quantities by bold-face type.

ditions. Addition shall have the same properties as the addition of numbers—commutative and associative rules, existence of a null-vector ($a + 0 = a$ for all vectors a), existence of the opposite vector $-a$ and the rule for subtraction. In multiplication by a scalar, λa and $a\lambda$ are the same vector; one associative and two distributive laws hold: $(\lambda\mu)a = \lambda(\mu a)$, $(\lambda + \mu)a = \lambda a + \mu a$, $\lambda(a + b) = \lambda a + \lambda b$. Also $1 \cdot a = a$ for all vectors a. It follows that $\lambda a = 0$ if and only if λ is the number 0 or a is the vector zero. Finally a ***dimension axiom***: there exist n linearly independent vectors, but any $n + 1$ vectors are linearly dependent. Here, as usual, we mean that m vectors $a_1, \ldots, a_m$ are linearly dependent when there exist m numbers $\lambda_1, \ldots, \lambda_m$, not all zero, such that $\lambda_1 a_1 + \ldots + \lambda_m a_m = 0$.

Vectors appear here as abstract algebraic objects. Their significance for geometry and physics lies in their being realized in different ways by concrete mathematical objects. Thus the intuitive concept of "directed line segments" in geometrical theorems coincides with the above definitions (for $n = 2$ or 3) when addition of line segments and their multiplication by a number are understood in the usual way. (One always thinks of vectors as being fastened to the origin; then just one vector belongs to each point P of the plane or space—namely, that one whose end point is at P—and conversely.) This realization can also be used for any n in so far as n-dimensional geometry (in itself quite an abstract construction) is intuitive and familiar. Since it is only one of many possible realizations, strict algebraists speak of a "K-module" rather than of a vector space. Nevertheless the name "vector space" (for the abstract object) enjoys widespread use, since it allows geometrical intuition in abstract considerations; I use it here for that reason. Another representation is given by the solutions of a homogeneous linear differential equation or of a system of differential equations. The whole significance of the development of this book for physics depends on this representation.

Every system $e_1, \ldots e_n$ of n linearly independent vectors forms a *basis* or *coordinate system* for R_n. If a is any vector, then there follows from the linear dependence of $a, e_1, \ldots, e_n$, together with the linear independence of the e_j, that we can write $a = \alpha_1 e_1 + \ldots + \alpha_n e_n$, and that this representation is unique. The numbers α_j are called the components of a with respect to the basis e_j. λa has components $\lambda\alpha_1, \ldots, \lambda\alpha_n$; $a + b$ has components $\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n$, if $\beta_1, \ldots, \beta_n$ are those of b. It follows from this that every n-dimensional vector space over K

is isomorphic to that vector space whose elements are defined as the n-tuples $(\alpha_1, \ldots, \alpha_n)$ for which products with scalars and sums are understood by the above formulae (i.e., there is a one-to-one correspondence between the two sets such that the sum of vectors maps into the sum of the image vectors and the product of a vector with a scalar maps into the product of the image vector with the same scalar). Thus, from the standpoint of abstract algebra, there is only one n-dimensional vector space over K.

A *linear subspace* is a collection of vectors with the following property: if $a, b \in r$, then $\lambda a + \mu b \in r$ for any scalars λ, μ.† The collection of all linear combinations of m fixed vectors is a linear subspace (one says: "the vectors span the linear subspace"). Conversely, in every linear subspace one can find a finite number of vectors which span it. If the m vectors are linearly independent, then the linear subspace r spanned by them has the dimension m; i.e., there exist m linearly independent vectors in r and any $m + 1$ vectors of r are linearly dependent. If the vectors are linearly dependent, then the dimension of r is less than m.

k linear subspaces $r_1, \ldots, r_k$ span a linear subspace r consisting of all sums

$$a = a_1 + \ldots + a_k \quad (a_j \in r_j) \ . \tag{1.1}$$

The sum is called *direct* (and one then writes $r = r_1 + \ldots + r_k$) when the decomposition (1.1) of the vectors $a \in r$ is unique. For this it is necessary and sufficient that "the decomposition of 0 is unique", i.e., $a = 0$ always implies $a_j = 0$ ($j = 1, \ldots, k$); an alternative condition is that the dimension m of r be the sum $m_1 + \ldots + m_k$ of the dimensions of $r_1, \ldots, r_k$; otherwise it is smaller. For $k = 2$ the sum is direct if and only if r_1 and r_2 have no vector in common other than 0. Subspaces whose sum is direct are called *linearly independent*. A basis $e_1, \ldots, e_n$ is said to be *adapted to a linear subspace* r (whose dimension is m) in R_n if the first m or the last m basis vectors span the subspace. One can always augment any and every basis of r to obtain such an adapted system.

§ 2. Linear Mappings. Matrices

A *linear transformation* of the n-dimensional vector space is a mapping of R_n into itself which maps every vector x into another vector x' such that $\lambda x + \mu y$ maps into $\lambda x' + \mu y'$. Let a basis $e_1, \ldots, e_n$ be

† The symbol $a \in r$ means "a is an element of r" or "a belongs to r".

given; let α_{ik} $(i = 1, \ldots, n)$ denote the components of the vector e'_k mapped from e_k, so that $e'_k = \sum_{i=1}^{n} e_i \alpha_{ik}$. Then $x = \xi_1 e_1 + \ldots + \xi_n e_n$ must obviously map into $x' = \xi_1 e'_1 + \ldots + \xi_n e'_n$. On the other hand $x' = \xi'_1 e_1 + \ldots + \xi'_n e_n$, where ξ'_j are the components of x'. Thus

$$\sum_{i=1}^{n} \xi'_i e_i = \sum_{k=1}^{n} \xi_k e'_k = \sum_{k=1}^{n} \sum_{i=1}^{n} \xi_k e_i \alpha_{ik} ,$$

whence, because of the linear independence of the e_j,

$$\xi'_i = \sum_{k=1}^{n} \alpha_{ik} \xi_k . \tag{2.1}$$

Thus for a given basis there is a *matrix*

$$A = \begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \ldots & \ldots & \ldots \\ \alpha_{n1} & \ldots & \alpha_{nn} \end{pmatrix} = \{\alpha_{ik}\} ,$$

corresponding to each linear transformation. For (2.1) one also writes for short†

$$x' = Ax . \tag{2.2}$$

As a useful theorem following from the above we have:

THEOREM 2.1. *The columns of a matrix A which performs a linear mapping consist of the components of the images of the basis vectors.*

To the successive performance of two linear transformations $\{\alpha_{ik}\} = A$ and $\{\beta_{ik}\} = B$ —first B, then A — corresponds the *matrix product*: $x' = A(Bx) = (AB)x$ yields for the elements γ_{ik} of $C = AB$

$$\gamma_{ik} = \sum_{j=1}^{n} \alpha_{ij} \beta_{jk} . \tag{2.3}$$

The *unit matrix* E†† which performs the identity transformation $x' = x$ has units on the principal diagonal and zeros elsewhere. Its

† Here and in the following—see e.g. eq. (2.7)—the letter x often not only stands for the vector (without reference to a particular basis) but also serves as an abbreviation for the n-tuple of its components for the given basis.

†† Often the dimension is attached to E as a subscript: E_r is the r-dimensional unit matrix.

elements are often denoted by the "Kronecker symbol" δ_{ik} (thus $\delta_{ik} = \{{1 \atop 0}{(i=k) \atop (i \neq k)}\}$). A linear transformation is *reversible* when the equations (2.1) are uniquely solvable for the ξ_i. As is well-known this is the case if and only if the determinant det A is not zero; the matrix A is then called *non-singular*. The inverse is denoted by A^{-1}; $AA^{-1} = A^{-1}A = E$. This rule permits one to calculate with positive and negative integral powers of non-singular matrices as with numbers.

A linear mapping of one vector space R_n into another S_m (we can have $n \neq m$) is performed for given bases by a rectangular matrix A with m rows and n columns. The image of R_n is a linear subspace $\mathbf{s}$ of S_m, and the set of the $x \in R_n$ which are mapped onto 0 is a linear subspace $\mathbf{r}$ of R_n. If we denote the dimension of $\mathbf{r}$ by r, that of $\mathbf{s}$ by s, then we have the relation $r + s = n$, which is fundamental for the theory of systems of linear equations. We call s the *rank* of A. The *matrix product* AB can also be defined for rectangular matrices when it is arranged that the row length of A is the same as the column length of B. To AB corresponds the successive application of two mappings, say from R_n into S_m (by B) and from S_m into T_l (by A).

Block matrix rules. A decomposition $n = n_1 + n_2 + \ldots + n_r$ o the number n into positive integral summands gives a decomposition of the sequence $1, \ldots, n$ into r segments: $1, \ldots, n_1$; $n_1 + 1, \ldots, n_1 + n_2$; etc. We denote the ρ-th segment by (ρ). By dividing up the rows and columns of a matrix A in such a way one can write A as a "block matrix":

$$A = \begin{pmatrix} A_{11} \ldots A_{1r} \\ \ldots\ldots\ldots \\ A_{r1} \ldots A_{rr} \end{pmatrix}; \quad A_{\rho\sigma} = \{\alpha_{ik}\} \quad (i \in (\rho), k \in (\sigma)) .$$

The blocks along the principal diagonal are square; the others are in general rectangular. The following useful "block rule" holds for multiplication of such matrices:

If $C = AB$, then

$$C_{\rho\sigma} = \sum_{\tau=1}^{r} A_{\rho\tau} B_{\tau\sigma} ; \tag{2.4}$$

i.e., one multiplies block matrices as if the blocks were numbers. It is easy to convince oneself that it is always possible to construct the products of rectangular matrices on the right hand side. Then

$$A_{\rho\tau}B_{\tau\sigma} = \left\{\sum_{j\in(\tau)} \alpha_{ij}\beta_{jk}\right\} \quad (i \in (\rho),\, k \in (\sigma))$$

and

$$\sum_{\tau=1}^{r} A_{\rho\tau}B_{\tau\sigma} = \left\{\sum_{j=1}^{n} \alpha_{ij}\beta_{jk}\right\} = C_{\rho\sigma} \quad (i \in (\rho),\, k \in (\sigma)) .$$

Change of basis in the vector space. If one moves from one basis $e_1, \ldots, e_n$ of the vector space to another, consisting of n linearly independent vectors $f_1, \ldots, f_n$, then for any vector

$$x = \xi_1 e_1 + \ldots + \xi_n e_n = \eta_1 f_1 + \ldots + \eta_n f_n ; \tag{2.5}$$

η_i are the components of the vector in the new basis. Let ρ_{ik} ($i = 1, \ldots, n$) be the components of f_k in the old basis of the e_j; $f_k = \sum_{i=1}^{n} e_i\rho_{ik}$. Then from (2.5)

$$\sum_i \xi_i e_i = \sum_{i,k} e_i \rho_{ik} \eta_k ,$$

so that

$$\xi_i = \sum_{k=1}^{n} \rho_{ik}\eta_k . \tag{2.6}$$

One also writes for short†

$$x = Ry . \tag{2.7}$$

There is an analogue to theorem 2.1:

THEOREM 2.2. *In the matrix of a coordinate transformation the columns contain the components (in the old system) of the new basis vectors.*

A coordinate transformation must be reversible; thus the matrix R must be non-singular. The reverse transformation gives $y = R^{-1}x$.

How does the matrix of the linear transformation given by formula (2.1) or (2.2) appear in the new coordinates? Since $x' = Ry'$ holds also for the image, we have $Ry' = ARy$ or $y' = R^{-1}ARy$. Thus one must replace the matrix A by $R^{-1}AR$, or, as one also says, *transform it with R.*

If r is a linear subspace of R_n, then there always exists a linear trans-

† Note the different meanings of formulae (2.2) and (2.7). There the left and right sides stand for different vectors—x and its image x'. Here the left and right sides stand for the same vector in different coordinate systems. Formulae (2.2) and (2.7) are to be understood merely as abbreviations for (2.1) and (2.6), respectively.

formation of R_n into itself—called the *projection* of the vector space onto the subspace—with the following properties: every vector of R_n maps into a vector of r, but the vectors of r are fixed:

$$
\begin{aligned}
&1.\ Ax \text{ lies in } r \quad (\text{any } x)\,,\\
&2.\ Ax = x \quad (x \in r)\,.
\end{aligned}
\tag{2.8}
$$

For if one has adapted the basis to r so that the first r basis vectors span r, then obviously the mapping

$$\xi_i' = \xi_i \quad (i = 1\,, \ldots\,, r), \qquad \xi_i' = 0 \quad (i = r + 1\,, \ldots\,, n)\,,$$

with the matrix

$$A = \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{bmatrix} \quad (r \text{ ones on the diagonal})$$

is a projection of R_n onto r. If we denote by r' the subspace spanned by the remaining basis vectors with numbers $r + 1\,, \ldots\,, n$, then r' maps into the null element. One speaks of "projection along r' onto r". From (2.8) follows: if one applies the mapping A twice in succession then the second transformation leaves everything unchanged. Thus $A^2 = A$. A quantity with this property is called *idempotent*; all higher powers are then equal to A. The above special matrix is obviously idempotent.

The converse of the previous paragraph is also true: *every idempotent matrix A performs a projection.* Let the rank of A be r. If $r = n$ then $A = E$, as is seen immediately on multiplying $A^2 = A$ on the right with A^{-1}. If $r < n$ then R_n is mapped onto a subspace r of r dimensions whose vectors remain fixed because $A^2 = A$. The vectors which are mapped into the null element constitute a subspace of $n - r$ dimensions. We now prove $R_n = r + r'$. For any x put $Ax = x_1$ and $x - x_1 = x_2$. Then $x_1 \in r$ and $x_2 \in r'$: $Ax_2 = Ax - Ax_1 = x_1 - x_1 = 0$. We do not need to show that the decomposition is unique, since we already know that the dimensions of r and r' add up to n.