Herman H. Goldstine

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To Ellen

"Shall I compare thee to a summer's day?"

Preface

In this book I have attempted to trace the development of numerical analysis during the period in which the foundations of the modern theory were being laid. To do this I have had to exercise a certain amount of selectivity in choosing and in rejecting both authors and papers. I have rather arbitrarily chosen, in the main, the most famous mathematicians of the period in question and have concentrated on their major works in numerical analysis at the expense, perhaps, of other lesser known but capable analysts. This selectivity results from the need to choose from a large body of literature, and from my feeling that almost by definition the great masters of mathematics were the ones responsible for the most significant accomplishments. In any event I must accept full responsibility for the choices.

I would particularly like to acknowledge my thanks to Professor Otto Neugebauer for his help and inspiration in the preparation of this book. This consisted of many friendly discussions that I will always value. I should also like to express my deep appreciation to the International Business Machines Corporation of which I have the honor of being a Fellow and in particular to Dr. Ralph E. Gomory, its Vice-President for Research, for permitting me to undertake the writing of this book and for helping make it possible by his continuing encouragement and support. I should also like to acknowledge the kindness of the Institute for Advanced Study in sustaining me intellectually through this task and for providing me with its facilities. I have been considerably helped by watching my colleagues here at their labors. They have served as exemplars of the highest standards of science and scholarship, and I hope this book reflects to some extent their inspiration. Since I bear the onus of responsibility for the contents of this work, I do not enumerate the names of these colleagues except to thank Professor Marshall Clagett for his many courtesies and kindnesses and Professor Bengt Strömgren of the University of Copenhagen for opening that university's libraries to me and providing me with other facilities as well.

In closing I wish especially to express my deep gratitude to Janet Sachs for her many kindnesses in helping to improve this book's style.

Finally my thanks are due to Springer–Verlag for its splendid work and help in making this material available in attractive form.

July 1977 HERMAN H. GOLDSTINE

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Introduction

In Chapter 1, I have attempted to indicate to a small extent the resurgence of interest by the Western world in science. During the sixteenth and early seventeenth century mathematical notation began to improve quite markedly. The rapid emergence of reasonable symbolisms contributed greatly to the development of mathematics and allied sciences such as mathematical astronomy.

I begin with a rather full account of both Bürgi's and Napier's discovery of logarithms in which I have tried to show in detail how Napier carried out out the laborious calculations he made in order to construct his table of logarithms by means of tables of geometrical progressions. It is fascinating to contrast this with the much more elegant and sophisticated techniques of his English successor, Henry Briggs. This man and a predecessor of his at Oxford, Thomas Harriot (1560–1621), were mainly responsible for the early developments of finite difference methods. They both understood and used interpolation formulas in general and subtabulation ones in particular. For some reason the accomplishments of these two men have not been sufficiently appreciated, perhaps because so many honors have been heaped upon Napier, who was certainly deserving of them.

Briggs noted, firstly that $\log(1 + x)$ is proportional to x for sufficiently small x; secondly that given any number y it is possible by repeated extractions of square roots to reduce it to a related number y' of the form $1 + x$ with x small; and thirdly that the logarithms of only relatively few numbers need be calculated since the others can be obtained either as sums of known logarithms or as subtabulated values. In fact Briggs not only devised subtabulation schemes, he also worked out a very ingenious difference method to eliminate some of the work of forming the square roots mentioned above.

In any case the ideas of Napier and Briggs spread rapidly across Europe, and we shall see Kepler calculating his own tables as soon as he heard of Napier's idea. From this point onwards the theory of finite differences was to be further developed with great artistry by such men as Newton, Euler, Gauss, Laplace, and Lagrange, among others. In fact we shall see that virtually all the great mathematicians of the seventeenth and eighteenth centuries had a hand in the subject.

Among Newton's predecessors one of the most extraordinary was François

Vieta. He had a considerable influence upon the young Newton and upon numerical analysis. I have given some illustrations of his work, in particular his solution of algebraic equations.

To illustrate the rapid advance of mathematics between Kepler and Newton, I have closed the chapter with the solution of the same problem in plane geometry as given by Kepler and Newton. This illustrates the progress in mathematical notation that took place over 75 years between the two men, and how mathematical analysis became easier as a result of that progress.

Chapter 2 is entitled "The Age of Newton". There I have discussed small portions of Newton's works: notably, his contributions to numerical techniques such as his method for solving equations iteratively, his interpolation and numerical integration formulas as well as his ideas on calculating tables of logarithms and of sines and cosines, In the part on interpolation there is included a brief account of how Wallis found the value of π by interpolation and how Newton generalized this by making the upper limit of an integral variable. As we shall see, this immediately led him to the binomial theorem.

Newton's friends and contemporaries quickly took up his ideas and published a great deal of work which is of interest in our field. Thus we find Halley developing series expansions for calculating logarithmic tables, and Roger Cotes systematically working out Newton's ideas on numerical integration formulas. Stirling and Maclaurin used Newton's techniques in developing important results on sums of series, and their names are still well known today and associated with fundamental series and approximation methods.

About the same time de Moivre and James Bernoulli worked at building the foundations for probability theory and the latter, while working in many other directions, was estimating the sums of powers of successive integers. This topic was also considered by Maclaurin and later by Euler. Their work resulted in a considerable body of literature important to numerical analysis under the general title of Bernoulli and Euler numbers and polynomials. This includes summation of functions and difference equations.

There is also some discussion of the extensive research of James Gregory, a Scot who worked in Edinburgh, more or less independently of Newton. Gregory had a method of using an interpolation formula involving finite differences and then of passing to the limit to find series expansions for a considerable variety of functions, essentially by Taylor's theorem. In passing we should note that Gregory's successor in the mathematics chair at Edinburgh was Maclaurin, a protégé of Newton.

The contribution of Euler, who did at least the ground work on virtually every topic in modern numerical analysis, is examined next. This work included the basic notions for the numerical integration of differential equations. Moreover, his development of lunar theory made possible the accurate calculation of the moon's position and the founding of the Nautical Almanac in Great Britain.

Lagrange worked on linear difference equations and introduced his now famous method of variation of parameters in this connection. He published extensively on the subject and must be considered as one of the founders of our field. He was very interested in interpolation theory, and he wrote several papers on the subject following up on Briggs's ideas. He introduced some quite elegant formalistic procedures which enabled him to develop many important results. He not only considered the more classical methods of interpolation, but he and Clairaut seem to have discovered trigonometric interpolation independently. He was deeply concerned with finding hidden periodicities in astronomical data and devised some interesting means for finding these periods.

Laplace used and developed the method of generating functions to study difference equations which came up in his work on probability theory. Using this apparatus, he was also able to develop various interpolation functions and to produce a calculus of finite differences. Out of his work on probabilities Laplace developed an elegant treatment of least squares. The subject, of course, was discovered by Gauss and later by Legendre. However, it is probably fair to attribute to Gauss and Laplace the real developments of the subject. But Gauss's treatment was both simpler and more elegant than Laplace's, which depended upon the Law of large numbers.

Gauss wrote much on numerical matters and obviously enjoyed calculating. He took the Newton–Cotes method of numerical integration and showed that by viewing the positions of the ordinates, taken to form the finite approximation to the integral, as parameters to be chosen he could materially improve the convergence. Jacobi reconsidered this result and gave a very elegant exposition of it. This was followed up later by Chebyshev who used another scheme to assign equal weights to the ordinates. In the Gaussian case the weights are unequal, and Gauss calculated a considerable number of them. He wrote penetratingly on interpolation and particularly on trigonometric interpolation. In fact he developed the entire subject of finite Fourier series, including what we now call the Cooley–Tukey algorithm or the fast Fourier transform.

Jacobi interested himself in a number of aspects of numerical analysis, including, as we mentioned above, the Gaussian method of numerical integration. He also gave an elegant analysis of the Euler–Maclaurin algorithm in the course of which he developed the Bernoulli polynomials. He wrote a paper on finding the characteristic values of a symmetric matrix which has given rise to the modern Jacobi method and its variants.

Cauchy was yet another great mathematician who worked on numerical methods. One of his most significant discoveries was a method for finding a rational function which passed through a sequence of given points. This idea of approximation by rational, rather than polynominal, functions is still important and in another connection — Padé approximations — is used today. In his usual way Jacobi gave a first-class exposition and analysis of this method of Cauchy. Cauchy also interested himself in trigonometric interpolation, as did Hermite, apparently in ignorance of Gauss's results.

In the course of investigating the Newton–Raphson method Cauchy came upon the well-known Cauchy–Schwarz inequality. He also made very skillful use of operational methods for solving both difference and differential equations. But probably his most important contribution to our field was made in the field of summation of functions. He based his beautiful results on his famous Residue theorem which precisely relates an integral and a sum. The exploitation of his ideas by Lindelöf and later by Nörlund has resulted in an elegant theory of considerable depth and beauty.

Another great advance Cauchy made in our subject was his method for showing the existence of the solutions of differential equations. This so-called Cauchy–Lipschitz method, as well as that of Picard, form the basis for some very important techniques for numerically integrating such equations. These theoretical methods were exploited by John Couch Adams, the astronomer, who used a successive approximation method numerically in a work with Bashforth on capillary action. This work was followed by that of F. R. Moulton who considerably improved upon it.

In a quite different direction K. Heun, W. Kutta and C. Runge developed a very pretty method for numerical integration of differential equations; and in fact one of the very first problems run on the ENIAC was done using Heun's method. Their ideas are current today.

Both J. C. Adams and Hermite wrote on Bernoulli's numbers making use of their exact form as given by Clausen and von Staudt, and Adams tabulated many of them. Hermite also studied the Bernoulli polynomials and the Euler–Maclaurin formula. Hermite was also one of the first to appreciate how Cauchy's Residue theorem could be used to obtain polynominal approximations to a function. In this he followed up on a remark of Cauchy that interpolation formulas properly should come out of the Residue theorem.

We are now on the threshold of the twentieth century, where I have quite arbitrarily decided to terminate this work. The closing section deals very superficially with the results of Cauchy and Lindelöf on the summation of functions and an asymptotic theorem of Poincaré and Perron on the relation of the zeros of an algebraic equation and certain quotients of solutions of a related linear difference equation.

1. The Sixteenth and Early Seventeenth Centuries

1.1. Introduction

One of the great discoveries of the sixteenth century was that of logarithms made independently by Bürgi and Napier. This marked a state in the development of mathematics where sufficiently sophisticated methods were finally made available for the understanding of exponentials and their inverses.

Before this time there had been considerable study of the trigonometric functions, made possible by the fact that their analysis can be undertaken purely geometrically. Greek mathematicians had already noted the importance of the relationship of a chord of a circle to the arc it subtends. Hipparchus (*circa* -140), was probably the first to introduce the chord function — in modern terms

$$\text{chord } 2\alpha = 2R \sin \alpha,$$

where R is the radius of the circle and 2α is the subtended angle. But the earliest extant chord table is that in the *Almagest* of Ptolemy (*circa* $+140$). Using elegant geometrical theorems, Ptolemy developed formulas for computing chd $(\alpha + \beta)$, chd $(\alpha - \beta)$ and chd $(\frac{1}{2}\alpha)$, given chd α and chd β. It is of interest to us to note that he calculated chd 1° by an approximation procedure, starting from chd 0°;45 and chd 1°;30; he found these two values by repeatedly using the half-angle formula beginning with chd 12°, which he calculated from a knowledge of the chords of 72° and 60°.

The problem of improving upon Ptolemy's method of finding chd 1° engaged many mathematicians, particularly in the Arab world, until the time of al-Kāshī (*circa* 1400), who worked at the observatory in Samarqand during the reigns of Tamerlane and his son Shahrukh. Al-Kāshī devised an elegant iterative scheme for solving the cubic

$$\sin 3\alpha = 3x - 4x^3.$$[1]

It is fair to assume that the great interest that was shown for many years in

[1] Aaboe [1954].

the theory of equations and in iterative methods for solving algebraic equations had its genesis in this problem of calculating sin 1° given sin 3°.

However, the study of logarithms is not a development stemming from early ideas on geometry but in a sense is a precursor of modern analysis. It was largely made possible by a series of sufficient developments in the understanding of algebraic processes and improvements in notation. As we shall see, in the hands of Briggs, it led very directly into the beginnings of numerical analysis.

1.2. Napier and Logarithms

It is interesting to trace the European origins of Napier's great discovery. In this search Michael Stifel's name is prominent. Stifel (1487–1567) was a German mathematician working at Jena, a generation before Napier. He studied the properties of exponents; in fact he seems to have coined the term "exponent" in his *Die Coss Christoffs Rudolffs* [1553]. He discussed properties of both positive and negative exponents in his *Arithmetica Integra* [1544], Book III, p. 377. There he considered the series

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6\ldots \\ 1 & x & x^2 & x^3 & x^4 & x^5 & x^6\ldots \end{matrix} .$$

He noted the intimate connection between these two sequences, the one arithmetic and the other geometric. In fact he remarked how addition (subtraction) of terms in the former corresponds to multiplication (division) in the latter. He also knew that multiplication (division) in the former corresponds to raising to a power (extracting a root). By 1600 these properties must have been fairly well understood. Compare, e.g., Simon Jacob, who was the inventor of an early geometrical or surveying instrument.[2] These were not by any means the first or only mathematicians to consider the problem of exponents.[3]

Archimedes, in his *Sand Reckoner*, was already aware of the notion of geometrical progression. He had a theorem which came very close to being a statement of one of the laws of exponents. It is given in a somewhat anachronistic form by Heath: "If there be any number of terms of a series in continued proportion, say $A_1, A_2, A_3, \ldots, A_m, \ldots, A_n, \ldots, A_{m+n-1}$ of which $A_1 = 1$, $A_2 = 10$ [so that the series forms the geometrical progression $1, 10^1, 10^2, \ldots, 10^{m-1}, \ldots, 10^{n-1}, \ldots, 10^{m+n-2}, \ldots$], and if any two terms as A_m, A_n be taken and multiplied, the product $A_m \cdot A_n$ will be a term in the same series and will be as many terms distant from A_n as A_m is distant from A_1; also

[2] Jacob [1600].

[3] The interested reader may wish to consult Tropfke, *GEM*, Vol. II, pp. 132–166, for an account of the early history of exponents.

it will be distant from A_1 by a number of terms less by one than the sum of the numbers of terms by which A_m and A_n respectively are distant from A_1."[4] Thus he seems to have had some feeling for the fundamental relation $a^m \cdot a^n = a^{m+n}$ in the third century B.C.

However the first two men who are major figures in the discovery of logarithms are Joost Bürgi, a Swiss (1552–1632/33), who worked in astronomy and mechanics both in Prague and Kassel, and John Napier, Laird of Merchiston (1550–1617). It was Napier who in 1614 published his work first in Edinburgh.[5] He had labored over the concept and the actual tabulations for about twenty years. He also wrote another book on the construction of these tables, which was published posthumously in 1619.[6] This is part one of the posthumous work; part two is in "Appendix as to the making of another and better kind of Logarithms"; part three contains "Propositiones for the solutions of spherical Triangles by an easier method." Part of this work, as the complete title indicates, is by Briggs. Moreover, in 1616 the *Descriptio* appeared in an English translation. It is interesting to note that in that edition Napier's name was spelled Nepair; it has been said it was a title conferred on an ancestor for his peerless bravery. It is not known if the story is true.

The translation by Macdonald is good and makes available Napier's work in an easily accessible form.[7] This contains a number of good expositions of Napier's works. (All future references to the *Constructio* by me are to this translation.) In considering Napier's text we should know that Napier prepared his so-called artificial table to make easy the calculation of products of sines. Of course it makes no difference whether we view the independent variable as x or $\sin x$ but to Napier it was the latter. Originally Napier referred to his numbers as *artificiales*; he then coined the term *logarithm* out of λόγων plus ἀριθμός, i.e., ratio number.

Napier says in his *Constructio*:

1. *A Logarithmic table is a small table by the use of which we can obtain a knowledge of all geometrical dimensions and motions in space, by a very easy calculation.*

It is deservedly called very small, because it does not exceed in size a table of sines; very easy, because by it all multiplications, divisions, and the more difficult extractions of roots are avoided; for by only a very few most easy additions, subtractions, and divisions by two, it measures quite generally all figures and motions.

It is picked out from numbers progressing in continuous proportion.[8]

[4] Heath, *ARC*, p. 230. A translation of the Greek text appears in Ver Eecke [1960], Vol. I, p. 366.

[5] Napier, *Descr.* (Cf. also, Napier, *NTV*.) This is usually referred to as the *Descriptio*.

[6] Napier, *Const.* Cf. also, Napier, *EC* for an English translation. This work is usually referred to as the *Constructio*.

[7] Napier, *NTV*. This contains a number of good expositions of Napier's works.

[8] Napier, *EC*, p. 7.

We find early on in this book Napier's invention of the period to signify a decimal fraction, i.e., the so-called decimal point. He also was aware of rounding errors and had a way to deal with them. He says in Propositions 5 and 6:

5. *In numbers distinguished thus by a period in their midst, whatever is written after the period is a fraction, the denominator of which is unity with as many cyphers after it as there are figures after the period.*

Thus

$$10000000.04 \text{ is the same as } 10000000\,\frac{4}{100};$$

also

$$25.803 \text{ is the same as } 25\,\frac{803}{1000};$$

also

$$9999998.0005021 \text{ is the same as } 9999998\,\frac{5021}{10000000};$$

and so of others.

6. *When the tables are computed, the fractions following the period may then be rejected without any sensible error. For in our large numbers, an error which does not exceed unity is insensible and as if it were none.*

Thus in the completed table, instead of

$$9987643.8213051, \text{ which is } 9987643\,\frac{8213051}{10000000},$$

we may put 9987643 without sensible error.

This use of the period was clearly an improvement on Stevin's notation [Stevin, e.g., wrote 8.937 as 8 ⓪ 9 ① 3 ② 7 ③], but it was not taken up for a long time by others. In fact Briggs, Napier's friend, wrote instead 9987643|8213051.[9] In Napier's Propositions 26 and 27 he defined his logarithm function. There he says:

26. *The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that which radius began to decrease geometrically, and in the same time as radius has decreased to the given sine.*

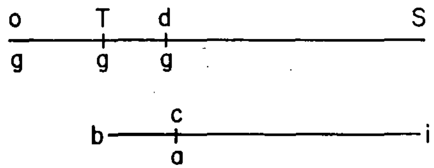


Figure 1.1.

Let the line TS be radius, and dS a given sine in the same line; let g move geometrically from T to d in certain determinate moments of time. Again, let bi be another line, infinite towards i, along which, from b, let a move arithmetically with the same velocity as g had at first when at T; and from the fixed point b in the direction of i let a advance in just the same moments of time up to the point c. The number measuring the line bc is called the logarithm of the given sine dS.

27. *Whence nothing is the logarithm of radius.*

For, referring to the figure, when g is at T making its distance from S radius, the arithmetical point d beginning at b has never proceeded thence. Whence by the definition of distance nothing will be the logarithm of radius.[10]

[9] Napier, *NTV*, p. 97.

We can now calculate what Napier's logarithms really are in our terms. The line TS is the radius r or *sinus totus* — in this case 10^7. Then if we designate by $(r - x)$ the length Td traveled by g and by y the length bc traveled by a in the same time, dS is x and

$$\frac{d}{dt}(r - x) = x, \qquad \frac{d}{dt}y = r, \tag{1.1}$$

$$x(0) = r, \qquad y(0) = 0. \tag{1.2}$$

From these relations we see that

$$x = re^{-y/r}.$$

But Napier has defined his logarithm, Nap. Log x, as y, i.e.,

$$y = \text{Nap. Log } x,$$

and so we have

$$\log_e x = \log_e r - \frac{1}{r}\text{Nap. Log } x$$

or

$$\text{Nap. Log } x = r \log_e \frac{r}{x} = 10^r \log_e \frac{10^r}{x}. \tag{1.3a}$$

Actually the line dS represented not x for Napier but $\text{Sin } x = r \sin x = 10^7 \sin x$ and so (1.3a) is

$$\text{Nap. Log Sin } x = r \log_e \frac{r}{\text{Sin } x} = 10^7 \log_e \frac{1}{\sin x} = -10^7 \log_e \sin x. \tag{1.3b}$$

He next showed in Proposition 28 of the *Constructio* that

$$\frac{r(r - \text{Sin } x)}{\text{Sin } x} > \text{Nap. Log Sin } x > r - \text{Sin } x. \tag{1.4}$$

His proof is very nice. In Figure 1.1 he extended the line $TS = r$ backwards to a point o such that oS is to TS as TS is to dS. He then showed that bc, the logarithm of the sine dS, is greater than Td and less than oT. His proof is this: "For in the same time that g is borne from o to T, g is borne from T to d; because . . . oT is such a part of oS as Td is of TS, and in the same time

[10] Napier, *EC*, p. 19.

(by the definition of a logarithm) is a borne from b to c; so that oT, Td, and bc are distances traversed in equal times. But since g when moving between T and o is swifter than at T, and between T and d slower, but at T is equally fast with a . . .; it follows that oT the distance traversed by g moving swiftly is greater, and Td the distance traversed by g moving slowly is less, than bc the distance traversed by the point a with its medium motion in just the same moments of time; the latter is, consequently, a certain mean between the two former."[11]

(It is clear in modern terms from the relations (1.1), (1.2) that $dS = re^{-t}$, $Td = r(1 - e^{-t})$, and since $oS/TS = TS/dS$, $oS = re^{t}$ and $oT = r(e^{t} - 1)$. Therefore $oT > bc > Td$. Thus Napier's inequalities are equivalent to the statement that $r(e^{t} - 1) > rt > r(1 - e^{-t})$.)

Napier gave his Canon, i.e., his table, in the *Descriptio*. In that work he gave no explanatory text, putting that off until he brought out his *Constructio*. The detailed calculations are of some interest because of the ingenuity of his approach.

He first constructed two ancillary tables. In this preliminary work Napier was at great pains to find easy ways to carry out his task. Thus he hit on the idea of forming geometric progressions whose terms are very easy to calculate. He realized that repeated shiftings of the decimal point and subsequent subtractions are quite simple to perform. He therefore decided to tabulate the logarithms of numbers lying between his "radius and half radius" using only such operations. To do this he constructed his so-called first table, which is a tabulation of the geometrical progression whose first term is $r = 10^7$ and whose common ratio is $\rho_1 = 1 - 10^{-7}$. It is shown below. We see there that his last term is $r\rho^{100} = 9999900.0004950$.

First table.
10000000.0000000
1.0000000
99999999.0000000
.9999999
99999998.0000001
.9999998
99999997.0000003
.9999997
99999996.0000006
to be continued
up to
9999900.0004950

Thus from radius, with seven cyphers added for greater accuracy, namely, 10000000.0000000, subtract 1.0000000, you get 9999999.0000000; from this subtract .9999999, you get 9999998.0000001; and proceed in this way, as shown at the side, until you create a hundred proportionals, the last of which, if you have computed rightly, will be 9999900.0004950.

In principle he could have continued this table until he reached half radius except for the fact that the amount of work would have been completely prohibitive. He therefore shifted next to a coarser ratio and formed his second

[11] Napier, *EC*, p. 20. Notice how easily Napier handled velocities. This shows quite clearly the degree of sophistication attained in the West by 1600 in coping with non-uniform motions.

table, which is again a tabulation of the geometrical progression with first term $r = 10^7$, ratio $\rho_2 = 1 - 10^{-5}$ and last term $r\rho_2{}^{50}$.

Second table
10000000.000000
100.000000
9999900.000000
99.999000
9999700.003000
99.997000
9999600.006000
&c., up to
9995001.222927

Thus the first and last numbers of the First table are 10000000.0000000 and 9999900.0004950, in which proportion it is difficult to form fifty proportional numbers. A near and at the same time an easy proportion is 100000 to 99999, which may be continued with sufficient exactness by adding six cyphers to radius and continually subtracting from each number its own 100000th part in the manner shown at the side; and this table contains, besides radius which is the first, fifty other proportional numbers, the last of which, if you have not erred, you will find to be 9995001.222927.

Notice that the second term $r\rho_2$ in the second table is the nearest "easy" number just below $r\rho_1^{100}$, the last entry in the first table. These tables connect very nicely with only a slight roughness at the transition point. In passing we might note that Napier made a small arithmetical blunder in forming the last entry in the second table. He gave it as 9995001.222927; in fact, as Delambre and others pointed out, it should be 9995001.224804.[12] This caused an error in the last place in Napier's table of logarithms. The Canon was also affected by errors in the table of sines he used. Indeed he knew this and remarked. ". . . it would seem that the table of sines is in some places faulty. Wherefore I advise the learned, who perchance may have plenty of pupils and computers, to publish a table of sines more reliable and with larger members, in which the radius is made 100000000 . . . ."[13]

Given the first and second tables Napier was now ready to form his third table which was more extensive than the others. It consisted of 69 separate geometrical progressions arranged in as many columns. Each had the same ratio $\rho = 1 - 5 \times 10^{-4}$. The rows were also geometrical progressions whose ratio was $1 - 10^{-2}$. The first term in column one was $r = 10^7$ and the last one in column 69 was 4998609.4034. Thus the third table covered the interval from radius to half radius, as Napier desired.

To assign logarithms to the quantities in the third table Napier first assigned a value to Nap. Log 9999999. This fixed all the others. By the result (1.4) above on upper and lower bounds,

$$1 + 10^{-7} + 10^{-14} + \cdots > \text{Nap. Log } 9999999 > 1.$$

He then remarks that the average of 1 and $1 + 10^{-7}$, 1.00000005, will be taken as Nap. Log 9999999. Now he has the logarithm of every term in the

[12] Napier, M. [1834]. This is an interesting biography by a descendant.
[13] Napier, *EC*, p. 46. He refers in the text to "Reinhold's common table of sines, or any other more exact." [Erasmus Reinhold (1511–1553) was a colleague of Rhaeticus at the University of Wittenberg. He was responsible for a famous table (1551) of motions of the planets based on Copernicus's *De Revolutionibus*. They were known as the *Tabulae Prutenicae* or Prussian tables after the Duke, Albert of Prussia, who patronized Reinhold.]

first table. Thus, e.g., "the logarithm of 9999998.0000001, the second sine after radius, will be contained between ...2.0000002 and 2.0000000; and the logarithm of 9999997.0000003, the third will be between the triple of the same, namely between 3.0000003 and 3.0000000." In this way he immediately found limits on the logarithms of each term in the first table.

To do the same for the second table he had first to interpolate for the logarithm of 9999900.[14] He first proved a quite interesting theorem. If Sin $a >$ Sin b, then

$$\frac{r(\text{Sin } a - \text{Sin } b)}{\text{Sin } b} > \text{Nap. Log Sin } b - \text{Nap. Log Sin } a > \frac{r(\text{Sin } a - \text{Sin } b)}{\text{Sin } a}. \quad (1.5)$$

This result is not hard to establish.[15] He did it as follows.

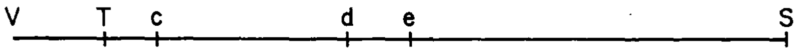


Figure 1.2.

In Figure 1.2 $TS = r$, $dS =$ Sin a, $eS =$ Sin b. Extend T backwards to V so that

$$\frac{TS}{TV} = \frac{eS}{de} = \frac{\text{Sin } b}{\text{Sin } a - \text{Sin } b}. \quad (1.6)$$

Next choose the point c so that

$$\frac{TS}{Tc} = \frac{dS}{de} = \frac{\text{Sin } a}{\text{Sin } a - \text{Sin } b}. \quad (1.7)$$

Therefore we can infer that $VS/TS = TS/cS = dS/eS$ and also

$$\begin{aligned}\text{Nap. Log } eS - \text{Nap. Log } dS &= \text{Nap. Log } cS - \text{Nap. } TS \\ &= \text{Nap. Log } cS\end{aligned} \quad (1.8)$$

since $TS = r$ and the logarithm of r is 0.

But by Napier's original result (1.4) we have

$$TV > \text{Nap. Log } cS > Tc,$$

and so with the help of (1.6), (1.7), and (1.8) we have (1.5) at once. These inequalities of Napier are quite nice and deserve a brief examination. His relations state that if $x > y > 0$

$$\frac{x}{y} - 1 > \text{Nap. Log } y - \text{Nap. Log } x = \log_e \frac{10^7}{y} - \log_e \frac{10^7}{x} = \log_e \frac{x}{y} > 1 - \frac{x}{y}.$$

It is easy to see that

$$\frac{1}{y}(x - y) = \int_y^x \frac{d\xi}{y} > \int_y^x \frac{d\xi}{\xi} > \int_y^x \frac{d\xi}{x} = \frac{1}{x}(x - y).$$

[14] Napier, *EC*, pp. 29ff.

[15] Napier, *EC*, pp. 26–27.

Thus his relations imply that the area from y to x under the curve $z = 1/x$ lies between the rectangle with height $1/y$ and width $x - y$ and the rectangle of height $1/x$ and the same width.

We can now apply the relations (1.5) to calculate the logarithm of 9999900.0000000, the second entry in the second table. We know that the logarithm of 9999900.0000000, the last one in the first table, is between 100.0000100 and 100.0000000. Then by the last theorem we see that Nap. Log 9999900 is bounded above by

$$100.0000100 + \frac{10^7 \times 0.0004950}{10^7 \times (1 - 10^{-5})} = 100.0005050$$

and below by

$$100.0000000 + \frac{10^7 \times 0.0004950}{10^7 \times (1 - 10^{-5} + 5 \times 10^{-11})} = 100.0004950.$$

Given this the logarithms of all other entries in the second table are now trivially calculable.

In the same fashion Napier proceeded to find the logarithms of the second entries in each column of the third table, and this is why all values were contaminated by his error in finding the last entry in the second table. He needed, e.g., the logarithm of 9995000.0000, the second entry in column one. To do this he used the relations (1.5), where Sin b was 9995001.222927 — instead of 9995001.224804 — and Sin a was 995000.0000. (He thereby introduced a very small error into all the logarithms of entries in the third table.) Having evaluated the logarithms of the quantities in the third table, Napier now found his so-called radical table, which was constructed by putting next to each entry in the third table its logarithm, keeping only one of the seven decimal places he previously had kept. Of this he said: "For shortness, however, two things should be borne in mind — First, that in these logarithms it is enough to leave one figure after the point, the remaining six being now rejected, which, however, if you had neglected at the beginning, the error arising thence by frequent multiplications in the previous tables would have grown intolerable in the third. Secondly, if the second figure after the point exceed the number four, the first figure after the point, which alone is retained, is to be increased by unity: thus for 10002.48 it is more correct to put 10002.5 than 10002.4; and for 1000.35001 we more fitly put 1000.4 than 1000.3. Now, therefore, continue the Radical table in the manner which has been set forth."[16]

Napier was now in a position to form his final table, his Canon of logarithms. To do this he needed not only to find an interpolatory technique so that he could form the logarithms of the sines of angles spaced 0°;1 (= 1 min.) apart, but also to find values for those outside the range of his radical table.

[16] Napier, *EC*, p. 35. Notice Napier's rounding-off procedure.

To accomplish his first end he gave a prescription for finding "*the logarithms of all sines embraced within the limits of the Radical table.*" It was this: "Multiply the difference of the given sine and table sine nearest it by radius. Divide the product by the easiest divisor, which may be either the given sine or the table sine nearest it, or a sine between both, however placed. By 39 there will be produced either the greater or less limit of the difference of the logarithms, or else something intermediate, no one of which will differ by a sensible error from the true difference of the logarithms on account of the nearness of the numbers in the table. Wherefore (by 35), add the result, whatever it may be, to the logarithm of the table sine, if the given sine be less than the table sine; if not, subtract the result from the logarithm of the table sine, and there will be produced the required logarithm of the given sine."[17]

To accomplish his second end he gave another prescription: namely, for finding "*the logarithms of all sines which are outside the limits of the radical table.*" He did this by writing out his so-called short table in which he recorded the logarithms of numbers (sines) in the ratios of 2, 4, 8, 10, 20, 40, 40, 80, 100, 200, . . . , 10^7. Then given any sine he multiplied it by one of these factors until the result was within the limits of the radical table.

His last result is then entitled, "*To form a logarithmic table.*"

Prepare forty-five pages, somewhat long in shape, so that besides margins at the top and bottom, they may hold sixty lines of figures. Divide each page into twenty equal spaces by horizontal lines, so that each space may hold three lines of figures. Then divide each page into seven columns by vertical lines, double lines being ruled between the fifth and sixth, but a single line only between the others.

Next write on the first page, at the top of the left, over the first three columns, "0 *degrees*." On the second page, above, to the left, "1 degree"; and below to the right, "88 *degrees*." On the third page above, "2 *degrees*"; and below, "87 *degrees*." Proceed thus with the other pages, so that the number written above, added to that written below, may always make up a quadrant, less one degree or 89 *degrees*.

Then, on each page write, at the head of the first column, "*Minutes of the degree written above*," at the head of the second column, "*Sines of the arcs to the left*"; at the head of the third column, "*Logarithms of the arcs to the left*"; at the head of the third column, "*Logarithms of the arcs to the left*"; at both the head and the foot of the third column, "*Difference between the logarithms of the complementary arcs*," at the foot of the fifth column, "*Logarithms of the arcs to the right*," at the foot of the sixth column, "*Sines of the arcs to the right*"; and at the foot of the seventh column, "*Minutes of the degree written beneath.*"

Then enter in the first column the numbers of minutes in ascending order from 0 to 60, and in the seventh column the number of minutes in descending order from 60 to 0; so that any of minutes placed opposite, in the first and seventh columns in the same line may make up a whole degree or 60 minutes.[18]

[17] Napier, *EC*, p. 36.
[18] Napier, *EC*, pp. 43–44.

To this book Napier added an Appendix, "On the construction of another and better kind of Logarithms, namely one in which the Logarithm of unity is 0." He evidently realized the awkwardness of his original system in which

$$\text{Nap. Log}\, xy = 10^7 \log_e \frac{10^7}{xy} = 10^7 \log_e \frac{10^7}{x} + 10^7 \log_e \frac{10^7}{y} - 10^7 \log_e 10^7$$

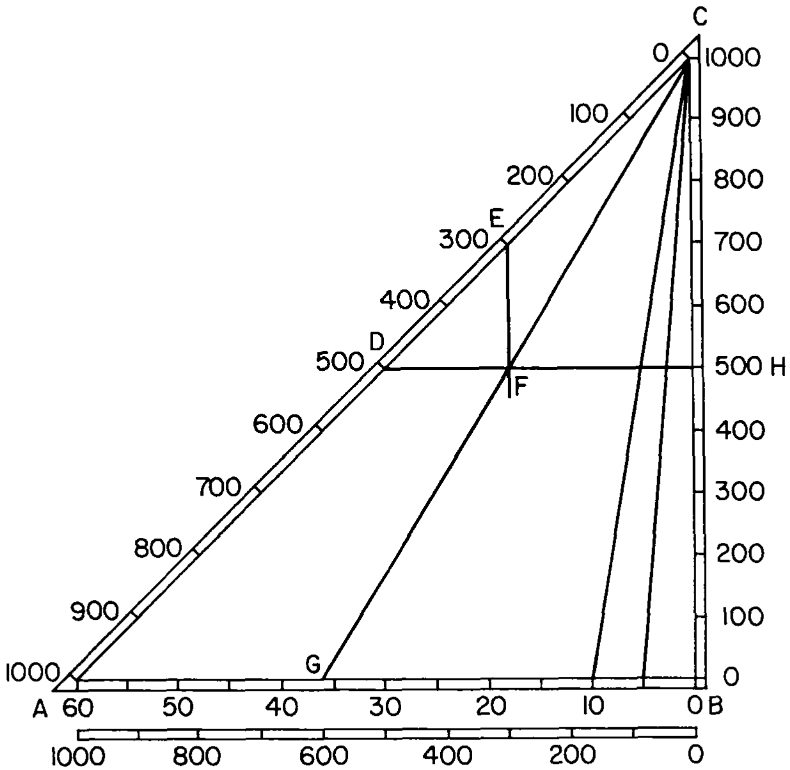
$$= \text{Nap. Log}\, x + \text{Nap. Log}\, y - \text{Nap. Log}\, 1.$$

To remedy this and also to have a system in which the logarithms of powers of 10 would be easy to calculate he proposed, but did not carry out, the construction of the logarithm where the logarithm of 1 is 0 and the logarithm of 10 (or 1/10) is 10^{10}. It is clear that the new logarithm of Napier is Log $x = 10^{10} \log_{10} x$, and he therefore discovered in a sense both the systems we know today: logarithms to the bases e and 10. In the same Appendix he gave several ways to calculate his new logarithms. To this is appended: "Some remarks by the learned Henry Briggs on the foregoing Appendix." In addition the 1616 English translation by Edward Wright of the *Descriptio* contained a graphical device for interpolating in the tables. It is shown in Figure 1.3. Wright himself died in 1615 before the work was published. It was published by Wright's son with the help of Briggs.[19] It should be remarked in passing that Edward Wright in 1599 calculated and published a text on *Certaine Errors in Navigation* . . . . This table corrected an error arising in the use of Gerhard Mercator's charts and is in essence a tabulation of $r \log \tan (45° - x/2)$. It gave the lengths of arcs on nautical meridians and was an important tool for navigators.[20]

The accounts of the meeting and collaboration of Napier and Briggs are so well known that they need not be repeated here. We will, however, quote Briggs's own account of their scientific interdependence in the Preface to his *Arithmetica Logarithmica* (1624).

"That these logarithms differ from those which that illustrious man, the Baron of Merchiston, published in his *Canon Mirificus* must not surprise you. For I myself, when expounding their doctrine publicly in London to my

[19] Henderson [1926], pp. 26–28. There we find how to use the graph: clearly $DC/ED = DH/FH = AB/BG$; thus if we know DE, EC, and DH, we can read off BG. Henderson gives as an example $60/x = 5/3$.
[20] Wright [1599] and Cajori, "Algebra in Napiers' Day and Alleged Prior Inventions of Logarithms," Napier, *NTV*, pp. 93–109. What Wright did was to approximate the value of the integral $\int \sec \theta \, d\theta$ by adding up the successive values of $\sec \theta$ starting at $\theta = 0$ and going by 0°;0,1 steps. The relation between the logarithmic tangents and the values in Wright's table was first noticed by a Henry Bond in 1645 but an actual proof was not given until 1668 when it was done by James Gregory in his *Exercitationes Geometricae*, p. 7. Cf. also, Gregory, *GTV*, p. 463. There is an account of this in Bourbaki [1960], pp. 203–204. (I am indebted to Prof. A. Weil for this reference.) It is also discussed in Cajori, Napier, *NTV*, pp. 189–190. Barrow, Wallis, and Halley all gave later proofs of the relation.



To determine x such that 60:x or 1000:x is any given ratio

Figure 1.3.

auditors in Gresham College, remarked that it would be much more convenient that 0 should be kept for the logarithm of the whole sine (as in the *Canon Mirificus*) but that the logarithm of the tenth part of the same whole sine, that is to say, 5 degrees 44 minutes and 21 seconds, should be 10000000000. And concerning that matter I wrote immediately to the author himself; and as soon as the season of the year and the vacation of my public duties of instruction permitted I journeyed to Edinburgh, where being most hospitably received by him, I lingered for a whole month. But as we talked over the change in the logarithms, he said that he had for some time been of the same opinion and had wished to accomplish it; he had however published those he had already prepared until he could construct more convenient ones if his affairs and his health would admit of it. But he was of opinion that the change should be affected in this manner, that 0 should be the logarithm of unity and 10000000000 that of the whole sine; which I could not but admit was by far the most convenient (*longe commodissimum*). So, rejecting those which I had previously prepared, I began at his exhortation to meditate

seriously about the calculation of these logarithms; and in the following summer I again journeyed to Edinburgh and showed him the principal part of the logarithms I here submit. I was about to do the same in the third summer also, had it pleased God to spare him to us so long."[21]

1.3. Briggs and His Logarithms

Briggs published in 1617 a table called *Logarithmorum Chilias Prima*. (This is the first appearance of logarithms to the base 10.) This was followed in 1624 by his *Arithmetica Logarithmica*.[22] These tables of Briggs were published to 14 places but may be in error in the last place. His original idea, as we saw above, had been to make the logarithm of the *sinus totus*, the radius = 10^{10}, zero and of $10^9 \sim 10^{10}\cdot\sin(5°;44,21)$, 10^{10}. Thus the original Briggsian logarithm was $10^{10}(10 - \log_{10} x)$. He changed his ideas and put out his tables so that with $r = 10^{10}$

$$\text{Bri. Log}\, x = 10^9 \log_{10} x,$$

hence Bri. Log 1 = 0 and Bri. Log $r = 10^{10}$.

Before discussing his work, let us say just a word about the man. Henry Briggs was born in Yorkshire in 1556 and died in 1630. He was at first a professor in Gresham's College, London, and then in 1619 he was called to one of the two chairs established by Sir Henry Savile (1549–1622) at Oxford. Much of Briggs's list was spent on the problem of making navigation safer and faster. This activity was evidently of the highest importance to England, particularly at this period when seapower was playing such a role in English history. It is therefore not surprising that Briggs took such a great interest in logarithms.

As we shall see, Briggs must be viewed as one of the great figures in numerical analysis. His ideas were far in advance of his time, and he has never been accorded the honor which is his due. This is probably because of the fallacious theory which grew up that he was merely the slavey or drudge who carried out the ideas of his master, Napier. Briggs's techniques were purely arithmetical and indicate that he must have been one of the very first, if not the first, to use the calculus of finite differences with great facility. His work is, however, difficult to read since he gave no proofs.

The main idea Briggs used (foreshadowed by a remark of Napier) was that for any number $a > 1$

$$a^{2^{-n}} \to 1.$$

Moreover for the numbers he was dealing with the convergence was not too

[21] Napier, *NTV*, pp. 126–127.

[22] Briggs, *LOG* and *ARITH*. The *Arithmetica* contains 88 pages of explanation and application.

slow. He first prepared a preliminary table for $a = 10$ and n ranging from 1 to 54, keeping up to 32 decimal places. A copy of part of this is given in Figure 1.4.[23] Let us examine the last row, which is of considerable interest.

1° Système logarithmique de Briggs
(Extrait)

	Nombres	Logarithmes
	10, 0000 00	1
1	3, 1622.77660.16387.93319.98893.54	0,5
2	1, 7782 79410 03892 28011 97304 13	0,25
3	1, 3335 21432 16322 40256 65389 308	0,125
4	1, 1547 81984 68945 81796 61918 213	0,0625
16	1, 0000 35135 27746 18566 08581 37077	0,00001 52587 89062 5
17	1, 0000 17567 48442 26738 33846 78274	0,00000 76293 94531 25
18	1, 0000 08783 70363 46121 46574 07431	0,00000 38146 97265 625
47	1, 0000 00000 00001 63608 51112 96427 283	»
48	1, 0000 00000 00000 81804 25556 48210 295	»
49	1, 0000 00000 00000 40902 12778 24104 311	»
52	1, 0000 00000 00000 05112 76597 28102 947	0,00000.00000.00002.2204.46059.25031
53	1, 0000 00000 00000 02556 38298 64006 470	0,00000.00000.00001.1102 .23024.62515
54	1, 0000 00000 00000 01278 19149 32003 235	0,00000.00000.00000.05561.11512.31257
	1, 0000 00000 00000 01	0,00000.00000.00000.04342.94481.90325.1804

Figure 1.4.

Briggs had somehow noticed that the decimal parts of the second column bear an interesting relation to each other. If $a = 1 + x$ with $x \ll 1$, then $a^{1/2} \sim 1 + x/2$. Briggs exploited this relationship most effectively, and indeed devised schemes which were used until fairly recently for making logarithmic tables.

He first calculated the function

$$10^{2^{-n}} \qquad (n = 1, 2, \ldots, 54),$$

as indicated in Figure 1.4, and noted that for n near 54

$$\log_{10} 10^{2^{-n}} = \log_{10}(1 + x_n) \sim k \cdot x_n.$$

That is, he observed that the logarithm of a number of the form $1 + x$ with x very small is essentially proportional to x. We recall that

$$\begin{aligned}\log_{10}(1 + x) &= (\log_{10} e) \log_e (1 + x)\\ &= (\log_{10} e)\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots\right).\end{aligned}$$

[23] This is extracted from an interesting work on logarithms; Naux [1966]. (The logarithmic entry corresponding to number 54 in that figure is in error; the digit 6 should be a 5.) A very good discussion of the history of the subject is contained in the "Large and Original History of the Discoveries and Writings Relating to these Subjects." This is by way of being an introduction to Hutton [1801] and is worth reading.

Thus if x is sufficiently small — say, $x < 10^{-16}$ — its higher powers are less than 10^{-32} and so up to the first power of x

$$\log_{10}(1 + x) = x \log_{10} e = k' \cdot x; \tag{1.9}$$

where

$$k' = \log_{10} e = 0.4342944819032518 2765\ldots .$$

Actually Briggs calculated the value of k by forming the proportion

$$\frac{2^{-53} - 2^{-54}}{10^{2^{-53}} - 10^{2^{-54}}} = \frac{0.00000\ 00000\ 00000\ 01}{10^{-16}k};$$

and he then found

$$10^{-16}k = 0.00000\ 00000\ 00000\ 04342\ 94481\ 90325\ 1804.^{24}$$

Notice that the two values k, k', given above, differ in the 17th decimal place. This is due to the fact that formula (1.9) is not correct to as many places as we have kept.

Clearly the more nearly correct formula is

$$\log_{10}(1 + x) = x(\log_{10} e)\left(1 - \frac{x}{2}\right). \tag{1.10}$$

Now for $x = 10^{-16}$ this gives for the expression $(\log e) \cdot (1 - x/2)$ the value

$$\begin{aligned}&0.4342\ 9448190325182765 - 2.1715 \times 10^{-17}\\ &= 0.4342\ 9448190325180594,\end{aligned}$$

which more nearly corresponds to Briggs's value.

To form a table of logarithms Briggs needed to consider only the prime numbers, for obvious reasons. If p is such a number, then for some n he would have

$$p^{2^{-n}} = 1 + x \tag{1.11}$$

with $x \sim 10^{-16}$. Thus by the formula (1.9) he had

$$\log_{10} p = 2^n \log_{10}(1 + x) = 2^n kx. \tag{1.12}$$

Thus he reduced his task to finding the successive square roots of a number down to the point where it was expressible in the form (1.9) above; then the relation (1.12) immediately gave the desired logarithm.

It is worth remarking on the high degree of ingenuity he displayed in finding the logarithms of the primes. Thus, e.g., he calculated log 2 by starting with the fact that $2^{10}/1000 = 1.024$. Then 47 extractions of square roots gave him his result; log 3 was formed by noting that $6^9 = 10077696$

[24] Delambre, *MOD*, Vol. I, pp. 536–537. (There is a systematic mistake in this text: one zero too many appears in some of the relevant formulas in this part of the discussion.)

(*Arithmetica Logarithmica*, p. 16). After 46 extractions of square roots he had his logarithm.

In practice this involved a tremendous amount of work, so Briggs ingeniously invented a number of labor-saving devices of a mathematical kind. Perhaps the most important is his discovery of the calculus of finite differences to expedite the extractions of square roots. He noticed in effect that when a number was of the form $1 + y$, its square root was expressible as

$$(1 + y)^{1/2} = 1 + \frac{1}{2}y - \frac{1}{8}y^2 \pm \cdots,$$

and he developed differences especially tailored to this situation. I have illustrated how it goes in the following table where superscripts indicate the order of the Briggsian difference, and where $1 + u_{i+1} = (1 + u_i)^{1/2}$.

ξ	B^1	B^2	B^3
$1 + u_1$			
	$\frac{1}{2}u_1 - u_2 = B_1^1$		
$1 + u_2$		$\frac{1}{4}B_1^1 - B_2^1 = B_1^2$	
	$\frac{1}{2}u_2 - u_3 = B_2^1$		$\frac{1}{8}B_1^2 - B_2^2 = B_1^3$
$1 + u_3$		$\frac{1}{4}B_2^1 - B_3^1 = B_2^2$	
	$\frac{1}{2}u_3 - u_4 = B_3^1$		
$1 + u_4$			

He then used these differences to extrapolate forward. Let us see how he did this by considering the example

n	ξ_n	B^1	B^2	B^3	B^4
1	1.00757 13453 69831				
		71386 59690			
2	1.00377 85340 25226		33 63766		
		17813 01156		693	
3	1.00188 74857 11457		4 19778		0
		04449 05511		43	
4	1.00094 32979 50217		52429		1
		01111 73949		4	
5	1.00047 15378 01159		6550		
		00277 86937			
6	1.00023 57411 13643				

We notice that the fourth Briggsian differences are very small. Proceed to find the entries for line 7 by means of the relations

$$B_{j+1}^4 = \frac{1}{32}B_j^4 - B_j^5, \qquad B_{j+2}^3 = \frac{1}{16}B_{j+1}^3 - B_{j+1}^4, \qquad B_{j+3}^2 = \frac{1}{8}B_{j+2}^2 - B_{j+2}^3,$$

$$B_{j+4}^1 = \frac{1}{4}B_{j+3}^1 - B_{j+3}^2, \qquad B_{j+5}^0 = u_{j+5} = \frac{1}{2}B_{j+4}^0 - B_{j+4}^1 = \frac{1}{2}u_{j+4} - B_{j+4}^1.$$

With their help, and the assumption that the B_j^5 are 0, we find

$$B_3^4 = 0.03, \qquad B_4^3 = 0.22, \qquad B_5^2 = 818.53, \qquad B_6^1 = 6945915.72,$$
$$B_7^0 = u_7 = 1178636\ 10905.78.$$

This gives us 1.00011 78636 10906 as the ξ entry in line 7. Since B_3^4 is essentially zero and $B_4^4 = B_3^4/32$, it is even less work to find u_8. In fact we have $B_6^2 = 102.38$, $B_7^1 = 1736376.62$, $u_8 = 589300\ 69076$,

$$\xi_8 = 1.00005\ 89300\ 69076.$$

We may proceed in the same way doing only divisions by low powers of 2 to achieve the successive square roots beyond this point. Let us extend the table further to see what happens:

n	ξ_n	B^1	B^2	B^3
		00069 45916		
7	1.00011 78636 10906		102	
		00017 36377		0
8	1.00005 89300 69076		13	
		4 34081		0
9	1.00002 94646 00457		2	
		1 08518		0
10	1.00001 47321 91710		0	
		27130		
11	1.00000 73660 68725		0	
		6782		
12	1.00000 36830 27580			
		1696		
13	1.00000 18415 12094			
		424		
14	1.00000 09207 55623			
		106		
15	1.00000 04603 77705			
		26		
16	1.00000 02301 88826			
		7		
17	1.00000 01150 94406			
		2		
18	1.00000 00575 47201			
		0		
19	1.00000 00287 73600			

We note that the various differences decrease very rapidly so that it becomes increasingly easy to extend the table.

Suppose we have already found the logarithms of the primes through 101 (there are 26 primes involved), and that we wish to find the logarithm of the prime 173. We have $173/170 = 1.01764\,70588\,23529$ and

$$1.01764\,70588\,23529/1.01 = 1.00757\,13453\,69831.$$

But this is the entry for ξ_1 in our table above. If we tentatively take the entry for ξ_{10} to evaluate $\log_{10} \xi_1$, we find

$$\log_{10} \xi_1 = 2^9 \cdot \log_{10} e \cdot 1.4732\,19171 \cdot 10^{-5} = 0.00327\,5832097$$

and therefore

$$\begin{aligned}\log_{10} 173 &= \log_{10} 170 + \log_{10} \xi_1 + \log 1.01\\ &= 2.230448921 + .00327\,5832 + .004321374\\ &= 2.238046127.\end{aligned}$$

Actually $\log_{10} 173 = 2.23804610$. Let us next try with ξ_{14}; we then find $\log_{10} 173 = 2.2384610$. Note the value of the first Briggsian difference B^1 in row 14 as compared to earlier ones. In fact Briggs's basic relation (1.9), $\log_{10}(1 + x) = x \log_{10} e$, requires the higher powers of x to vanish to the number of places involved.

Let us stop for a moment to see another aspect of Briggs's differences for the square-root function.[25] As before let the ith Briggsian difference be written as B_j^i where $j = 1, 2, \ldots$ indicates the row in which B_j^i appears. We recall that

$$B_j^0 = u_j, \qquad B_0^0 = u \tag{1.13}$$

$$B_j^{i+1} = \frac{1}{2^{i+1}} B_j^i - B_{j+1}^i \qquad (i, j = 0, 1, \ldots), \tag{1.14}$$

where $1 + u$ is the quantity whose successive square roots we wish to determine. Now Briggs recorded his differences as powers of u.[26]

We do this as follows:

$$u_{j+1} = \frac{1}{2} u_j - B_j^1, \ldots, \qquad B_{j+1}^i = \frac{1}{2^{i+1}} B_j^i - B_j^{i+1}, \ldots;$$

this may be written as

$$u_{j+1} = \frac{1}{2} u_j - \frac{1}{2^2} B_{j-1}^1 + \frac{1}{2^3} B_{j-2}^2 - \frac{1}{2^4} B_{j-3}^3 + \cdots + \frac{(-1)^i}{2^{i+1}} B_{j-i}^i + \cdots \tag{1.15}$$

[25] Whiteside, *Patterns*, p. 234. This is a very elegant paper and well worth reading in its entirety. Also Hutton [1801], pp. 67–68. The Briggsian differences appear in Briggs, *ARITH*, p. 16 of his Introduction.

[26] He went up to B_{j-9}^9. Whiteside, *Patterns*, p. 234.

However, Briggs was well aware that eventually his differences became very small — *perexiguus* — and may be neglected. Thus for some i, say I, we have

$$B_{j+1}^I = \frac{1}{2^{I+1}} B_j^I \qquad (I < j),$$

and the series (1.15) may be terminated when $i = I$. Briggs chose $I = 9$.

Finally, Briggs evaluated the successive B_{j-1}^i for $i = 1, 2, \ldots, 9$ with the help of the fact that $(1 + u_j) = (1 + u_{j-1})^{1/2}$ and found quite correctly that, if $x = u_j$,

$$B_{j-1}^1 = \frac{1}{2} B_{j-1}^0 - B_j^0 = \frac{1}{2} x^2$$

$$B_{j-2}^2 = \frac{1}{4} B_{j-2}^1 - B_{j-1}^1 = \frac{1}{2} x^3 + \frac{1}{8} x^4$$

$$B_{j-3}^3 = \frac{1}{8} B_{j-3}^2 - B_{j-2}^2 = \frac{7}{8} x^4 + \frac{7}{8} x^5 + \frac{7}{16} x^6 + \frac{1}{8} x^7 + \frac{1}{64} x^8$$

$$\vdots$$

$$B_{j-9}^9 = \frac{1}{512} B_{j-9}^8 - B_{j-8}^8 = \qquad 2805527 x^{10}.$$

If these are now substituted into (1.15), there results — *mirabile dictu* — the Binomial theorem for $n = \frac{1}{2}$, $u_j = x$, namely:

$$(1 + x)^{1/2} = 1 + \frac{1}{2} x - \frac{1 \cdot 1}{2 \cdot 4} x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} x^4 + \cdots .$$

This must be regarded as the first time the Binomial theorem was developed for a noninteger exponent. There are two relevant comments: first, the calculating labor involved in finding the Briggsian differences as functions of u_j is not at all trivial and one can only wonder at Briggs's prowess; second, it is curious that he perceived the essential value of the formula *a priori*. Whiteside points out the curious fact that the first use of a series approximation to find logarithms was not of the logarithm but of the square root.[27]

Not only did Briggs use these Briggsian differences, he also was facile with ordinary differences and used them to subtabulate his tables. In fact his general *modus operandi* was this: firstly he found the logarithms of the first 25 primes, 2 through 97, using his method of repeated square roots plus his Briggsian differences together with other clever devices to simplify the calculation of these roots; secondly he tabulated the logarithms of about 20 percent of his table with the help of these primes; and thirdly he subtabulated, i.e., he filled in intermediate values, to find the rest of his entries. To do this he had to discover one of the now well-known interpolation formulas, which we discuss in Section 1.5.[28]

[27] Whiteside, *Patterns*, p. 234.

[28] Whittaker, *WR*, p. 11.