Graduate Series in Analysis

Chaos in Partial Differential Equations

Y. Charles Li
University of Missouri, Columbia

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Chaos in Partial Differential Equations

Preface

The area: Chaos in Partial Differential Equations, is at its fast developing stage. Notable results have been obtained in recent years. The present book aims at an overall survey on the existing results. On the other hand, we shall try to make the presentations introductory, so that beginners can benefit more from the book.

It is well-known that the theory of chaos in finite-dimensional dynamical systems has been well-developed. That includes both discrete maps and systems of ordinary differential equations. Such a theory has produced important mathematical theorems and led to important applications in physics, chemistry, biology, and engineering etc.. For a long period of time, there was no theory on chaos in partial differential equations. On the other hand, the demand for such a theory is much stronger than for finite-dimensional systems. Mathematically, studies on infinite-dimensional systems pose much more challenging problems. For example, as phase spaces, Banach spaces possess much more structures than Euclidean spaces. In terms of applications, most of important natural phenomena are described by partial differential equations – nonlinear wave equations, Maxwell equations, Yang-Mills equations, and Navier-Stokes equations, to name a few. Recently, the author and collaborators have established a systematic theory on chaos in nonlinear wave equations.

Nonlinear wave equations are the most important class of equations in natural sciences. They describe a wide spectrum of phenomena – motion of plasma, nonlinear optics (laser), water waves, vortex motion, to name a few. Among these nonlinear wave equations, there is a class of equations called soliton equations. This class of equations describes a variety of phenomena. In particular, the same soliton equation describes several different phenomena. Mathematical theories on soliton equations have been well developed. Their Cauchy problems are completely solved through inverse scattering transforms. Soliton equations are integrable Hamiltonian partial differential equations which are the natural counterparts of finite-dimensional integrable Hamiltonian systems. We have established a standard program for proving the existence of chaos in perturbed soliton equations, with the machineries: 1. Darboux transformations for soliton equations, 2. isospectral theory for soliton equations under periodic boundary condition, 3. persistence of invariant manifolds and Fenichel fibers, 4. Melnikov analysis, 5. Smale horseshoes and symbolic dynamics, 6. shadowing lemma and symbolic dynamics.

The most important implication of the theory on chaos in partial differential equations in theoretical physics will be on the study of turbulence. For that goal, we chose the 2D Navier-Stokes equations under periodic boundary conditions to begin a dynamical system study on 2D turbulence. Since they possess Lax pair and Darboux transformation, the 2D Euler equations are the starting point for an analytical study. The high Reynolds number 2D Navier-Stokes equations are viewed as a singular perturbation of the 2D Euler equations through the perturbation parameter $\epsilon = 1/Re$ which is the inverse of the Reynolds number.

Our focus will be on nonlinear wave equations. New results on shadowing lemma and novel results related to Euler equations of inviscid fluids will also be presented. The chapters on figure-eight structures and Melnikov vectors are written in great details. The readers can learn these machineries without resorting to other references. In other chapters, details of proofs are often omitted. Chapters 3 to 7 illustrate how to prove the existence of chaos in perturbed soliton equations. Chapter 9 contains the most recent results on Lax pair structures of Euler equations of inviscid fluids. In chapter 12, we give brief comments on other related topics.

The monograph will be of interest to researchers in mathematics, physics, engineering, chemistry, biology, and science in general. Researchers who are interested in chaos in high dimensions, will find the book of particularly valuable. The book is also accessible to graduate students, and can be taken as a textbook for advanced graduate courses.

I started writing this book in 1997 when I was at MIT. This project continued at Institute for Advanced Study during the year 1998-1999, and at University of Missouri - Columbia since 1999. In the Fall of 2001, I started to rewrite from the old manuscript. Most of the work was done in the summer of 2002. The work was partially supported by an AMS centennial fellowship in 1998, and a Guggenheim fellowship in 1999.

Finally, I would like to thank my wife Sherry and my son Brandon for their strong support and appreciation.

Contents

CHAPTER 1

General Setup and Concepts

We are mainly concerned with the Cauchy problems of partial differential equations, and view them as defining flows in certain Banach spaces. Unlike the Euclidean space $\mathbb{R}^n$, such Banach spaces admit a variety of norms which make the structures in infinite dimensional dynamical systems more abundant. The main difficulty in studying infinite dimensional dynamical systems often comes from the fact that the evolution operators for the partial differential equations are usually at best C^0 in time, in contrast to finite dimensional dynamical systems where the evolution operators are C^1 smooth in time. The well-known concepts for finite dimensional dynamical systems can be generalized to infinite dimensional dynamical systems, and this is the main task of this chapter.

1.1. Cauchy Problems of Partial Differential Equations

The types of evolution equations studied in this book can be casted into the general form,

$$\partial_t Q = G(Q, \partial_x Q, \ldots, \partial_x^\ell Q) \,, \tag{1.1}$$

where $t \in \mathbb{R}^1$ (time), $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $Q = (Q_1, \ldots, Q_m)$ and $G = (G_1, \ldots, G_m)$ are either real or complex valued functions, and ℓ, m and n are integers. The equation (1.1) is studied under certain boundary conditions, for example,

- periodic boundary conditions, e.g. Q is periodic in each component of x with period 2π,
- decay boundary conditions, e.g. $Q \to 0$ as $x \to \infty$.

Thus we have Cauchy problems for the equation (1.1), and we would like to pose the Cauchy problems in some Banach spaces $\mathcal{H}$, for example,

- $\mathcal{H}$ can be a Sobolev space H^k,
- $\mathcal{H}$ can be a Solobev space $H^k_{e,p}$ of even periodic functions.

We require that the problem is well-posed in $\mathcal{H}$, for example,

- for any $Q_0 \in \mathcal{H}$, there exists a unique solution $Q = Q(t, Q_0) \in C^0[(-\infty, \infty); \mathcal{H}]$ or $C^0[[0; \infty), \mathcal{H}]$ to the equation (1.1) such that $Q(0, Q_0) = Q_0$,
- for any fixed $t_0 \in (-\infty, \infty)$ or $[0, \infty)$, $Q(t_0, Q_0)$ is a C^r function of Q_0, for $Q_0 \in \mathcal{H}$ and some integer $r \geq 0$.

Example: Consider the integrable cubic nonlinear Schrödinger (NLS) equation,

$$iq_t = q_{xx} + 2\left[|\, q \,|^2 - \omega^2 \right] q \,, \tag{1.2}$$

where $i = \sqrt{-1}$, $t \in \mathbb{R}^1$, $x \in \mathbb{R}^1$, q is a complex-valued function of (t, x), and ω is a real constant. We pose the periodic boundary condition,

$$q(t, x + 1) = q(t, x)\,.$$

The Cauchy problem for equation (1.2) is posed in the Sobolev space H^1 of periodic functions,

$$\mathcal{H} \;\equiv\; \Big\{ Q = (q, \bar{q}) \;\Big|\; q(x+1) = q(x),\ q \in H^1_{[0,1]} : \text{the Sobolev space } H^1 \text{ over the period interval } [0,1] \Big\},$$

and is well-posed **[38]** **[32]** **[33]**.

Fact 1: For any $Q_0 \in \mathcal{H}$, there exists a unique solution $Q = Q(t, Q_0) \in C^0[(-\infty, \infty), \mathcal{H}]$ to the equation (1.2) such that $Q(0, Q_0) = Q_0$.

Fact 2: For any fixed $t_0 \in (-\infty, \infty)$, $Q(t_0, Q_0)$ is a C^2 function of Q_0, for $Q_0 \in \mathcal{H}$.

1.2. Phase Spaces and Flows

For finite dimensional dynamical systems, the phase spaces are often $\mathbb{R}^n$ or $\mathbb{C}^n$. For infinite dimensional dynamical systems, we take the Banach space $\mathcal{H}$ discussed in the previous section as the counterpart.

DEFINITION 1.1. We call the Banach space $\mathcal{H}$ in which the Cauchy problem for (1.1) is well-posed, a *phase space*. Define an operator F^t labeled by t as

$$Q(t, Q_0) = F^t(Q_0)\,;$$

then $F^t : \mathcal{H} \to \mathcal{H}$ is called the *evolution operator* (or flow) for the system (1.1).

A point $p \in \mathcal{H}$ is called a *fixed point* if $F^t(p) = p$ for any t. Notice that here the fixed point p is in fact a function of x, which is the so-called stationary solution of (1.1). Let $q \in \mathcal{H}$ be a point; then $\ell_q \equiv \{F^t(q), \text{for all } t\}$ is called the orbit with initial point q. An orbit ℓ_q is called a *periodic orbit* if there exists a $T \in (-\infty, \infty)$ such that $F^T(q) = q$. An orbit ℓ_q is called a *homoclinic orbit* if there exists a point $q_* \in \mathcal{H}$ such that $F^t(q) \to q_*$, as $| t | \to \infty$, and q_* is called the asymptotic point of the homoclinic orbit. An orbit ℓ_q is called a *heteroclinic orbit* if there exist two different points $q_\pm \in \mathcal{H}$ such that $F^t(q) \to q_\pm$, as $t \to \pm\infty$, and $q_\pm$ are called the asymptotic points of the heteroclinic orbit. An orbit ℓ_q is said to be homoclinic to a submanifold W of $\mathcal{H}$ if $\inf_{Q \in W} \| F^t(q) - Q \| \to 0$, as $| t | \to \infty$.

Example 1: Consider the same Cauchy problem for the system (1.2). The fixed points of (1.2) satisfy the second order ordinary differential equation

$$q_{xx} + 2\left[| q |^2 - \omega^2\right] q = 0\,. \tag{1.3}$$

In particular, there exists a circle of fixed points $q = \omega e^{i\gamma}$, where $\gamma \in [0, 2\pi]$. For simple periodic solutions, we have

$$q = a e^{i\theta(t)}, \quad \theta(t) = -\left[2(a^2 - \omega^2)t - \gamma\right]\,; \tag{1.4}$$

where $a > 0$, and $\gamma \in [0, 2\pi]$. For orbits homoclinic to the circles (1.4), we have

$$q \;=\; \frac{1}{\Lambda}\left[\cos 2p - \sin p \operatorname{sech} \tau \cos 2\pi x - i \sin 2p \tanh \tau\right] a e^{i\theta(t)}\,, \tag{1.5}$$

$$\Lambda = 1 + \sin p \operatorname{sech} \tau \cos 2\pi x\,,$$

where $\tau = 4\pi\sqrt{a^2-\pi^2}\ t+\rho$, $p = \arctan\left[\frac{\sqrt{a^2-\pi^2}}{\pi}\right]$, $\rho \in (-\infty, \infty)$ is the Bäcklund parameter. Setting $a = \omega$ in (1.5), we have heteroclinic orbits asymptotic to points on the circle of fixed points. The expression (1.5) is generated from (1.4) through a Bäcklund-Darboux transformation [**137**].

Example 2: Consider the sine-Gordon equation,

$$u_{tt} - u_{xx} + \sin u = 0 ,$$

under the decay boundary condition that u belongs to the Schwartz class in x. The well-known "breather" solution,

$$u(t,x) = 4\arctan\left[\frac{\tan\nu\cos[(\cos\nu)t]}{\cosh[(\sin\nu)x]}\right] , \tag{1.6}$$

where ν is a parameter, is a periodic orbit. The expression (1.6) is generated from trivial solutions through a Bäcklund-Darboux transformation [**59**].

1.3. Invariant Submanifolds

Invariant submanifolds are the main objects in studying phase spaces. In phase spaces for partial differential equations, invariant submanifolds are often submanifolds with boundaries. Therefore, the following concepts on invariance are important.

DEFINITION 1.2 (Overflowing and Inflowing Invariance). A submanifold W with boundary ∂W is

- overflowing invariant if for any $t > 0$, $\bar{W} \subset F^t \circ W$, where $\bar{W} = W \cup \partial W$,
- inflowing invariant if any $t > 0$, $F^t \circ \bar{W} \subset W$,
- invariant if for any $t > 0$, $F^t \circ \bar{W} = \bar{W}$.

DEFINITION 1.3 (Local Invariance). A submanifold W with boundary ∂W is locally invariant if for any point $q \in W$, if $\bigcup_{t\in[0,\infty)} F^t(q) \not\subset W$, then there exists $T \in (0,\infty)$ such that $\bigcup_{t\in[0,T)} F^t(q) \subset W$, and $F^T(q) \in \partial W$; and if $\bigcup_{t\in(-\infty,0]} F^t(q) \not\subset W$, then there exists $T \in (-\infty, 0)$ such that $\bigcup_{t\in(T,0]} F^t(q) \subset W$, and $F^T(q) \in \partial W$.

Intuitively speaking, a submanifold with boundary is locally invariant if any orbit starting from a point inside the submanifold can only leave the submanifold through its boundary in both forward and backward time.

Example: Consider the linear equation,

$$iq_t = (1+i)q_{xx} + iq , \tag{1.7}$$

where $i = \sqrt{-1}$, $t \in \mathbb{R}^1$, $x \in \mathbb{R}^1$, and q is a complex-valued function of (t,x), under periodic boundary condition,

$$q(x+1) = q(x) .$$

Let $q = e^{\Omega_j t + ik_j x}$; then

$$\Omega_j = (1-k_j^2) + i\,k_j^2 ,$$

where $k_j = 2j\pi$, $(j \in \mathbb{Z})$. $\Omega_0 = 1$, and when $|\, j \,| > 0$, $R_e\{\Omega_j\} < 0$. We take the H^1 space of periodic functions of period 1 to be the phase space. Then the submanifold

$$W_0 = \left\{ q \in H^1 \;\middle|\; q = c_0,\ c_0 \text{ is complex and } \|\, q \,\| < 1 \right\}$$

is an outflowing invariant submanifold, the submanifold

$$W_1 = \left\{ q \in H^1 \;\middle|\; q = c_1 e^{ik_1 x}, \quad c_1 \text{ is complex, and } \| q \| < 1 \right\}$$

is an inflowing invariant submanifold, and the submanifold

$$W = \left\{ q \in H^1 \;\middle|\; q = c_0 + c_1 e^{ik_1 x}, \quad c_0 \text{ and } c_1 \text{ are complex, and } \| q \| < 1 \right\}$$

is a locally invariant submanifold. The *unstable subspace* is given by

$$W^{(u)} = \left\{ q \in H^1 \;\middle|\; q = c_0, \quad c_0 \text{ is complex} \right\},$$

and the *stable subspace* is given by

$$W^{(s)} = \left\{ q \in H^1 \;\middle|\; q = \sum_{j \in Z / \{0\}} c_j \, e^{ik_j x}, \quad c_j\text{'s are complex} \right\}.$$

Actually, a good way to view the partial differential equation (1.7) as defining an infinite dimensional dynamical system is through Fourier transform, let

$$q(t, x) = \sum_{j \in Z} c_j(t) e^{ik_j x};$$

then $c_j(t)$ satisfy

$$\dot{c}_j = \left[(1 - k_j^2) + i k_j^2 \right] c_j, \quad j \in \mathbb{Z};$$

which is a system of infinitely many ordinary differential equations.

1.4. Poincaré Sections and Poincaré Maps

In the infinite dimensional phase space $\mathcal{H}$, Poincaré sections can be defined in a similar fashion as in a finite dimensional phase space. Let l_q be a periodic or homoclinic orbit in $\mathcal{H}$ under a flow F^t, and q_* be a point on l_q, then the Poincaré section Σ can be defined to be any codimension 1 subspace which has a transversal intersection with l_q at q_*. Then the flow F^t will induce a Poincaré map P in the neighborhood of q_* in Σ_0. Phase blocks, e.g. Smale horseshoes, can be defined using the norm.

CHAPTER 2

Soliton Equations as Integrable Hamiltonian PDEs

2.1. A Brief Summary

Soliton equations are integrable Hamiltonian partial differential equations. For example, the Korteweg-de Vries (KdV) equation

$$u_t = -6uu_x - u_{xxx} \ ,$$

where u is a real-valued function of two variables t and x, can be rewritten in the Hamiltonian form

$$u_t = \partial_x \frac{\delta H}{\delta u} \ ,$$

where

$$H = \int \left[\frac{1}{2} u_x^2 - u^3 \right] dx \ ,$$

under either periodic or decay boundary conditions. It is integrable in the classical Liouville sense, i.e., there exist enough functionally independent constants of motion. These constants of motion can be generated through isospectral theory or Bäcklund transformations [**8**]. The level sets of these constants of motion are elliptic tori [**178**] [**154**] [**153**] [**68**].

There exist soliton equations which possess level sets which are normally hyperbolic, for example, the focusing cubic nonlinear Schrödinger equation [**137**],

$$iq_t = q_{xx} + 2|q|^2 q \ ,$$

where $i = \sqrt{-1}$ and q is a complex-valued function of two variables t and x; the sine-Gordon equation [**157**],

$$u_{tt} = u_{xx} + \sin u \ ,$$

where u is a real-valued function of two variables t and x, etc.

Hyperbolic foliations are very important since they are the sources of chaos when the integrable systems are under perturbations. We will investigate the hyperbolic foliations of three typical types of soliton equations: (i). (1+1)-dimensional soliton equations represented by the focusing cubic nonlinear Schrödinger equation, (ii). soliton lattices represented by the focusing cubic nonlinear Schrödinger lattice, (iii). (1+2)-dimensional soliton equations represented by the Davey-Stewartson II equation.

REMARK 2.1. For those soliton equations which have only elliptic level sets, the corresponding representatives can be chosen to be the KdV equation for (1+1)-dimensional soliton equations, the Toda lattice for soliton lattices, and the KP equation for (1+2)-dimensional soliton equations.

Soliton equations are canonical equations which model a variety of physical phenomena, for example, nonlinear wave motions, nonlinear optics, plasmas, vortex dynamics, etc. [**5**] [**1**]. Other typical examples of such integrable Hamiltonian partial differential equations are, e.g., the defocusing cubic nonlinear Schrödinger equation,

$$iq_t = q_{xx} - 2|q|^2 q \ ,$$

where $i = \sqrt{-1}$ and q is a complex-valued function of two variables t and x; the modified KdV equation,

$$u_t = \pm 6u^2 u_x - u_{xxx} \ ,$$

where u is a real-valued function of two variables t and x; the sinh-Gordon equation,

$$u_{tt} = u_{xx} + \sinh u \ ,$$

where u is a real-valued function of two variables t and x; the three-wave interaction equations,

$$\frac{\partial u_i}{\partial t} + a_i \frac{\partial u_i}{\partial x} = b_i \bar{u}_j \bar{u}_k \ ,$$

where $i, j, k = 1, 2, 3$ are cyclically permuted, a_i and b_i are real constants, u_i are complex-valued functions of t and x; the Boussinesq equation,

$$u_{tt} - u_{xx} + (u^2)_{xx} \pm u_{xxxx} = 0 \ ,$$

where u is a real-valued function of two variables t and x; the Toda lattice,

$$\partial^2 u_n / \partial t^2 = \exp\{-(u_n - u_{n-1})\} - \exp\{-(u_{n+1} - u_n)\} \ ,$$

where u_n's are real variables; the focusing cubic nonlinear Schrödinger lattice,

$$i \frac{\partial q_n}{\partial t} = (q_{n+1} - 2q_n + q_{n-1}) + |q_n|^2 (q_{n+1} + q_{n-1}) \ ,$$

where q_n's are complex variables; the Kadomtsev-Petviashvili (KP) equation,

$$(u_t + 6uu_x + u_{xxx})_x = \pm 3u_{yy} \ ,$$

where u is a real-valued function of three variables t, x and y; the Davey-Stewartson II equation,

$$\begin{cases} i\partial_t q = [\partial_x^2 - \partial_y^2] q + [2|q|^2 + u_y] q \ , \\ [\partial_x^2 + \partial_y^2] u = -4\partial_y |q|^2 \ , \end{cases}$$

where $i = \sqrt{-1}$, q is a complex-valued function of three variables t, x and y; and u is a real-valued function of three variables t, x and y. For more complete list of soliton equations, see e.g. [**5**] [**1**].

The cubic nonlinear Schrödinger equation is one of our main focuses in this book, which can be written in the Hamiltonian form,

$$iq_t = \frac{\delta H}{\delta \bar{q}} \ ,$$

where

$$H = \int [-|q_x|^2 \pm |q|^4] dx \ ,$$

under periodic boundary conditions. Its phase space is defined as

$$\mathcal{H}^k \equiv \left\{ \vec{q} = \begin{pmatrix} q \\ r \end{pmatrix} \middle| \ r = -\bar{q}, \ q(x+1) = q(x), \right.$$

$$\left. q \in H^k_{[0,1]} : \text{ the Sobolev space } H^k \text{ over the period interval } [0,1] \right\} .$$

REMARK 2.2. It is interesting to notice that the cubic nonlinear Schrödinger equation can also be written in Hamiltonian form in spatial variable, i.e.,

$$q_{xx} = iq_t \pm 2|q|^2 q \ ,$$

can be written in Hamiltonian form. Let $p = q_x$; then

$$\frac{\partial}{\partial x}\begin{pmatrix} q \\ \bar{q} \\ \bar{p} \\ p \end{pmatrix} = J \begin{pmatrix} \frac{\delta H}{\delta q} \\ \frac{\delta H}{\delta \bar{q}} \\ \frac{\delta H}{\delta \bar{p}} \\ \frac{\delta H}{\delta p} \end{pmatrix} ,$$

where

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} ,$$

$$H = \int [|p|^2 \mp |q|^4 - \frac{i}{2}(q_t \bar{q} - \bar{q}_t q)]dt \ ,$$

under decay or periodic boundary conditions. We do not know whether or not other soliton equations have this property.

2.2. A Physical Application of the Nonlinear Schrödinger Equation

The cubic nonlinear Schrödinger (NLS) equation has many different applications, i.e. it describes many different physical phenomena, and that is why it is called a canonical equation. Here, as an example, we show how the NLS equation describes the motion of a vortex filament – the beautiful Hasimoto derivation [**82**]. Vortex filaments in an inviscid fluid are known to preserve their identities. The motion of a very thin isolated vortex filament $\vec{X} = \vec{X}(s,t)$ of radius ϵ in an incompressible inviscid unbounded fluid by its own induction is described asymptotically by

$$\partial \vec{X}/\partial t = G\kappa \vec{b} \ , \tag{2.1}$$

where s is the length measured along the filament, t is the time, κ is the curvature, $\vec{b}$ is the unit vector in the direction of the binormal and G is the coefficient of local induction,

$$G = \frac{\Gamma}{4\pi}[\ln(1/\epsilon) + O(1)] \ ,$$

which is proportional to the circulation Γ of the filament and may be regarded as a constant if we neglect the second order term. Then a suitable choice of the units of time and length reduces (2.1) to the nondimensional form,

$$\partial \vec{X}/\partial t = \kappa \vec{b} \ . \tag{2.2}$$

Equation (2.2) should be supplemented by the equations of differential geometry (the Frenet-Seret formulae)

$$\partial \vec{X}/\partial s = \vec{t} \ , \ \ \partial \vec{t}/\partial s = \kappa \vec{n} \ , \ \ \partial \vec{n}/\partial s = \tau \vec{b} - \kappa \vec{t} \ , \ \ \partial \vec{b}/\partial s = -\tau \vec{n} \ , \tag{2.3}$$

where τ is the torsion and $\vec{t}$, $\vec{n}$ and $\vec{b}$ are the tangent, the principal normal and the binormal unit vectors. The last two equations imply that

$$\partial(\vec{n} + i\vec{b})/\partial s = -i\tau(\vec{n} + i\vec{b}) - \kappa\vec{t}\,, \tag{2.4}$$

which suggests the introduction of new variables

$$\vec{N} = (\vec{n} + i\vec{b})\exp\left\{i\int_0^s \tau ds\right\}, \tag{2.5}$$

and

$$q = \kappa\exp\left\{i\int_0^s \tau ds\right\}. \tag{2.6}$$

Then from (2.3) and (2.4), we have

$$\partial\vec{N}/\partial s = -q\vec{t}\,,\quad \partial\vec{t}/\partial s = \mathrm{Re}\{q\overline{\vec{N}}\} = \frac{1}{2}(\bar{q}\vec{N} + q\overline{\vec{N}})\,. \tag{2.7}$$

We are going to use the relation $\frac{\partial^2\vec{N}}{\partial s\partial t} = \frac{\partial^2\vec{N}}{\partial t\partial s}$ to derive an equation for q. For this we need to know $\partial\vec{t}/\partial t$ and $\partial\vec{N}/\partial t$ besides equations (2.7). From (2.2) and (2.3), we have

$$\begin{aligned}\partial\vec{t}/\partial t &= \frac{\partial^2\vec{X}}{\partial s\partial t} = \partial(\kappa\vec{b})/\partial s = (\partial\kappa/\partial s)\vec{b} - \kappa\tau\vec{n} \\ &= \kappa\ \mathrm{Re}\{(\frac{1}{\kappa}\partial\kappa/\partial s + i\tau)(\vec{b} + i\vec{n})\}\,,\end{aligned}$$

i.e.

$$\partial\vec{t}/\partial t = \mathrm{Re}\{i(\partial q/\partial s)\overline{\vec{N}}\} = \frac{1}{2}i[(\partial q/\partial s)\overline{\vec{N}} - (\partial q/\partial s)^{-}\vec{N}]\,. \tag{2.8}$$

We can write the equation for $\partial\vec{N}/\partial t$ in the following form:

$$\partial\vec{N}/\partial t = \alpha\vec{N} + \beta\overline{\vec{N}} + \gamma\vec{t}\,, \tag{2.9}$$

where α, β and γ are complex coefficients to be determined.

$$\begin{aligned}\alpha + \bar{\alpha} &= \frac{1}{2}[\partial\vec{N}/\partial t\cdot\overline{\vec{N}} + \partial\overline{\vec{N}}/\partial t\cdot\vec{N}] \\ &= \frac{1}{2}\partial(\vec{N}\cdot\overline{\vec{N}})/\partial t = 0\,,\end{aligned}$$

i.e. $\alpha = iR$ where R is an unknown real function.

$$\begin{aligned}\beta &= \frac{1}{2}\partial\vec{N}/\partial t\cdot\vec{N} = \frac{1}{4}\partial(\vec{N}\cdot\vec{N})/\partial t = 0\,, \\ \gamma &= -\vec{N}\cdot\partial\vec{t}/\partial t = -i\partial q/\partial s\,.\end{aligned}$$

Thus

$$\partial\vec{N}/\partial t = i[R\vec{N} - (\partial q/\partial s)\vec{t}\,]\,. \tag{2.10}$$