Analysis of Heat Equations on Domains

El Maati Ouhabaz

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Analysis of Heat Equations on Domains

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A mes parents,
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Preface

The influence of the theory of linear evolution equations upon developments in other branches of mathematics, as well as physical sciences, would be hard to exaggerate. The theory has a rich interplay with other subjects in functional analysis, stochastic analysis and mathematical physics. Of particular interest are evolution equations associated with second-order elliptic operators in divergence form. Such equations arise in many models of physical phenomena; the classical heat equation is a prototype example. They are also of interest for nonlinear analysis; the proof of existence of local solutions to many nonlinear partial differential equations uses linear theory.

The theory for self-adjoint second-order elliptic operators is well documented, and there is an increasing interest in the non-self-adjoint case. It is one of the aims of the present book to give a systematic study of L^p theory of evolution equations associated with non-self-adjoint operators

$$A = -\sum_{k,j} \frac{\partial}{\partial x_j}\left(a_{kj}\frac{\partial}{\partial x_k}\right) + \sum_k b_k \frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_k}(c_k \cdot) + a_0.$$

We consider operators with bounded measurable coefficients on arbitrary domains of Euclidean space. The sesquilinear form technique provides the right tool to define such operators, and associates them with analytic semigroups on L^2. We are interested in obtaining contractivity properties of these semigroups as well as Gaussian upper bounds on their associated heat kernels. Gaussian upper bounds are then used to prove several results in the L^p-spectral theory.

A special feature of the present book is that several important properties of semigroups are characterized in terms of verifiable inequalities concerning their sesquilinear forms. The operators under consideration are subject to various boundary conditions and do not need to be self-adjoint. We also consider second-order elliptic operators with possibly complex-valued coefficients. Such operators have attracted attention in recent years as their associated heat kernels do not have the same properties as those of their analogues with real-valued coefficients. This book is also motivated by new developments and applications of Gaussian upper bounds to spectral theory. A large number of the results given here have been proved during the last decade.

The approach using sesquilinear form techniques avoids heavy use of sophisticated results from the theory of partial differential equations or Sobolev embedding properties for which smoothness of the boundary is required. On the other hand, as we consider heat equations on arbitrary domains, we shall not address regularity properties (with respect to the space variable) of their solutions.

This book is for researchers and graduate students who require an introductory text to sesquilinear form technique, semigroups generated by second-order elliptic operators in divergence form, heat kernel bounds, and their applications. It should also be of value for mathematical physicists. We tried to keep the text self-contained and most of the material needed is introduced here. A few standard results are stated without proofs, but we provide the reader with several references.

We now give an outline of the content of each chapter. Chapter 1 is devoted to sesquilinear forms and their associated operators and semigroups. It provides the necessary background from functional analysis and evolution equations. Most of the material on sesquilinear forms is known, but our presentation differs from that in other books on this topic. We give a systematic account on the interplay between forms, operators, and semigroups. Chapter 2 is devoted to contractivity properties of semigroups associated with sesquilinear forms. We give criteria in terms of forms for positivity, irreducibility, L^∞-contractivity, and domination of semigroups. These criteria are obtained as simple consequences of a result on invariance of closed convex sets under the action of the semigroup (see Theorems 2.2 and 2.3). We also include a section on semigroups acting on vector-valued functions. All the results in this chapter are in the spirit of the famous Beurling-Deny criteria for sub-Markovian semigroups. Chapter 3 contains Kato type inequalities for generators of sub-Markovian semigroups. For symmetric sub-Markovian semigroups, a partial description of the domain of the corresponding generator in L^p is given. Chapter 4 is devoted to uniformly elliptic operators of type A as above. We discuss some examples of boundary conditions and apply the criteria of Chapter 2 to describe precisely, in terms of the boundary conditions and the coefficients, when the semigroup generated by $-A$ is positive, irreducible, or L^p-contractive. Chapter 2 also gives the right tools to compare (in the pointwise sense) semigroups associated with two different divergence form operators. Some results are extended in Chapter 5 to the case of degenerate-elliptic operators. Gaussian upper bounds for heat kernels of uniformly elliptic operators are proved in Chapter 6. We prove sharp bounds for operators with real-valued symmetric principal coefficients a_{kj}. Gaussian upper bounds are derived from L^p-contractivity results together with a well-known perturbation argument due to E.B. Davies.

We also derive bounds for the time derivatives as well as weighted gradient estimates for heat kernels. In Chapter 7, we use Gaussian upper bounds to prove several spectral properties. This includes L^p-analyticity of the semigroup, p-independence of the spectrum, L^p-estimates for Schrödinger and wave type equations. Although the book is devoted to uniformly elliptic operators on domains of Euclidean space, this chapter is written in a general setting of abstract operators on domains of metric spaces. The framework includes uniformly elliptic operators on domains of Euclidean space or more general Riemannian manifolds, sub-Laplacians on Lie groups, or Laplacians on fractals. In the last chapter we review the Kato square root problem for uniformly elliptic operators. We include at the end of each chapter a section of notes where the reader can find references to the literature and supplementary information.

Acknowledgments: I wish to express my hearty thanks to the many colleagues and friends who have contributed to my understanding of the subject of this book. I want to thank Wolfgang Arendt, Pascal Auscher, Sonke Blunck, Thierry Coulhon, Brian Davies, Xuan Thinh Duong, Alan McIntosh, and Rainer Nagel for their help and encouragement. I'm grateful to Philippe Depouilly for his unstinting help with the many tasks involved in typing the manuscript.

Notation

$C_c(\Omega)$: The space of continuous functions with compact support in Ω.

$C_c^\infty(\Omega)$: The space of C^∞-functions with compact support in Ω.

$(C_c^\infty(\Omega))'$: The space of distributions on Ω.

$\text{supp}(u)$: The support of the function u.

$\Sigma(\psi) := \{z \in \mathbb{C}, z \neq 0, |\arg z| < \psi\}$, $\mathbb{C}^+ := \Sigma(\frac{\pi}{2})$.

$u^+ := \sup(u, 0)$ the positive part of u, $u^- := \sup(-u, 0)$ the negative part.

$f \wedge g := \inf(f, g)$, $f \vee g := \sup(f, g)$.

$$\text{sign } u(x) = \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

$\Re$: Real part, $\Im$: Imaginary part.

χ_Ω: Characteristic function of Ω.

$L^p(X, \mu, \mathbb{K})$: The classical Lebesgue spaces of functions with values in $\mathbb{K}$.

$\|.\|_p$: The norm of $L^p(X, \mu, \mathbb{K})$.

dx: Lebesgue measure.

$W^{s,p}$: Sobolev spaces.

$H^1(\Omega) := W^{1,2}(\Omega)$, $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

$D_i := \frac{\partial}{\partial x_i}$ and $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ is the Laplacian.

$\mathcal{L}(E, F)$: The space of bounded linear operators from E into F. $\mathcal{L}(E) := \mathcal{L}(E, E)$.

$\|T\|_{\mathcal{L}(E,F)}$: The operator norm of T in $\mathcal{L}(E, F)$.

$\rho(A)$: Resolvent set of the operator A. $\sigma(A)$: Spectrum of A.

Contents

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Chapter One

SESQUILINEAR FORMS, ASSOCIATED OPERATORS, AND SEMIGROUPS

1.1 BOUNDED SESQUILINEAR FORMS

Let H be a Hilbert space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We denote by $(.;.)$ the inner product of H and by $\|.\|$ the corresponding norm. Let $\mathfrak{a}$ be a sesquilinear form on H, i.e., $\mathfrak{a}$ is an application from $H \times H$ into $\mathbb{K}$ such that for every $\alpha \in \mathbb{K}$ and $u, v, h \in H$:

$$\mathfrak{a}(\alpha u + v, h) = \alpha\mathfrak{a}(u, h) + \mathfrak{a}(v, h) \text{ and } \mathfrak{a}(u, \alpha v + h) = \overline{\alpha}\mathfrak{a}(u, v) + \mathfrak{a}(u, h). \tag{1.1}$$

Here $\overline{\alpha}$ denotes the conjugate number of α. Of course, $\overline{\alpha} = \alpha$ if $\mathbb{K} = \mathbb{R}$ and in this case the form $\mathfrak{a}$ is then bilinear. For simplicity, we will not distinguish the two cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. We will use the sesquilinear term in both cases and also write conjugate, real part, imaginary part, and so forth of elements in $\mathbb{K}$ as if we had $\mathbb{K} = \mathbb{C}$. These quantities have their obvious meaning if $\mathbb{K} = \mathbb{R}$.

DEFINITION 1.1 *A sesquilinear form $\mathfrak{a} : H \times H \to \mathbb{K}$ is continuous if there exists a constant M such that*

$$|\mathfrak{a}(u, v)| \le M\|u\|\|v\| \text{ for all } u, v \in H.$$

Every continuous form can be represented by a unique bounded linear operator. More precisely,

PROPOSITION 1.2 *Assume that $\mathfrak{a} : H \times H \to \mathbb{K}$ is a continuous sesquilinear form. There exists a unique bounded linear operator T acting on H such that*

$$\mathfrak{a}(u, v) = (Tu; v) \text{ for all } u, v \in H.$$

Proof. Fix $u \in H$ and consider the linear continuous functional

$$\phi(v) := \overline{\mathfrak{a}(u, v)}, \ v \in H.$$

By the Riesz representation theorem, there exists a unique vector $Tu \in H$, such that

$$\phi(v) = (v; Tu) \text{ for all } v \in H.$$

The fact that T is a linear and continuous operator on H follows easily from the linearity and continuity of the form $\mathfrak{a}$. The uniqueness of T is obvious. □

The bounded operator T is the operator associated with the form $\mathfrak{a}$. One can study the invertibility of T (or its adjoint T^*) using the form. More precisely, the following basic result holds.

LEMMA 1.3 *(Lax-Milgram) Let $\mathfrak{a}$ be a continuous sesquilinear form on H. Assume that $\mathfrak{a}$ is coercive, that is, there exists a constant $\delta > 0$ such that*

$$\Re\mathfrak{a}(u, u) \geq \delta \|u\|^2 \text{ for all } u \in H.$$

Let ϕ be a continuous linear functional on H. Then there exists a unique $v \in H$ such that

$$\phi(u) = \mathfrak{a}(u, v) \quad \text{for all } u \in H.$$

Proof. It suffices to prove that the adjoint operator T^* is invertible on H. Indeed, by the Riesz representation theorem, there exists a unique $g \in H$ such that

$$\phi(u) = (u; g) \quad \text{for all } u \in H,$$

and hence by writing $g = T^*v$ for some $v \in H$, it follows that

$$\phi(u) = (u; T^*v) = (Tu; v) = \mathfrak{a}(u, v) \text{ for all } u \in H.$$

Now we prove that T^* is invertible. Let $v \in H$ be such that $T^*v = 0$. Thus,

$$0 = (v; T^*v) = (Tv; v) = \Re\mathfrak{a}(v, v) \geq \delta \|v\|^2.$$

Hence $v = 0$ and so T^* is injective.

It remains to show that T^* has range $R(T^*) = H$. We first prove that $R(T^*)$ is dense. If $u \in H$ is such that

$$(u; T^*v) = 0 \text{ for all } v \in H,$$

then by taking $v = u$ and using again the coercivity assumption, we obtain $u = 0$. Finally, we prove that $R(T^*)$ is closed. For this, let $v_k = T^*u_k$ be a sequence which converges to v in H. We have

$$\begin{aligned} \delta \|u_k - u_j\|^2 &\leq \Re\mathfrak{a}(u_k - u_j, u_k - u_j) \\ &\leq |(u_k - u_j; T^*u_k - T^*u_j)| \\ &\leq \|u_k - u_j\| \|v_k - v_j\|. \end{aligned}$$

From this, it follows that $(u_k)_k$ is a Cauchy sequence and hence it converges in H. If u denotes the limit, then $v = T^*u$ by continuity of T^*. This proves that $R(T^*)$ is closed. □

1.2 UNBOUNDED SESQUILINEAR FORMS AND THEIR ASSOCIATED OPERATORS

1.2.1 Closed and closable forms

In this section, we consider sesquilinear forms which do not act on the whole space H, but only on subspaces of H. These forms are unbounded sesquilinear forms. They play an important role in the study of elliptic or parabolic equations (cf. Chapters 4 and 5). We will say, for simplicity, sesquilinear forms rather than "unbounded sesquilinear forms."

Let H be as in the previous section and consider a sesquilinear form $\mathfrak{a}$ defined on a linear subspace $D(\mathfrak{a})$ of H, called the domain of $\mathfrak{a}$. That is,

$$\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{K}$$

is a map which satisfies (1.1) for $u, v, h \in D(\mathfrak{a})$.

DEFINITION 1.4 *Let $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{K}$ be a sesquilinear form. We say that:*
1) $\mathfrak{a}$ is densely defined if

$$D(\mathfrak{a}) \text{ is dense in } H. \tag{1.2}$$

2) $\mathfrak{a}$ is accretive if

$$\Re\mathfrak{a}(u, u) \geq 0 \text{ for all } u \in D(\mathfrak{a}). \tag{1.3}$$

3) $\mathfrak{a}$ is continuous if there exists a non-negative constant M such that

$$|\mathfrak{a}(u, v)| \leq M\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}} \text{ for all } u, v \in D(\mathfrak{a}) \tag{1.4}$$

where $\|u\|_{\mathfrak{a}} := \sqrt{\Re\mathfrak{a}(u, u) + \|u\|^2}$.
4) $\mathfrak{a}$ is closed if

$$(D(\mathfrak{a}), \|.\|_{\mathfrak{a}}) \text{ is a complete space.} \tag{1.5}$$

If $\mathfrak{a}$ satisfies (1.2)–(1.5), one checks easily that $\|.\|_{\mathfrak{a}}$ is a norm on $D(\mathfrak{a})$. It is called the norm associated with the form $\mathfrak{a}$.

DEFINITION 1.5 *Let $\mathfrak{a}$ be a sesquilinear form on H. The adjoint form of $\mathfrak{a}$ is the sesquilinear form $\mathfrak{a}^*$ defined by:*

$$\mathfrak{a}^*(u, v) := \overline{\mathfrak{a}(v, u)} \text{ with domain } D(\mathfrak{a}^*) = D(\mathfrak{a}).$$

The symmetric part of $\mathfrak{a}$ *is defined by*

$$\mathfrak{b} := \frac{1}{2}(\mathfrak{a} + \mathfrak{a}^*), \; D(\mathfrak{b}) = D(\mathfrak{a}).$$

We say that $\mathfrak{a}$ *is a symmetric form if* $\mathfrak{a}^* = \mathfrak{a}$, *that is,*

$$\mathfrak{a}(u, v) = \overline{\mathfrak{a}(v, u)} \; for \; all \; u, v \in D(\mathfrak{a}).$$

Let $\mathfrak{a}$ be a sesquilinear form which satisfies (1.2)–(1.5). Then $D(\mathfrak{a})$ is a Hilbert space. The inner product is given by

$$(u; v)_{\mathfrak{a}} := \frac{1}{2}[\mathfrak{a}(u, v) + \mathfrak{a}^*(u, v)] + (u; v) \text{ for all } u, v \in D(\mathfrak{a}).$$

The norm $\|.\|_{\mathfrak{a}}$ is the same as $\|.\|_{\mathfrak{b}}$, where $\mathfrak{b}$ is the symmetric part of $\mathfrak{a}$.

On a complex Hilbert space H, every sesquilinear form $\mathfrak{a}$ can be written in terms of symmetric forms $\mathfrak{b}$ and $\mathfrak{c}$ as follows:

$$\mathfrak{a} = \mathfrak{b} + i\mathfrak{c}, \; D(\mathfrak{a}) = D(\mathfrak{b}) = D(\mathfrak{c}). \tag{1.6}$$

It suffices indeed to take $\mathfrak{b} := \frac{1}{2}(\mathfrak{a}+\mathfrak{a}^*)$ and $\mathfrak{c} := \frac{1}{2i}(\mathfrak{a}-\mathfrak{a}^*)$. In this way, the symmetric part $\mathfrak{b}$ is seen as the real part of the form $\mathfrak{a}$ and $\mathfrak{c}$ as the imaginary part.

In the present chapter we will consider only accretive forms (i.e., forms that satisfy (1.3)). We could instead consider forms that are merely bounded from below, that is,

$$\Re\mathfrak{a}(u, u) \geq -\gamma(u; u) \text{ for all } u \in D(\mathfrak{a})$$

for some positive constant γ. The general theory of such forms does not differ much from that of accretive ones. A simple perturbation argument (which consists of considering the form $\mathfrak{a} + \gamma$, defined by $(\mathfrak{a} + \gamma)(u, v) := \mathfrak{a}(u, v) + \gamma(u; v)$ for $u, v \in D(\mathfrak{a})$) allows us to consider only accretive forms. According to Section 1.2.3 below, if B denotes the operator associated with the accretive form $\mathfrak{a} + \gamma$, then $A = B - \gamma I$ is the operator associated with $\mathfrak{a}$. Here I denotes the identity operator on H.

If $\mathfrak{a}$ is a symmetric form, the accretivity property (1.3) means that $\mathfrak{a}$ is non-negative, that is,

$$\mathfrak{a}(u, u) \geq 0 \text{ for all } u \in D(\mathfrak{a}).$$

Thus, for symmetric forms, we use both terms non-negative or accretive to refer to the property (1.3).

The condition (1.4) means that the sesquilinear form $\mathfrak{a}$ is continuous on the space $(D(\mathfrak{a}), \|.\|_{\mathfrak{a}})$. The smallest possible constant M for which (1.4) holds is of some interest (see, e.g., Theorem 1.52).

PROPOSITION 1.6 *Let $\mathfrak{a} : H \times H \to \mathbb{K}$ be a closed accretive sesquilinear form. Then the norms $\|.\|$ and $\|.\|_{\mathfrak{a}}$ are equivalent on H.*

Proof. We have for every $u \in H$

$$\|u\| \leq \|u\|_{\mathfrak{a}} = [\|u\|^2 + \Re\mathfrak{a}(u,u)]^{1/2}.$$

In other words, the identity operator $I : (H, \|.\|_{\mathfrak{a}}) \to H$ is continuous. Since I is bijective, its inverse $I^{-1} = I$ is continuous by the closed graph theorem. Hence, there exists a non-negative constant C such that

$$\|u\|_{\mathfrak{a}} \leq C\|u\| \text{ for all } u \in H.$$

This shows that the two norms are equivalent. □

A stronger assumption than continuity is sectoriality, which we introduce in the following definition.

DEFINITION 1.7 *A sesquilinear form $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{C}$, acting on a complex Hilbert space H, is called sectorial if there exists a non-negative constant C, such that*

$$|\Im\mathfrak{a}(u,u)| \leq C\Re\mathfrak{a}(u,u) \textit{ for all } u \in D(\mathfrak{a}). \tag{1.7}$$

The numerical range of $\mathfrak{a}$ is the set

$$\mathcal{N}(\mathfrak{a}) := \{\mathfrak{a}(u,u),\ u \in D(\mathfrak{a}) \textit{ with } \|u\| = 1\}. \tag{1.8}$$

Clearly, $\mathfrak{a}$ satisfies (1.7) if and only if the numerical range $\mathcal{N}(\mathfrak{a})$ is contained in the closed sector $\{z \in \mathbb{C}, |\arg z| \leq \arctan C\}$.

PROPOSITION 1.8 *Every sectorial form acting on a complex Hilbert space H is continuous. More precisely, if*

$$|\Im\mathfrak{a}(u,u)| \leq C\Re\mathfrak{a}(u,u) \textit{ for all } u \in D(\mathfrak{a}),$$

where $C \geq 0$ is a constant, then

$$|\mathfrak{a}(u,v)| \leq (1 + C)(\Re\mathfrak{a}(u,u))^{1/2}(\Re\mathfrak{a}(v,v))^{1/2} \textit{ for all } u, v \in D(\mathfrak{a}).$$

Proof. By (1.6) we have $\mathfrak{a} = \mathfrak{b} + i\mathfrak{c}$, where $\mathfrak{b}$ and $\mathfrak{c}$ are symmetric forms and $\mathfrak{b}$ is non-negative. By the Cauchy-Schwarz inequality,

$$|\mathfrak{b}(u,v)| \leq \mathfrak{b}(u,u)^{1/2}\mathfrak{b}(v,v)^{1/2}.$$

It remains to estimate $|\mathfrak{c}(u,v)|$. Changing v into $e^{i\psi}v$ for some ψ, we may assume without loss of generality that $\mathfrak{c}(u,v)$ is real. In this case, we have

$$\mathfrak{c}(u,v) = \frac{1}{4}[\mathfrak{c}(u+v,u+v) - \mathfrak{c}(u-v,u-v)].$$

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