TRANSFORMATION METHODS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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FORWARD

The use of transformation methods to solve systems of partial differential equations in the classical manner (i.e., by symmetry and similarity methods) is severely limited by the Bäcklund theorem. This theorem states that a group or pseudogroup of transformation that carries solutions of a system of partial differential equations into solutions of that system is the prolongation of a group of point transformation that acts on the Cartesian product of the domain space of the independent variables and the range space of the dependent variables. One way to circumvent this limitation is to discard the notion that the transformations form a group or pseudogroup. This has been successfully pursued in published studies of what are now termed Bäcklund transformations. The purpose of this work is to present several essentially different alternatives.

The approach we will take is primarily geometric in nature. Full use is therefore made of E. Cartan's exterior calculus. We will use the standard notation of the exterior calculus:

$\wedge$ for exterior multiplication,
d for exterior differentiation,
$\rfloor$ for inner multiplication,
£ for Lie differentiation.

These operations lead to remarkable simplifications, both in the theory and in actual calculations. The reader is referred to [1 - 7] for proofs of the following results involving these operations. Let M be a given manifold of finite dimension, let α, β and γ be exterior differential forms over M (i.e., elements of the graded algebra $\Lambda(M)$ of exterior differential forms over M), let U and V be vector fields over M (i.e., derivations on the ring $\Lambda^0(M)$ of C^∞ functions over M), let f belong to $\Lambda^0(M)$, let a = degree of α, and let b = degree of β. We have

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma, \quad \alpha \wedge \beta = (-1)^{ab} \beta \wedge \alpha \,,$$

$$d(\alpha + \beta) = d\alpha + d\beta \,, \quad d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{a} \alpha \wedge d\beta \,,$$
$$dd\alpha = 0 \,,$$

$$(U + V) \rfloor \alpha = U \rfloor \alpha + V \rfloor \alpha \,, \quad U \rfloor (\alpha + \beta) = U \rfloor \alpha + U \rfloor \beta \,,$$
$$U \rfloor f = 0 \,, \quad U \rfloor (\alpha \wedge \beta) = (U \rfloor \alpha) \wedge \beta + (-1)^{a} \alpha \wedge (U \rfloor \beta) \,,$$

$$\pounds_U \alpha = U \rfloor d\alpha + d(U \rfloor \alpha) \,, \quad \pounds_U (\alpha + \beta) = \pounds_U \alpha + \pounds_U \beta \,,$$
$$\pounds_{U+V} \alpha = \pounds_U \alpha + \pounds_V \alpha \,, \quad \pounds_U (\alpha \wedge \beta) = (\pounds_U \alpha) \wedge \beta + \alpha \wedge \pounds_U \beta \,,$$
$$\pounds_U d\alpha = d\pounds_U \alpha \,, \quad \pounds_{fU} \alpha = f \pounds_U \alpha + df \wedge (U \rfloor \alpha) \,,$$
$$\pounds_U (V \rfloor \alpha) = \llbracket U, V \rrbracket \rfloor \alpha + V \rfloor \pounds_U \alpha \,,$$
$$\pounds_{\llbracket U, V \rrbracket} \alpha = \pounds_U \pounds_V \alpha - \pounds_V \pounds_U \alpha \,,$$

$$\int_{J_{a+1}} d\alpha = \int_{\partial J_{a+1}} \alpha \,,$$

where $\llbracket U, V \rrbracket$ is the commutator or Lie product of the vector fields U and V that is defined by $\llbracket U, V \rrbracket\langle f\rangle = U\langle V\langle f\rangle\rangle - V\langle U\langle f\rangle\rangle$, and where J_{a+1} is a smooth domain of dimension a+1 in M with boundary ∂J_{a+1} (remember that a = degree of α).

Let $\Lambda(M)$ denote the graded algebra of exterior differential forms over M and let $\Lambda^k(M)$ denote the module of exterior differential forms of degree k. The *ideal* of $\Lambda(M)$ that is algebraically generated by the exterior differential forms α, β, ... is denoted by $I\{\alpha, \beta, \ldots\}$ (see [7], chapter 4). A collective symbolic name for an ideal will usually be designated by use of an upper case script letter. Thus, we write $\mathcal{I} = I\{\alpha, \beta, \ldots\}$ for the ideal of $\Lambda(M)$ that is algebraically generated by the differential forms α, β, ... An ideal $\mathcal{I}$ is said to be *closed* (i.e., $d\mathcal{I} \subset \mathcal{I}$) if and only if $d\eta \in \mathcal{I}$ for every $\eta \in \mathcal{I}$. An ideal $\mathcal{I}$ is said to be *stable* under transport by a vector field V (i.e., $\pounds_V \mathcal{I} \subset \mathcal{I}$) if and only if $\pounds_V \eta \in \mathcal{I}$ for each $\eta \in \mathcal{I}$, in which case $\exp(s\pounds_V)\mathcal{I} \subset \mathcal{I}$. Such vector fields are referred to as *isovectors* of the ideal $\mathcal{I}$. A vector field V is said to be a *Cauchy characteristic* of an ideal $\mathcal{I}$ (i.e., $V \rfloor \mathcal{I} \subset \mathcal{I}$) if and only if $V \rfloor \eta \in \mathcal{I}$ for each $\eta \in \mathcal{I}$.

If $\Phi: N \to M$ is a differentiable mapping, we write $\Phi^*: \Lambda(M) \to \Lambda(N)$ for the induced mapping of the graded algebras of exterior differential forms. Thus, if $\Phi: N \to M$, then Φ^* maps any exterior differential form of degree k on M onto an exterior differential form of degree k on N; that is, Φ^* maps in the direction opposite to that of Φ. We note in particular that

$$(\Phi \circ \Psi)^* \alpha = \Psi^* \circ \Phi^* \alpha \ , \quad \Phi^*(d\alpha) = d(\Phi^* \alpha) \ ,$$

and that

$$\Phi^* \{ (\Phi_* V) \rfloor \alpha \} = V \rfloor \Phi^* \alpha \ ,$$

where $\Phi_* V$ denotes the image of the vector field V over M that is induced by the map Φ from the vector field V over N. A map Φ is said to *solve* (annihilate) an ideal $\mathfrak{I}$ of $\Lambda(M)$ (i.e., $\Phi^* \mathfrak{I} = 0$) if and only if $\Phi^* \eta = 0$ for each $\eta \in \mathfrak{I}$. Since ideals of $\Lambda(M)$ can be used to encode systems of partial differential equations when M is chosen in an appropriate manner, a solving map of an ideal provides a geometric presentation of a solution of a system of partial differential equations.

A vector field V over M induces a flow $T_V(s): M \to M$ (i.e., a 1-parameter pseudogroup of point transformations of M) by solving the orbital equations associated with the vector field V with generic initial data (see [7]). This flow can be represented symbolically by its induced action on C^∞-functions:

$$T_V(s)\langle f \rangle = \exp(s V)\langle f \rangle$$

for any f in $\Lambda^0(M)$. We then have (see [7], chapter 4)

$$T_V(s)^* \alpha = \exp(s \pounds_V) \alpha$$

for any $\alpha \in \Lambda(M)$, and hence we obtain the fundamental transport relation

$$(T_V(s) \circ \Phi)^* \alpha = \Phi^* \{ \exp(s \pounds_V) \alpha \} \ .$$

If V is an isovector of an ideal $\mathcal{I}$, then $T_V(s)^*\mathcal{I} = \exp(s\,\pounds_V)\mathcal{I} \subset \mathcal{I}$. Let Φ be a solving map of the ideal $\mathcal{I}$, that is $\Phi^*\mathcal{I} = 0$, then $(T_V(s)\circ\Phi)^*\mathcal{I} = \Phi^*\{\exp(s\,\pounds_V)\mathcal{I}\} = 0$. Thus, if Φ is a solving map of an ideal $\mathcal{I}$ and V is a isovector of $\mathcal{I}$, then $\Phi_V(s) = T_V(s)\circ\Phi$ is a solving map of $\mathcal{I}$ for all s in a neighborhood of $s = 0$.

This elementary result is the basis upon which classical transformation methods (symmetry and similarity methods) for partial differential equations have been developed. Chapters one and two present and extend these notions in the context of contact manifolds of finite order. In particular, the Bäcklund theorem and its implied restrictions are explicitly established for systems of partial differential equations with more than one dependent variable. Chapters one and two thus act as a basis for the pursuit of alternatives to the Bäcklund theorem that are presented in the remaining chapters. The basic idea underlying these alternatives is to use explicitly constructed systems of Cartan annihilating vector fields of a subideal of the fundamental ideal for a given system of partial differential equations. Such systems of vector fields can then be identified with Cauchy characteristic vector fields of a modified subideal of the fundamental ideal. The demand that this modified subideal be completely integrable leads to a foliation of the underlying manifold M by n-dimensional submanifolds, where n is the number of independent variables for the given system of PDE. Certain leaves of these foliations can then be identified with the graphs of solution maps of the fundamental ideal (i.e., with solutions of the given system of PDE) under suitable additional hypotheses. In fact, it will be shown that every smooth (C^3) solution of the given system of PDE can be realized as the graph of a leaf of a foliation of M that is constructed in this manner.

A number of technical questions arise in this analysis that have to do with the construction and properties of completely integrable subideals of appropriate modifications of the fundamental ideal. These questions are answered in the last chapter by the study of *extended canonical transformations*; namely, transformations that map completely integrable horizontal ideals onto completely integrable horizontal ideals. Such transformations are not restricted by the Bäcklund theorem. They thus provide an extensive body of new results of both theoretical and practical importance.

We have tried to provide explicit examples throughout the text in order to illustrate the scope and/or restrictions of the methods under discussion, or to compare and contrast our results with those available in the current literature. Whether these efforts prove to be adequate is left to the evaluation of the reader.

Citations of equations in a given chapter will be by equation numbers of that chapter. A citation of an equation from a previous chapter will by indicated by prefixing the equation number with the chapter number.

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CHAPTER 1

CONTACT MANIFOLDS, IDEALS, AND PARTIAL DIFFERENTIAL EQUATIONS

The purpose of this chapter is to set an appropriate foundation for the study of partial differential equations (PDE) by transformation-theoretic methods. One of the central issues in such studies hinges on the mode of representation of the PDE. Most of the current literature elects to represent the PDE through the introduction of an appropriate jet bundle. In contrast, this study represents the PDE by introducing a contact manifold of appropriate order. When the number of dependent variables is greater than one, the two methods lead to the same groups of symmetry transformations for a given system of PDE. On the other hand, when there is only one dependent variable, the group of symmetry transformations computed by the jet bundle formalism for certain PDE is only a proper subgroup of the group of symmetry transformations computed on the appropriate contact manifold. In addition, the contact manifold formulation provides a particularly clear means for understanding the limitations imposed by the Bäcklund theorem, and is suggestive of how to overcome these limitations.

1. THE SPACE OF INDEPENDENT VARIABLES

The principal topic of this monograph is the structure that can be associated with the solution sets of systems of partial differential equations with n independent variables. Since it is the structure of the solution sets rather than the solutions themselves that is of interest, realization of the solution sets in terms of the geometric structures of their graphs proves to be useful. Now, the geometry of the graphs of solution sets of systems of PDE can and should be distinguished from the geometry of the base manifold of the independent variables. It is therefore sufficient to take the space of independent variables to be an open, simply connected, n-dimensional set D_n that is an element of an atlas of open sets of an n-dimensional differentiable manifold M_n. If, as is the case with most applications in the quantified sciences, $M_n = \mathbb{R}^n$, then we can take D_n to be any open, simply connected set in $\mathbb{R}^n$. On the other hand, if M_n is a general n-dimensional differentiable manifold, then our considerations are necessarily of a local nature since they will apply only to one element of an atlas of M_n at a time. The difficult global problems of piecing together the results between different elements of an atlas of M_n will not be discussed here. In fact, it is precisely because $M_n = \mathbb{R}^n$ avoids these difficult problems associated with the intrinsic geometry of a general n-dimensional differentiable base manifold, that $M_n = \mathbb{R}^n$ is often assumed in the study of PDE. We therefore take the domain set D_n of the independent variables to be an open, simply connected subset of $\mathbb{R}^n$ with a fixed coordinate cover $\{x^i \mid 1 \leq i \leq n\}$. Other coordinate covers can be introduced at the convenience of the reader by standard techniques of differential geometry.

The volume element of D_n is denoted by μ; that is

$$\mu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \tag{1.1}$$

is a basis for $\Lambda^n(D_n)$ in the coordinate cover $\{x^i\}$. The natural basis for the module $T(D_n)$ of derivations of $\Lambda^0(D_n)$ (i.e., for vector fields on D_n) is given by

$$\partial_i = \frac{\partial}{\partial x^i}, \quad 1 \leq i \leq n \tag{1.2}$$

in the coordinate cover $\{x^i\}$. Any $V \in T(D_n)$ can thus be written as $V = v^i(x^j)\,\partial_i$, where the standard summation convention is adopted. It is then immediate that

$$\mu_i = \partial_i \rfloor \mu, \quad 1 \leq i \leq n \tag{1.3}$$

is a basis for $\Lambda^{n-1}(D_n)$ (the *conjugate* basis) and exhibits the following properties:

$$d\mu_i = 0 \,, \quad dx^j \wedge \mu_i = \delta^j_i \mu \,. \tag{1.4}$$

Similarly,

$$\mu_{ji} = \partial_j \rfloor \mu_i = \partial_j \rfloor \partial_i \rfloor \mu = -\mu_{ij} \tag{1.5}$$

exhibits the properties

$$d\mu_{ji} = 0 \,, \quad dx^k \wedge \mu_{ji} = \delta^k_j \mu_i - \delta^k_i \mu_j \,, \tag{1.6}$$

and hence $\{\mu_{ji} \mid 1 \le j < i \le n\}$ is a basis form $\Lambda^{n-2}(D_n)$. Thus, if $\alpha \in \Lambda^{n-1}(D_n)$ and $\beta \in \Lambda^{n-2}(D_n)$, then

$$\alpha = a^i(x^k)\mu_i \,, \quad \beta = \tfrac{1}{2} b^{ij}(x^k)\mu_{ij} \,, \quad b^{ij} = -b^{ji} \,, \tag{1.7}$$

and

$$d\alpha = (\partial_i a^i)\mu \,, \quad d\beta = (\partial_i b^{ij})\mu_j \,. \tag{1.8}$$

2. GRAPH SPACE

The simplest geometric structure that can be associated with the solution set of a system of PDE with N dependent variables is that of the graphs of a solution set. We therefore introduce a *graph space* $G = D_n \times \mathbb{R}^N$ with local coordinates $\{x^i, q^\alpha \mid 1 \le i \le n \,, 1 \le \alpha \le N\}$ and realize the solution set in terms of mappings from $D_n \subset \mathbb{R}^n$ into G.

Let Φ denote a smooth map from an n-dimensional set $J_n \subset \mathbb{R}^n$ into G. Such a map is said to be *regular* if

$$\Phi^*\mu \ne 0 \tag{2.1}$$

throughout J_n. The collection of all regular maps of any $J_n \subset \mathbb{R}^n$ into G is denoted by

$$\text{(2.2)} \qquad RG = \{\Phi : J_n \to G \mid \Phi^*\mu \neq 0\} .$$

A regular map Φ is realized by

$$\text{(2.3)} \qquad \Phi \mid x^i = \phi^i(\tau^k) , \quad q^\alpha = \phi^\alpha(\tau^k) ,$$

where $\{\tau^k \mid 1 \leq k \leq n\}$ denotes the coordinates of points in $J_n \subset \mathbb{R}^n$ relative to a fixed coordinate cover of $\mathbb{R}^n$. Regularity of Φ requires

$$\Phi^*\mu = (\partial(\mathbf{x})/\partial(\tau))\, d\tau^1 \wedge \cdots \wedge d\tau^n \neq 0 ,$$

and hence $\dfrac{\partial(\mathbf{x})}{\partial(\tau)} \neq 0$ on J_n. The x's thus remain independent on the range of Φ, and the implicit function theorem shows that we can solve for the parameters $\{\tau^k\}$ in terms of the x's, at least locally, so as to obtain $\tau^k = m^k(x^i)$. Composition of this map with the second of (2.3) yields the relations

$$\text{(2.4)} \qquad q^\alpha = \phi^\alpha(m^k(x^i)) = \Phi^\alpha(x^i) ,$$

and hence any regular map from J_n to G can also be realized by

$$\text{(2.5)} \qquad \tilde{\Phi} : d_n \subset D_n \to G \mid x^i = x^i , \quad q^\alpha = \Phi^\alpha(x^i) ,$$

where d_n is the image of J_n in D_n under the map Φ. From this point of view, the parameters $\{\tau^k\}$ are superfluous. They will be retained in the subsequent discussions, however. This is because solution sets for systems of PDE are most often obtained in the implicit parametric form (2.3), rather than in the explicit form (2.5). For theoretical purposes, however, we may use the explicit representation (2.5) without loss of generality, and this will often simplify many of the detailed calculations with which we must contend.

3. THE FIRST ORDER CONTACT MANIFOLD

Studies of systems of PDE require that at least the first order partial derivatives of the dependent variables appear as a new system of dependent variables. This is most easily done by embedding graph space in a larger space. The analysis will follow that presented in [7, 8].

Let $K_1 = G \times \mathbb{R}^{nN}$ be an $(n+N+nN)$-dimensional space with the local coordinate cover $\{x^i, q^\alpha, r_i^\alpha \mid 1 \le i \le n, 1 \le \alpha \le N\}$. We refer to K_1 as a *first order contact manifold.* Since K_1 has the product structure $G \times \mathbb{R}^{nN}$, K_1 can be viewed as a trivial fiber space with projection

$$\pi_G : K_1 \to G \mid (x^i, q^\alpha, r_i^\alpha) \mapsto (x^i, q^\alpha) \tag{3.1}$$

and fibers $\mathbb{R}^{nN}$ over G. The new variables $\{r_i^\alpha\}$ of K_1 allow us to introduce N nontrivial 1-forms

$$C^\alpha = dq^\alpha - r_i^\alpha dx^i, \quad 1 \le \alpha \le N \tag{3.2}$$

which are referred to as the *contact 1-forms* of K_1. Since

$$C^1 \wedge C^2 \wedge \cdots \wedge C^N = dq^1 \wedge dq^2 \wedge \cdots \wedge dq^N + \cdots \neq 0, \tag{3.3}$$

the N contact 1-forms of K_1 are independent. Further, an elementary calculation based on (3.2) shows that each of the contact 1-forms has Darboux class $2n + 1$ (see [7], chapter 4).

The fiber coordinates $\{r_i^\alpha\}$ of K_1 over G are arbitrary nN-tuples of real numbers. They are identified with place holders for first partial derivatives through the following construction. Let $\Phi : J_n \to G \mid x^i = \phi^i(\tau^k),\ q^\alpha = \phi^\alpha(\tau^k)$ be a regular map. We lift Φ to a map of $J_n \to K_1$ by the requirements

$$\Phi^* C^\alpha = 0, \quad 1 \le \alpha \le N; \tag{3.4}$$

that is, *the section of* K_1 *that is generated by any regular section of* G *annihilates the contact 1-forms of* K_1. A substitution of (3.2) into (3.4) yields the explicit