RELIABILITY AND LIFE TESTING

S K SINHA

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TO

MY MOTHER

AMIYA BALA SINHA

Foreword

Assessment of reliability of an equipment is of paramount importance in the context of modern technology and its future developments. Special statistical methods have been developed for this purpose during the last few decades, but there has been no comprehensive and up-to-date account of the various techniques to promote their application and to stimulate further research. Professor S.K. Sinha has done a valuable service in writing a book bringing together all the available results and presenting them in the logical framework of statistical estimation and testing of hypotheses.

"Reliability and Life Testing" by Professor Sinha is a welcome contribution in many ways. It is based on the author's own outstanding contributions in this special area of applied statistics and experience in giving courses on this subject at a number of Universities throughout the world. It discusses the basic theory and gives detailed proofs of the various results, which make it an ideal text book for students. It uses live data in illustrating the statistical techniques, which makes it a valuable guide for practicing engineers. It reviews current developments in life testing and reliability estimation and lists the latest references, which are of value to research scholars. An attractive feature of the book is the extensive discussion of Bayesian techniques in reliability estimation.

The present volume is a revised and enlarged version of the author's earlier book. "Life Testing and Reliability Estimation" co-authored with Professor B.K. Kale. Judging from the success of the previous book, I am sure the revised edition with several additional attractive features will be well received by students, research scholars, statistical consultants and practicing engineers.

C.R. Rao

Pittsburgh
August 1, 1985

Preface

This book is a thoroughly revised and a considerably extended version of my earlier work "Life Testing and Reliability Estimation" (Wiley Eastern/Halsted Press, 1980) co-authored with Professor B. K. Kale, University of Poona, India.

The present edition includes several new sections, examples and exercises, two new chapters, an additional appendix and extensive discussions on newer statistical techniques. The project was motivated by the increasing interests of Academics and Professionals working with life length distributions and the need for a comprehensive and consolidated text on the basics of reliability estimation/testing. The book is intended for honours and first year graduate students of Statistics, Mathematics and Engineering Sciences. Practising Reliability Engineers, Researchers, Instructors and Consultant Statisticians will also find this publication a very helpful guide.

The text has nine chapters. Chapters 1-5 deal with a thorough analysis of the point and interval estimation (Classical) for the well-known life distributions based on complete/censored samples, mixture distributions, competing risks and tests of hypotheses. Chapters 6, 7 and 8 cover Bayesian methods and Chapter 9 briefly discusses reliability estimation and reliability bounds for series and parallel systems—Classical and Bayesian. The Classical methods of estimation considered are: (i) Maximum Likelihood (MLE), (ii) Uniformly Minimum Variance Unbiased (UMVUE), (iii) Method of Moments, (iv) Linear Combinations of Selected Order Statistics and (v) Regression Approach. A number of illustrative examples have been worked out at the end of each section and a set of exercises given at the end of each chapter.

Three appendices review the results used in the text. Appendix I deals with the properties of Gamma and Beta distributions and distributions of order statistics, Appendix II covers the Theory of Estimation and Tests of Hypotheses and Appendix III, which is particularly interesting, treats Bayesian Approximation. The general approach is introductory but rigorous, with an excellent list of references ranging from 1927 to 1985 which may serve to stimulate readers for further studies along this line.

The major part of this project was completed while I was visiting the Institute of Mathematical Statistics, University of Umeå, Sweden, and Indian Statistical Institute, Calcutta during the spring of 1985. I am grateful to Professor Gunnar Kulldorff, the members of the Institute and Professor Ashok Maitra, the Director of Indian Statistical Institute for their warm hospitality, the excellent Library, Office and Secretarial facilities and their overall interest in my work. I offer grateful acknowledgement to Professor B.K. Kale for his encouragement, assistance and advice never to be forgotten and sincere thanks to Dr. Yoshisada Shibata for the Japanese translation of the Sinha-Kale edition.

I am indebted to Ingrid Westerberg, Christina Karlberg and Christina Holmström, University of Umeå, Sharon Henderson and Mabel Davies, University of Manitoba and Eva Lowen, Winnipeg, Manitoba, for their patience, care and skill in typing the manuscript, Professor H.L. Harter for permission to use the data in Exercises 2.14 and 2.15, Mr. Jeffrey Sloan and Mr. William Mortimer for computing assistance and Wiley Eastern for their interest in this project.

I owe a great debt to my family for their affectionate support throughout the trying periods of research and writing, without which this publication would never have been possible.

Winnipeg, Manitoba,
Canada
April, 1986

S.K. SINHA

Contents

Introduction

When we buy a transistor radio or a scooter or even a simple product such as an electric bulb or a battery we expect it to function properly for a reasonable period of time. When a manufacturer floats a new brand of light bulb in the market, he would like his customers to have some information about the average life of his product. Life testing experiments are designed to measure the average life of the component or to answer such questions as 'what is the probability that the item will fail in the time interval $[t_0, t_0 + t]$ given that it was working at time t_0'?

In a simple life testing experiment a number of items are subjected to tests and the data consist of the recorded lives of all or some of the items. No matter how efficient the manufacturing process is, one or more failures may occur. This failure may be due to:

(i) careless planning, substandard equipment and raw material used, lack of proper quality control, etc.;

(ii) random or chance causes. Random failures occur quite unpredictably at random intervals and cannot be eliminated by taking necessary steps at the planning, production or inspection stage;

(iii) wear-out or fatigue, caused by the aging of the item.

Since the item is likely to fail at any time, it is quite customary to assume that the life of the item is a random variable with a distribution function $F(t)$ which is the probability that the item fails before time t. Many of the questions raised above can be answered if we know $F(t)$. For example, the average life could be defined as the mean of the distribution $F(t)$ while the probability of failure-free operation between $[t_0, t_0 + t]$, given that the item was 'alive' or working at time t_0 is given by

$$\frac{F(t_0 + t) - F(t_0)}{1 - F(t_0)}.$$

Another very important function associated with the failure distribution $F(t)$ is the hazard rate denoted by $\mu(t)$. Consider the probability of failure-free operation within the interval $[t, t + h]$, where h is infinitesimal. If $f(t)$ denotes the probability density function (p.d.f.) corresponding to $F(t)$, then the hazard rate or the 'instantaneous failure rate' is given by

$$\mu(t) = \lim_{h \to 0} \frac{F(t + h) - F(t)}{h\{1 - F(t)\}} = \frac{f(t)}{1 - F(t)}.$$

The function $\mu(t)$ is also known as the force of mortality in actuarial and life contingency problems.

Let $R(t) = 1 - F(t)$ denote the probability of failure-free operation until time t, or survival until time t. Then it is quite evident that the stochastic behaviour of the failure time can be studied through either of the four functions $f(t)$, $F(t)$, $R(t)$ or $\mu(t)$. Note that

$$R(t) = 1 - F(t),\ f(t) = dF(t)/dt \text{ and } \mu(t) = -\, d/dt \log\{1 - F(t)\}.$$

From simple integration it follows that

$$F(t) = 1 - \exp\left\{-\int_0^t \mu(w)\,dw\right\} \text{and } f(t) = \mu(t)\exp\left\{-\int_0^t \mu(w)\,dw\right\}.$$

We list below several common forms of $f(t)$ that are generally assumed in life testing experiments and reliability problems. The simplest form is the exponential distribution with

$$f(t \mid \sigma) = \frac{1}{\sigma}\exp\left(-\frac{t}{\sigma}\right),\ t > 0,\ \sigma > 0$$

for which $F(t) = 1 - \exp(-t/\sigma)$, $R(t) = \exp(-t/\sigma)$, $\mu(t) = 1/\sigma$.

Davis (1952) examined different types of data and the exponential distribution appears to fit most of the situations quite well. Indeed Epstein (1958) remarks that the exponential distribution plays as important a role in life testing experiments as the part played by the normal distribution in agricultural experiments on effects of different treatments on the yield. We will consider the exponential distribution in detail in Chapter 1. Different sampling schemes such as censoring and truncation will be discussed in this and subsequent chapters.

In Chapter 2 we will introduce more complex forms. First, we consider the gama distribution with p.d.f.

$$f(t \mid \sigma, p) = \frac{1}{\Gamma(p)\,\sigma^p} t^{p-1} \exp\left(-\frac{t}{\sigma}\right),\ t > 0,\ p > 0,\ \sigma > 0$$

where $\Gamma(p)$ is the well-known gamma function. Note that for $p = 1$, the gamma distribution reduces to the exponential distribution. For the gamma distribution,

$$F(t) = \frac{1}{\Gamma(p)}\int_0^t \frac{t^{p-1}}{\sigma^p}\exp\left(-\frac{t}{\sigma}\right)dt$$

and there is no explicit formula (closed expression) for $F(t)$. It may be noted that $F(t)$ is the well known incomplete gamma integral and has been studied extensively. Similarly, there is no simple formula for the instantaneous failure rate $\mu(t)$ although it can be shown that for $p > 1$, $\mu(t)$ is an increasing function of t. This implies the 'aging effect', i.e., the failure rate increases with the time (age) t. This property makes the gamma distribution applicable to many life testing experiments in which the 'aging effect' is expected. [See Birnbaum and Saunders (1958), Gupta (1960), Greenwood and Durand (1960), Harter (1969), Kendall and Stuart (1972)].

Next, we will consider the Weibull distribution for which the p.d.f.

$$f(t \mid \sigma, k) = (k/\sigma)\, t^{k-1} \exp(-t^k/\sigma),\ t > 0,\ k > 0,\ \sigma > 0.$$

Again note that for $k = 1$, the Weibull distribution reduces to the exponential distribution. We also note that the Weibull distribution arises in a natural way from the exponential distribution if we ussume that the kth power of the failure time has exponential distribution. For the Weibull distribution we have

$$F(t) = 1 - \exp\left(-\frac{t^k}{\sigma}\right),\ \mu(t) = \frac{kt^{k-1}}{\sigma}.$$

Thus, for the Weibull distribution the hazard rate is an increasing function of time and increases as a power of t for $k > 1$; [See Weibull (1939, 1951), Mendenhall and Lehman (1960), Mennon (1963), Cohen (1965), Harter and Moore (1965), Mann (1968), Lawless (1972), Sinha (1982, 1984)].

In Chapter 3 we consider the situation where the failure time follows either the normal or the log-normal distribution. Here the densities are, for the

(i) normal distribution

$$f(t \mid \mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(t-\mu)^2}{2\sigma^2}\right\},\ -\infty < t < \infty,$$
$$-\infty < \mu < \infty,\ \sigma > 0.$$

(ii) lognormal distribution

$$f(t \mid \mu,\ \sigma) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left\{\frac{-(\log t - \mu)^2}{2\sigma^2}\right\},\ t > 0,\ \sigma > 0,$$
$$-\infty < \mu < \infty.$$

The expressions for $\mu(t)$ or $F(t)$ for these distributions cannot be obtained in a closed form. For the normal we can show that $\mu(t)$ is an increasing function of time and, therefore, the model, implies the aging-effect.

The lognormal would be an appropriate model when the failure rate is rather high initially and then decreases as t increases. We refer to papers by Gupta (1952), Aitchison and Brown (1957), Plackett (1959), Bazovsky (1961), Larson (1969), Sinha (1980, 1981).

In Chapter 4 we would consider mixture distribution [Mendenhall and Hader (1958), Swamy and Doss (1961), Sinha (1982)] and competing risk model (Kalbfleisch and Prentice, 1980).

In Chapter 5 we will discuss the problem of testing hypotheses and construction of confidence intervals for various life testing models [Lehman and Scheffe (1950; 1955), Lehman (1959), Mood, Graybill and Boes (1974), Hogg and Craig (1978)].

In the next three chapters we discuss a Bayesian approach to life testing and reliability estimation.

In Chapter 6 we consider Bayes estimators of the parameters and reliability functions for the exponential, Weibull and normal failure time distributions. We refer to papers by Bhattacharya (1967), Draper and Guttman (1972), Pierce (1973), Sinha and Guttman (1976; 1984), Howlader and Sinha (1984), and Sinha (1985), which consider some situations where Bayesian approach is suitable and appropriate.

In Chapter 7 we discuss Bayesian approximations and its application to reliability estimation for Weibull, normal and Inverse Gaussian failure distributions. [See Lindley (1980), Sinha (1983; 1984)].

Chapter 8 deals with Credible and Highest-posterior-density intervals for the parameters and reliability functions of normal, one and two-parameter exponentials, Weibull and Rayleigh distributions. [See Box and Tiao (1973); Sinha and Guttman (1976; 1984), Sinha and Howlader (1983) and Howlader and Sinha (1984)].

Finally in Chapter 9 we briefly discuss the reliability of series and parallel systems—Classical and Bayesian. [See Lentner and Buchler (1963), Lieberman and Ross (1971), Sarkar (1971), Draper and Guttman (1972)].

The book concludes with three appendices. Appendix I outlines the basic distribution theory and some important results used. Appendix II covers the basic theory of estimation and Tests of hypotheses. Appendix III deals with the details with the detailed derivation of Lindley's (1980) approximation to the ratio of two integrals which cannot be expressed in simple/closed form.

CHAPTER 1

Exponential Failure Model

1.1 Introduction

Any inference about the average life is based on the data that are assumed to be drawn from a universe or population specified by a distribution function (d.f.), $F(x)$. Before one approaches the general problem of life testing, one has to make some assumptions about the underlying $F(x)$ or its corresponding p.d.f., $f(x)$. In life testing research the simplest and the most widely exploited model is the one-parameter exponential distribution with p.d.f.

$$f(x \mid \sigma) = (1/\sigma) \exp(-x/\sigma),\ x \geqslant 0,\ \sigma > 0. \qquad (1)$$

Here σ is the average or the mean life of the item and it also acts as a scale parameter.

Exponential distribution plays an important part in life testing problems as mentioned in the 'Introduction'. For a situation where the failure rate appears to be more or less constant, the exponential distribution would be an adequate choice but not all items satisfy the condition that 'it does not age'. There are several situations where the failure rate may be increasing or decreasing and Weibull, gamma or log-normal would be a more realistic choice. Given the data, perhaps the best one can do is to apply some transformation which will support the assumption that the transformed observations are exponentially distributed (Draper and Guttman, 1965), or check the assumption of exponentiality by some appropriate statistical test (Epstein, 1960). Exponential distribution also occurs in several other contexts, such as the waiting time problems. Maguire, Pearson and Wynn (1952) studied mine accidents and showed that time intervals between accidents follow exponential distribution. Let X be the life of an item under test. The exponential distribution may be easily derived by using the relationship

$$f(x) = \mu(x) \exp\left\{-\int_0^x \mu(w)\, dw\right\}.$$

A constant failure-rate λ yields

$$f(x \mid \lambda) = \lambda \exp(-\lambda x),\ x, \lambda > 0.$$

There are, however, some other elementary considerations which lead to an exponential distribution. These considerations may be formally stated as assumptions:

(1) The failure of the item in a given interval of time $[t_0, t_1]$ on the condition that the item works until time t_0 depends only on $(t_1 - t_0)$, the length of the time interval and not on t_0, the position of the time interval.
(2) On the condition that the item works until time t_0, the probability that the item will fail in an infinitesimal interval $[t, t+h]$ is proportional to h except for higher order.
(3) The probability of failure at $t = 0$ i.e. the instant the test started is zero.

Let $R(t) = P(X \geqslant t)$
= Probability that the item survives for at least time t.

In view of the assumptions (1) and (2) we write

$$R(t+h) = R(t)(1 - \lambda h) + 0(h)$$

where λ is a constant.

$$[R(t+h) - R(t)]/h = -\lambda R(t) + 0(h)/h.$$

Taking limits as $h \to 0$ we get a simple differential equation

$$dR(t)/dt = -\lambda R(t)$$

or

$$d/dt \log R(t) = -\lambda$$

the solution of which is

$$R(t) = A \exp(-\lambda t)$$

where A is an arbitrary constant.

From the assumption (3),

$$R(0) = 1 = A.$$

Hence

$$R(t) = \exp(-\lambda t) \tag{2}$$

$$F(t/\lambda) = 1 - R(t) = 1 - \exp(-\lambda t)$$

and

$$f(t/\lambda) = \lambda \exp(-\lambda t),\ t, \lambda > 0.$$

1.2 Some Properties of Exponential Distribution

The exponential distribution has several interesting properties. We mention a few below:

(i) The distribution is 'forgetful' or 'has no memory'. What it means, however, is that if a unit has survived t hours, then the probability of its surviving an additional h hours is exactly the same as the probability of surviving h hours of a new item.

Consider the one-parameter exponential density with mean life σ, viz.,

$$f(x \mid \sigma) = (1/\sigma) \exp(-x/\sigma), \quad x, \sigma > 0.$$

$$P(X \geqslant t+h \mid X \geqslant t) = \frac{\int_{t+h}^{\infty} \frac{1}{\sigma} \exp\left(-\frac{x}{\sigma}\right) dx}{\int_{t}^{\infty} \frac{1}{\sigma} \exp\left(-\frac{x}{\sigma}\right) dx} = \exp\left(-\frac{h}{\sigma}\right) = P(X \geqslant h).$$

(ii) Suppose n items are under test with replacements and the failure time distribution is exponential with mean life σ; then the between failure times are independent and identically distributed as exponential with mean life σ/n. (We will frequently use the terms 'with or without replacement' to refer to respective situations where the items that fail 'are or are not replaced by similar new items').

Let $X_{(1)} < X_{(2)} \ldots < X_{(n)}$ be the ordered failure times of n items under test and let the failure times be exponentially distributed with mean life σ. The test starts at time $X_{(0)} = 0$ and the system operates till $X_{(1)} = x_{(1)}$ when the first failure occurs. Suppose the failed item is immediately replaced by a new item and the system operates till the second failure occurs at time $X_{(2)} = x_{(2)}$ and the item failed is immediately replaced. The process continues till all items fail and let

$$W_1 = X_{(1)},\ W_2 = X_{(2)} - X_{(1)}, \ldots,\ W_n = X_{(n)} - X_{(n-1)}.$$

Now W_1 is distributed as the first order statistic $X_{(1)}$ in a sample of n from the exponential p.d.f. (1). Hence the p.d.f. of W_1 may be written down as

$$g(w_1 \mid \sigma) = (n/\sigma) \exp(-nw_1/\sigma), \quad 0 < w_1 < \infty \tag{3}$$

which implies that W_1 is exponentially distributed with mean life (σ/n). Since the items that fail are immediately replaced, $W_2, W_3, \ldots, W_n$ are independent and identically distributed (i.i.d.) as W_1. (See Appendix I for discussion on the distribution of order statistics.)

(iii) If n times are put up to test without replacement and $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$ are ordered failure times from an exponential population with mean life σ, then $(Z_1, Z_2, \ldots, Z_n)$ are i.i.d. as

$$g(z \mid \sigma) = (1/\sigma) \exp(-z/\sigma), \quad z, \sigma > 0$$

where $Z_i = (n - i + 1)\{X_{(i)} - X_{(i-1)}\}, \quad i = 1, 2, \ldots, n;\ X_{(0)} = 0.$

Note the difference between (ii) and (iii). Under the 'with replacement' plan, the number of items exposed at any time is n and the total time on the test till the kth failure time $x_{(k)}$ is $n\{x_{(1)} + x_{(2)} - x_{(1)} + x_{(3)} - x_{(2)} + \ldots + x_{(k)} - x_{(k-1)}\} = nx_{(k)}$ whereas, when the failed times are not replaced, the number of items exposed at $x_{(0)}$ is n, at $x_{(1)}$ is $(n-1)$, at $x_{(k)}$ it is $(n-k)$, etc., and the total time on the test up to $x_{(1)}$ is $nx_{(1)}$, up to $x_{(2)}$ it is $(n-1)\{x_{(2)} - x_{(1)}\}$, up to $x_{(k)}$ it is $(n-k+1)\{x_{(k)} - x_{(k-1)}\}$.

The joint distribution of $\{X_{(1)}, X_{(2)}, \ldots, X_{(n)}\}$ is given by

$$g(x_{(1)}, x_{(2)}, \ldots, x_{(n)} \mid \sigma) = \frac{n!}{\sigma^n} \exp\left\{\frac{\sum_1^n x_{(i)}}{\sigma}\right\}, 0 < x_{(1)} < x_{(2)} < \ldots < x_{(n)} < \infty. \tag{4}$$

Let

$$\begin{aligned} z_1 &= nx_{(1)} \\ z_2 &= (n-1)\{x_{(2)} - x_{(1)}\} \\ z_3 &= (n-2)\{x_{(3)} - x_{(2)}\} \\ &\vdots \\ z_k &= (n-k+1)\{x_{(k)} - x_{(k-1)}\}, \quad k = 1, 2, \ldots, n;\ x_{(0)} = 0. \end{aligned} \tag{5}$$