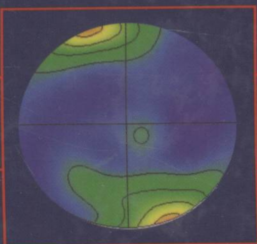
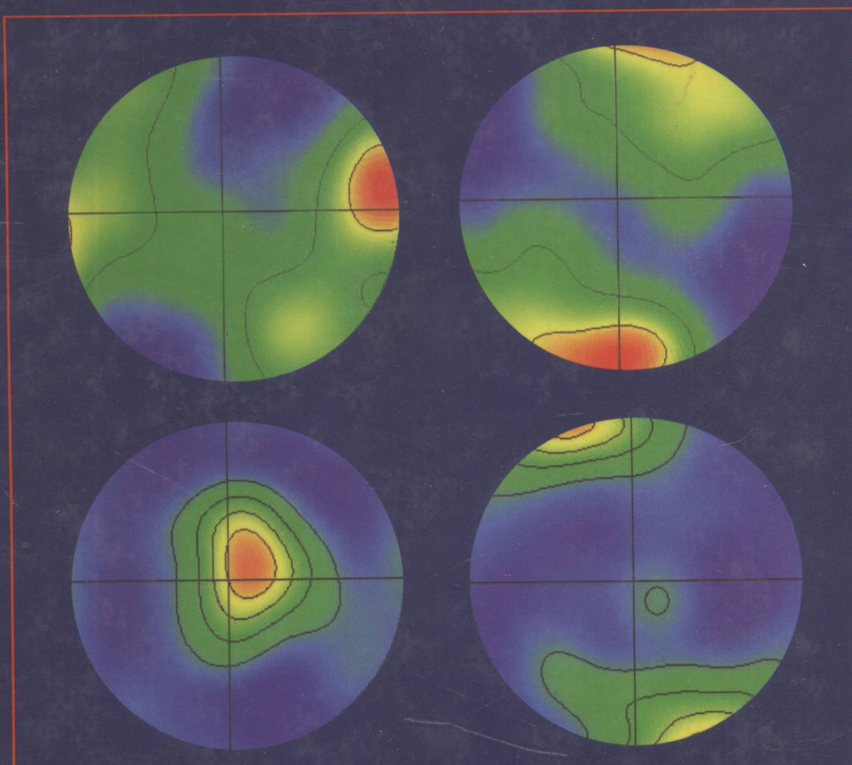
Deformation of Earth Materials



An Introduction to the Rheology of Solid Earth



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Deformation of Earth Materials

An Introduction to the Rheology of Solid Earth

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Deformation of Earth Materials

Much of the recent progress in the solid Earth sciences is based on the interpretation of a range of geophysical and geological observations in terms of the properties and deformation of Earth materials. One of the greatest challenges facing geoscientists in achieving this lies in finding a link between physical processes operating in minerals at the smallest length scales to geodynamic phenomena and geophysical observations across thousands of kilometers.

This graduate textbook presents a comprehensive and unified treatment of the materials science of deformation as applied to solid Earth geophysics and geology. Materials science and geophysics are integrated to help explain important recent developments, including the discovery of detailed structure in the Earth's interior by high-resolution seismic imaging, and the discovery of the unexpectedly large effects of high pressure on material properties, such as the high solubility of water in some minerals. Starting from fundamentals such as continuum mechanics and thermodynamics, the materials science of deformation of Earth materials is presented in a systematic way that covers elastic, anelastic, and viscous deformation. Although emphasis is placed on the fundamental underlying theory, advanced discussions on current debates are also included to bring readers to the cutting edge of science in this interdisciplinary area.

Deformation of Earth Materials is a textbook for graduate courses on the rheology and dynamics of the solid Earth, and will also provide a much-needed reference for geoscientists in many fields, including geology, geophysics, geochemistry, materials science, mineralogy, and ceramics. It includes review questions with solutions, which allow readers to monitor their understanding of the material presented.

Shun-ichiro Karato is a Professor in the Department of Geology and Geophysics at Yale University. His research interests include experimental and theoretical studies of the physics and chemistry of minerals, and their applications to geophysical and geological problems. Professor Karato is a Fellow of the American Geophysical Union and a recipient of the Alexander von Humboldt Prize (1995), the Japan Academy Award (1999), and the Vening Meinesz medal from the Vening Meinesz School of Geodynamics in The Netherlands (2006). He is the author of more than 160 journal articles and has written/edited seven other books.

Preface

Understanding the microscopic physics of deformation is critical in many branches of solid Earth science. Long-term geological processes such as plate tectonics and mantle convection involve plastic deformation of Earth materials, and hence understanding the plastic properties of Earth materials is key to the study of these geological processes. Interpretation of seismological observations such as tomographic images or seismic anisotropy requires knowledge of elastic, anelastic properties of Earth materials and the processes of plastic deformation that cause anisotropic structures. Therefore there is an obvious need for understanding a range of deformation-related properties of Earth materials in solid Earth science. However, learning about deformation-related properties is challenging because deformation in various geological processes involves a variety of microscopic processes. Owing to the presence of multiple deformation mechanisms, the results obtained under some conditions may not necessarily be applicable to a geological problem that involves deformation under different conditions. Therefore in order to conduct experimental or theoretical research on deformation, one needs to have a broad knowledge of various mechanisms to define conditions under which a study is to be conducted. Similarly, when one attempts to use results of experimental or theoretical studies to understand a geological problem, one needs to evaluate the validity of applying particular results to a given geological problem. However, there was no single book available in which a broad range of the physics of deformation of materials was treated in a systematic manner that would be useful for a student (or a scientist) in solid Earth science. The motivation of writing this book was to fulfill this need.

In this book, I have attempted to provide a unified, interdisciplinary treatment of the science of deformation of Earth with an emphasis on the materials science (microscopic) approach. Fundamentals of the materials science of deformation of minerals and rocks over various time-scales are described in addition to the applications of these results to important geological and geophysical problems. Properties of materials discussed include elastic, anelastic (viscoelastic), and plastic properties. The emphasis is on an *interdisciplinary approach*, and, consequently, I have included discussions on some advanced, controversial issues where they are highly relevant to Earth science problems. They include the role of hydrogen, effects of pressure, deformation of two-phase materials, localization of deformation and the link between viscoelastic deformation and plastic flow. This book is intended to serve as a textbook for a course at a graduate level in an Earth science program, but it may also be useful for students in materials science as well as researchers in both areas. No previous knowledge of geology/geophysics or of materials science is assumed. The basics of continuum mechanics and thermodynamics are presented as far as they are relevant to the main topics of this book.

Significant progress has occurred in the study of deformation of Earth materials during the last ~30 years, mainly through experimental studies. Experimental studies on synthetic samples under well-defined chemical conditions and the theoretical interpretation of these results have played an important role in understanding the microscopic mechanisms of deformation. Important progress has also been made to expand the pressure range over which plastic deformation can be investigated, and the first low-strain anelasticity measurements have been conducted. In addition, some large-strain deformation experiments have been performed that have provided important new insights into the microstructural evolution during deformation. However, experimental data are always obtained under limited conditions and their applications to the Earth involve large extrapolation. It is critical to understand the *scaling laws* based on the physics and chemistry of deformation of materials in order to properly apply experimental data to Earth. A number of examples of such scaling laws are discussed in this book.

This book consists of three parts: Part I (Chapters 1–3) provides a general background including basic continuum mechanics, thermodynamics and phenomenological theory of deformation. Most of this part, particularly Chapters 1 and 2 contain material that can be found in many other textbooks. Therefore those who are familiar with basic continuum mechanics and thermodynamics can skip this part. Part II (Chapters 4–16) presents a detailed account of materials science of time-dependent deformation, including elastic, anelastic and plastic deformation with an emphasis on anelastic and plastic deformation. They include, not only the basics of properties of materials characterizing deformation (i.e., elasticity and viscosity (creep strength)), but also the physical principles controlling the microstructural developments (grain size and lattice-preferred orientation). Part III (Chapters 17–21) provides some applications of the materials science of deformation to important geological and geophysical problems, including the rheological structure of solid Earth and the interpretation of the pattern of material circulation in the mantle and core from geophysical observations. Specific topics covered include the lithosphere–asthenosphere structure, rheological stratification of Earth's deep mantle and a geodynamic interpretation of anomalies in seismic wave propagation. Some of the representative experimental data are summarized in tables. However, the emphasis of this book is on presenting basic theoretical concepts and consequently references to the data are not exhaustive. Many problems (with solutions) are provided to make sure a reader understands the content of this book. Some of them are advanced and these are shown by an asterisk.

The content of this book is largely based on lectures that I have given at the University of Minnesota and Yale University as well as at other institutions. I thank students and my colleagues at these institutions who have given me opportunities to improve my understanding of the subjects discussed in this book through inspiring questions. Some parts of this book have been read/reviewed by A. S. Argon, D. Bercovici, H. W. Green, S. Hier-Majumder, G. Hirth, I. Jackson, D. L. Kohlstedt, J. Korenaga, R. C. Liebermann, J.-P. Montagner, M. Nakada, C. J. Spiers, J. A. Tullis and J. A. Van Orman. However, they do not always agree with the ideas presented in this book and any mistakes are obviously my own. W. Landuyt, Z. Jiang and P. Skemer helped to prepare the figures. I should also thank the editors at Cambridge University Press for their patience. Last but not least, I thank my family, particularly my wife, Yoko, for her understanding, forbearance and support during the long gestation of this monograph. Thank you all.

Contents

The colour plates are between pages 118 and 119.

Part I
General background

1 Stress and strain

The concept of stress and strain is key to the understanding of deformation. When a force is applied to a continuum medium, stress is developed inside it. Stress is the force per unit area acting on a given plane along a certain direction. For a given applied force, the stress developed in a material depends on the orientation of the plane considered. Stress can be decomposed into hydrostatic stress (pressure) and deviatoric stress. Plastic deformation (in non-porous materials) occurs due to deviatoric stress. Deformation is characterized by the deformation gradient tensor, which can be decomposed into rigid body rotation and strain. Deformation such as simple shear involves both strain and rigid body rotation and hence is referred to as rotational deformation whereas pure shear or tri-axial compression involves only strain and has no rigid body rotation and hence is referred to as irrotational deformation. In rotational deformation, the principal axes of strain rotate with respect to those of stress whereas they remain parallel in irrotational deformation. Strain can be decomposed into dilatational (volumetric) strain and shear strain. Plastic deformation (in a non-porous material) causes shear strain and not dilatational strain. Both stress and strain are second-rank tensors, and can be characterized by the orientation of the principal axes and the magnitude of the principal stress and strain and both have three invariants that do not depend on the coordinate system chosen.

Key words stress, strain, deformation gradient, vorticity, principal strain, principal stress, invariants of stress, invariants of strain, normal stress, shear stress, Mohr's circle, the Flinn diagram, foliation, lineation, coaxial deformation, non-coaxial deformation.

1.1. Stress

1.1.1. Definition of stress

This chapter provides a brief summary of the basic concept of stress and strain that is relevant to understanding plastic deformation. For a more comprehensive treatment of stress and strain, the reader may consult Malvern (1969), Mase (1970), Means (1976).

In any deformed or deforming continuum material there must be a force inside it. Consider a small block of a deformed material. Forces acting on the material can be classified into two categories, i.e., a short-range force due to atomic interactions and the long-range force due to an external field such as the gravity field. Therefore the forces that act on this small block include (1) short-range forces due to the displacement of atoms within this block, (2) long-range forces such as gravity that act equally on each atom and (3) the forces that act on this block through the surface from the neighboring materials. The (small) displacements of each atom inside this region cause forces to act on surrounding atoms, but by assumption these forces are short range. Therefore one can consider them as forces between a pair of atoms A and B. However, because of Newton's law of action and counter-action, the forces acting between two atoms are anti-symmetric: $f_{AB} = -f_{BA}$ where $f_{AB\ (BA)}$

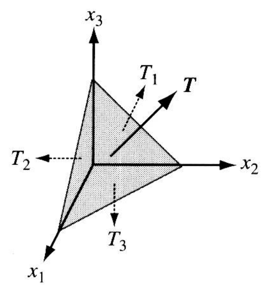

FIGURE 1.1 Forces acting on a small pyramid.

are the force exerted by atom A (B) to B (A). Consequently these forces caused by atomic displacement within a body must cancel. The long-range force is called a *body force*, but if one takes this region as small, then the magnitude of this body force will become negligible compared to the *surface force* (i.e., the third class of force above). Therefore the net force acting on the small region must be the forces across the surface of that region from the neighboring materials. To characterize this force, let us consider a small piece of block that contains a plane with the area of dS and whose normal is $\boldsymbol{n}$ ($\boldsymbol{n}$ is the unit vector). Let $\boldsymbol{T}$ be the force (per unit area) acting on the surface dS from outside this block (positive when the force is compressive) and consider the force balance (Fig. 1.1). The force balance should be attained among the force $\boldsymbol{T}$ as well as the forces $\boldsymbol{T}^{1,2,3}$ that act on the surface $dS_{1,2,3}$ respectively ($dS_{1,2,3}$ are the projected area of dS on the plane normal to the $x_{1,2,3}$ axis). Then the force balance relation for the block yields,

$$\boldsymbol{T}\,dS = \sum_{j=1}^{3} \boldsymbol{T}^j\,dS_j. \tag{1.1}$$

Now using the relation $dS_j = n_j\,dS$, one obtains,

$$T_i = \sum_{j=1}^{3} T_i^j n_j = \sum_{j=1}^{3} \sigma_{ij} n_j \tag{1.2}$$

where T_i is the ith component of the force $\boldsymbol{T}$ and σ_{ij} is the ith component of the traction $\boldsymbol{T}^j$, namely the ith component of force acting on a plane whose normal is the jth direction ($n_{ij} = T_i^j$). This is the definition of *stress*. From the balance of torque, one can also show,

$$\sigma_{ij} = \sigma_{ji}. \tag{1.3}$$

The values of stress thus defined depend on the coordinate system chosen. Let us denote quantities in a new coordinate system by a tilda, then the new coordinate and the old coordinate system are related to each other by,

$$\tilde{x}_i = \sum_{j=1}^{3} a_{ij} x_j \tag{1.4}$$

where a_{ij} is the transformation matrix that satisfies the orthonormality relation,

$$\sum_{j=1}^{3} a_{ij} a_{jm} = \delta_{im} \tag{1.5}$$

where δ_{im} is the Kronecker delta ($\delta_{im} = 1$ for $i = m$, $\delta_{im} = 0$ otherwise). Now in this new coordinate system, we may write a relation similar to equation (1.2) as,

$$\tilde{T}_i = \sum_{j=1}^{3} \tilde{\sigma}_{ij} \tilde{n}_j. \tag{1.6}$$

Noting that the traction ($\boldsymbol{T}$) transforms as a vector in the same way as the coordinate system, equation (1.4), we have,

$$\tilde{T}_i = \sum_{j=1}^{3} a_{ij} T_j. \tag{1.7}$$

Inserting equation (1.2), the relation (1.7) becomes,

$$\tilde{T}_i = \sum_{j,k=1}^{3} \sigma_{jk} a_{ij} n_k. \tag{1.8}$$

Now using the orthonormality relation (1.5), one has,

$$n_i = \sum_{j=1}^{3} a_{ji} \tilde{n}_i. \tag{1.9}$$

Inserting this relation into equation (1.8) and comparing the result with equation (1.6), one obtains,[1]

$$\tilde{\sigma}_{ij} = \sum_{k,l=1}^{3} \sigma_{kl} a_{ik} a_{jl}. \tag{1.10}$$

The quantity that follows this transformation law is referred to as *a second rank tensor*.

1.1.2. Principal stress, stress invariants

In any material, there must be a certain orientation of a plane on which the direction of traction ($\boldsymbol{T}$) is normal to it. For that direction of $\boldsymbol{n}$, one can write,

$$T_i = \sigma n_i \tag{1.11}$$

[1] In the matrix notation, $\tilde{\sigma} = A \cdot \sigma \cdot A^T$ where $A = (a_{ij})$ and $A^T = (a_{ji})$.

where σ is a scalar quantity to be determined. From equations (1.11) and (1.2),

$$\sum_{j=1}^{3}(\sigma_{ij} - \sigma\delta_{ij})n_j = 0. \tag{1.12}$$

For this equation to have a non-trivial solution other than $\boldsymbol{n}=0$, one must have,

$$\left|\sigma_{ij} - \sigma\delta_{ij}\right| = 0 \tag{1.13}$$

where $\left|X_{ij}\right|$ is the determinant of a matrix X_{ij}. Writing equation (1.13) explicitly, one obtains,

$$\begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = -\sigma^3 + I_\sigma\sigma^2 + II_\sigma\sigma + III_\sigma = 0 \tag{1.14}$$

with

$$I_\sigma = \sigma_{11} + \sigma_{22} + \sigma_{33} \tag{1.15a}$$

$$II_\sigma = -\sigma_{11}\sigma_{22} - \sigma_{11}\sigma_{33} - \sigma_{33}\sigma_{22} + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 \tag{1.15b}$$

$$III_\sigma = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2. \tag{1.15c}$$

Therefore, there are three solutions to equation (1.14), $\sigma_1, \sigma_2, \sigma_3 (\sigma_1 > \sigma_2 > \sigma_3)$. These are referred to as the *principal stresses*. The corresponding $\boldsymbol{n}$ is *the orientation of principal stress*. If the stress tensor is written using the coordinate whose orientation coincides with the orientation of principal stress, then,

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}. \tag{1.16}$$

It is also seen that because equation (1.14) is a scalar equation, the values of I_σ, II_σ and III_σ are independent of the coordinate. These quantities are called the *invariants of stress tensor*. These quantities play important roles in the formal theory of plasticity (see Section 3.3). Equations (1.15a–c) can also be written in terms of the principal stress as,

$$I_\sigma = \sigma_1 + \sigma_2 + \sigma_3 \tag{1.17a}$$

$$II_\sigma = -\sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 \tag{1.17b}$$

and

$$III_\sigma = \sigma_1\sigma_2\sigma_3. \tag{1.17c}$$

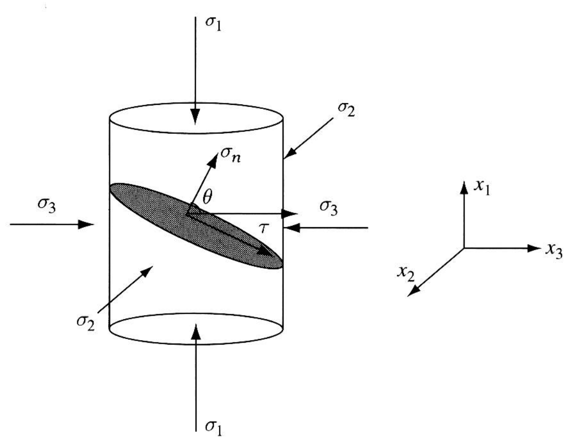

FIGURE 1.2 Geometry of normal and shear stress on a plane.

1.1.3. Normal stress, shear stress, Mohr's circle

Now let us consider the normal and shear stress on a given plane subjected to an external force (Fig. 1.2). Let x_1 be the axis parallel to the maximum compressional stress σ_1 and x_2 and x_3 be the axes perpendicular to x_1. Consider a plane whose normal is at the angle θ from x_3 (positive counterclockwise). Now, we define a new coordinate system whose x'_1 axis is normal to the plane, but the x'_2 axis is the same as the x_2 axis. Then the transformation matrix is,

$$[a_{ij}] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{1.18}$$

and hence,

$$[\tilde{\sigma}_{ij}] = \begin{bmatrix} \frac{\sigma_1+\sigma_3}{2} + \frac{\sigma_1-\sigma_3}{2}\cos 2\theta & 0 & \frac{\sigma_1-\sigma_3}{2}\sin 2\theta \\ 0 & \sigma_2 & 0 \\ \frac{\sigma_1-\sigma_3}{2}\sin 2\theta & 0 & \frac{\sigma_1+\sigma_3}{2} - \frac{\sigma_1-\sigma_3}{2}\cos 2\theta \end{bmatrix}. \tag{1.19}$$

Problem 1.1

Derive equation (1.19).

Solution

The stress tensor (1.16) can be rotated through the operation of the transformation matrix (1.18) using equation (1.10),

$$[\tilde{\sigma}_{ij}] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sigma_1+\sigma_3}{2} + \frac{\sigma_1-\sigma_3}{2}\cos 2\theta & 0 & \frac{\sigma_1-\sigma_3}{2}\sin 2\theta \\ 0 & \sigma_2 & 0 \\ \frac{\sigma_1-\sigma_3}{2}\sin 2\theta & 0 & \frac{\sigma_1+\sigma_3}{2} - \frac{\sigma_1-\sigma_3}{2}\cos 2\theta \end{bmatrix}.$$

Therefore the shear stress τ and normal stress σ_n on this plane are

$$\tilde{\sigma}_{13} \equiv \tau = \frac{\sigma_1 - \sigma_3}{2}\sin 2\theta \tag{1.20}$$

and

$$\tilde{\sigma}_{33} \equiv \sigma_n = \frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2}\cos 2\theta \tag{1.21}$$

respectively. It follows that the maximum shear stress is on the two conjugate planes that are inclined by $\pm\pi/4$ with respect to the x_1 axis and its absolute magnitude is $(\sigma_1 - \sigma_3)/2$. Similarly, the maximum compressional stress is on a plane that is normal to the x_1 axis and its value is σ_1. It is customary to use $\sigma_1 - \sigma_3$ as (differential (or deviatoric)) stress in rock deformation literature, but the shear stress, $\tau \equiv (\sigma_1 - \sigma_3)/2$, is also often used. Eliminating θ from equations (1.20) and (1.21), one has,

$$\tau^2 + \left(\sigma_n - \frac{\sigma_1 + \sigma_3}{2}\right)^2 = \frac{1}{4}(\sigma_1 - \sigma_3)^2. \tag{1.22}$$

Thus, the normal and shear stress on planes with various orientations can be visualized on a two-dimensional plane (τ–σ_n space) as a circle whose center is located at $(0, (\sigma_1 + \sigma_3)/2)$ and the radius $(\sigma_1 - \sigma_3)/2$ (Fig. 1.3). This is called a *Mohr's circle* and plays an important role in studying the brittle fracture that is controlled by the stress state (shear–normal stress ratio; see Section 7.3).

When $\sigma_1 = \sigma_2 = \sigma_3 (= P)$, then the stress is isotropic (hydrostatic). The hydrostatic component of stress does not cause plastic flow (this is not true for porous materials, but we do not discuss porous materials here), so it is useful to define *deviatoric stress*

$$\sigma'_{ij} \equiv \sigma_{ij} - \delta_{ij}P. \tag{1.23}$$

When we discuss plastic deformation in this book, we use σ_{ij} (without prime) to mean deviatoric stress for simplicity.

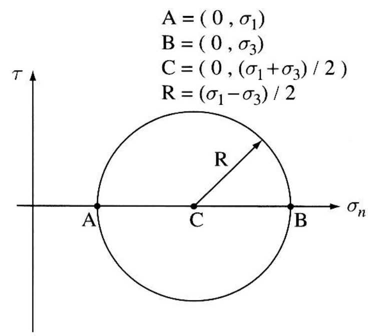

FIGURE 1.3 A Mohr circle corresponding to two-dimensional stress showing the variation of normal, σ_n, and shear stress, τ, on a plane.

Problem 1.2

Show that the second invariant of deviatoric stress can be written as $II_{\sigma'} = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$.

Solution

If one uses a coordinate system parallel to the principal axes of stress, from equation (1.15), one has $II_{\sigma'} = -\sigma'_1\sigma'_2 - \sigma'_1\sigma'_3 - \sigma'_3\sigma'_2$. Using $I_{\sigma'} = \sigma'_1 + \sigma'_2 + \sigma'_3 = 0$, one finds $I^2_{\sigma} = \sigma'^2_1 + \sigma'^2_2 + \sigma'^2_3 + 2(\sigma'_1\sigma'_2 + \sigma'_2\sigma'_3 + \sigma'_3\sigma'_1) = 0$. Therefore $II_{\sigma'} = \frac{1}{2}(\sigma'^2_1 + \sigma'^2_2 + \sigma'^2_3)$. Now, inserting $\sigma'_1 = \sigma_1 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ etc., one obtains $II_{\sigma'} = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$.

Problem 1.3

Show that when the stress has axial symmetry with respect to the x_1 axis (i.e., $\sigma_2 = \sigma_3$), then $\sigma_n = P + (\sigma_1 - \sigma_3)(\cos^2\theta - \frac{1}{3})$.

Solution

From (1.21), one obtains, $\sigma_n = (\sigma_1 + \sigma_3)/2 + ((\sigma_1 - \sigma_3)/2)\cos 2\theta$. Now $\cos 2\theta = 2\cos^2\theta - 1$ and $P = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(\sigma_1 + 2\sigma_3) = \sigma_1 - \frac{2}{3}(\sigma_1 - \sigma_3)$. Therefore $\sigma_n = P + (\sigma_1 - \sigma_3)(\cos^2\theta - \frac{1}{3})$.

Equations similar to (1.15)–(1.17) apply to the deviatoric stress.

1.2. Deformation, strain

1.2.1. Definition of strain

Deformation refers to a change in the shape of a material. Since homogeneous displacement of material points does not cause deformation, deformation must be related to *spatial variation* or *gradient* of displacement. Therefore, deformation is characterized by a displacement gradient tensor,

$$d_{ij} \equiv \frac{\partial u_i}{\partial x_j}. \tag{1.24}$$

where u_i is the displacement and x_j is the spatial coordinate (after deformation). However, this displacement gradient includes the rigid-body rotation that has nothing to do with deformation. In order to focus on deformation, let us consider two adjacent material points $P_0(\boldsymbol{X})$ and $Q_0(\boldsymbol{X}+d\boldsymbol{X})$, which will be moved to $P(\boldsymbol{x})$ and $Q(\boldsymbol{x}+d\boldsymbol{x})$ after deformation (Fig. 1.4). A small vector connecting P_0 and Q_0, $d\boldsymbol{X}$, changes to $d\boldsymbol{x}$ after deformation. Let us consider how the length of these two segments changes. The difference in the squares of the length of these small elements is given by,

$$\begin{aligned}(dx)^2 - (dX)^2 &= \sum_{i=1}^{3}(dx_i)^2 - \sum_{i=1}^{3}(dX_i)^2 \\ &= \sum_{i,j,k=1}^{3}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) dx_i\, dx_j. \end{aligned} \tag{1.25}$$

Therefore deformation is characterized by a quantity,

$$\varepsilon_{ij} \equiv \frac{1}{2}\left(\delta_{ij} - \sum_{k=1}^{3}\frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) \tag{1.26}$$

which is the definition of *strain*, ε_{ij}. With this definition, the equation (1.25) can be written as,

$$(dx)^2 - (dX)^2 \equiv 2\sum_{i,j}\varepsilon_{ij}\, dx_i\, dx_j. \tag{1.27}$$

From the definition of strain, it immediately follows that the strain is a symmetric tensor, namely,

$$\varepsilon_{ij} = \varepsilon_{ji}. \tag{1.28}$$

Now, from Fig. 1.4, one obtains,

$$du_i = dx_i - dX_i \tag{1.29}$$

hence

$$\frac{\partial u_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j}. \tag{1.30}$$

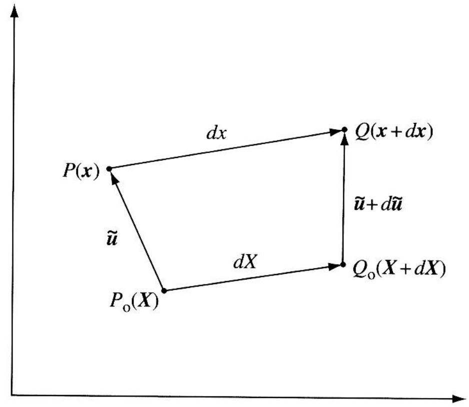

FIGURE 1.4 Deformation causes the change in relative positions of material points.

Inserting equation (1.30) into (1.26) one finds,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \sum_{k=1}^{3}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right). \tag{1.31}$$

This definition of strain uses the deformed state as a reference frame and is called the *Eulerian strain*. One can also define strain using the initial, undeformed reference state. This is referred to as the *Lagrangian strain*. For small strain, there is no difference between the Eulerian and Lagrangian strain and both are reduced to[2]

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.32}$$

1.2.2. Meaning of strain tensor

The interpretation of strain is easier in this linearized form. The displacement gradient can be decomposed into two components,

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{1.33}$$

The first component is a symmetric part,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ji} \tag{1.34}$$

which represents the strain (as will be shown later in this chapter).

[2] Note that in some literature, another definition of shear strain is used in which $\varepsilon_{ij} = \partial u_i/\partial x_j + \partial u_j/\partial x_i$ for $i \neq j$ and $\varepsilon_{ii} = \partial u_i/\partial x_i$; e.g., Hobbs *et al.* (1976). In such a case, the symbol γ_{ij} is often used for the non-diagonal ($i \neq j$) strain component instead of ε_{ij}.