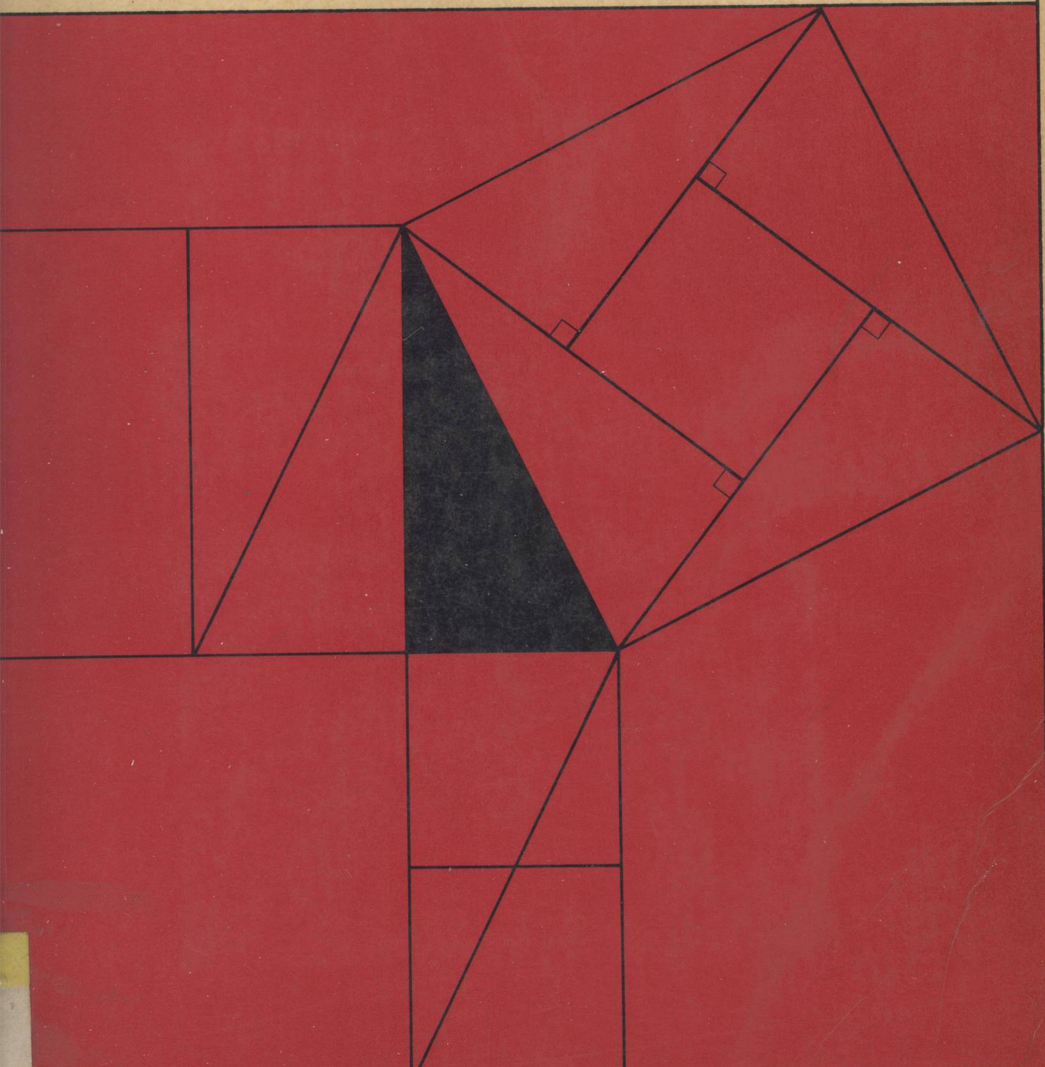
How to Solve Problems

ELEMENTS OF A THEORY OF PROBLEMS AND PROBLEM SOLVING

WICKELGREN



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ELEMENTS OF A THEORY OF PROBLEMS AND PROBLEM SOLVING

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How to Solve Problems

For as long as I can remember, I have been more interested in reflecting on what I was doing or thinking and in thinking about ways to improve my methods than I have been in the particular things I was doing or thinking about. This emphasis on self-analysis and improvement reflects the influence of my mother and father, Alma and Herman Wickelgren, to whom this book is dedicated and whose values and practical principles have contributed so much to my life.

Preface

In the mathematics and science courses I took in college, I was enormously irritated by the hundreds of hours that I wasted staring at problems without any good idea about what approach to try next in attempting to solve them. I thought at the time that there was no educational value in those "blank" minutes, and I see no value in them today. The general problem-solving methods described in this book virtually guarantee that you will never again have a blank mind in such circumstances. They should also help you solve many more problems and solve them faster. But whether or not you solve any particular problem, you will always have lots of ideas about ways to attack the problem. Also, the use of general problem-solving methods often indicates the properties of the principles you need to know from the subject matter that the problem is attempting to teach and test. Thus, whether you succeed or fail in solving any particular problem, the effort will be interesting and educational.

The theoretical and practical analyses of problems and problem solving presented here were heavily influenced by advances made over the last 20 years in the fields of artificial intelligence and computer simulation of thought. My greatest intellectual debts are to Allen Newell, Herbert Simon, and George Polya. Newell and Simon's analyses of problems and problem solving constituted my starting point for working in this area, and many of the best ideas in the book are ideas they have already presented in one form or another. Many other good ideas were taken more or less directly from Polya, whose books on mathematical problem solving are a rich source of methods and a stimulus for thought.

My efforts to understand and organize problem-solving methods began in 1959 when, as an undergraduate at Harvard, I first became aware of the pioneering work of Allen Newell, Cliff Shaw, and Herb Simon on the computer simulation of thinking. During graduate school at the University of California, Berkeley, I regarded problem solving as my major research area. I do not think that my experimental studies of human problem solving ever amounted to much. However, I thought at the time (and think today) that my theoretical (mathematical) understanding of problems and problem solving was immeasurably increased and that this greatly enhanced my ability to solve all kinds of mathematical problems. Shortly after coming to MIT as a new faculty member in the Psychology Department, I decided that one contribution I could make to the undergraduates there was to teach them this newly acquired skill of mathematical problem solving. The students enjoyed the course and, more important, reported back to me in later years that they thought that their problem-solving ability in mathematics, science, and engineering courses had been greatly increased by learning these general problem-solving methods. Enrollment in the course went from 20 to 80 in three years, when I stopped giving it because my primary research interest had shifted to human memory. Some years later, after moving to the University of Oregon, I decided that I now had the time to write a book containing all the ideas that I had acquired from others and generated myself concerning problems and problem solving.

The purpose of the book is to improve your ability to solve all kinds of mathematical problems whether in mathematics, science, engineering, business, or purely recreational mathematical problems (puzzles, games, and so on). This book is primarily intended for college students who are currently taking elementary mathematics, science, or engineering courses. However, I hope that students with less mathematical background can read the book and master the methods without an undue degree of additional effort and also that more advanced readers will profit from it without being bored. I believe that almost everyone who solves mathematical problems can profit substantially from learning the general problem-solving methods described here, and I have tried to write in a way that will communicate effectively to all such people. The approach is to define each general problem-solving method and illustrate its application to simple recreational mathematics problems that require no more mathematical background than that possessed by someone with a year of high school algebra and a year of plane geometry. An elementary knowledge of "new mathematics" (sets, relations, functions, probability, and so on) would be helpful, and some of this is briefly taught in Chapter 10.

The solutions to example problems are presented gradually, usually in the form of hints to give the reader more and more chances to go back and solve the problem. This technique is founded on the belief that you will remember best what you discover for yourself. The book aims to guide you to discovering how to apply general problem-solving methods to a rich variety of problems. I believe that if you read this book and try to apply the methods to around 50 or 100 of your own problems, you will improve substantially in problem-solving ability, with consequent benefits in job performance, school grades, and "intelligence" test scores (including SAT college entrance exams, and The Graduate Record Exam).

Finally, I would like to make a negative acknowledgment. This book was written in spite of my four-year-old son, Abraham, and my six-year-old daughter, Ingrid, who are such delightful people that I cannot resist spending vast amounts of time with them.

October 1973 *Wayne A. Wickelgren*

How to Solve Problems

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Introduction

The purpose of this book is to help you improve your ability to solve mathematical, scientific, and engineering problems. With this in mind, I will describe certain elementary concepts and principles of the theory of problems and problem solving, something we have learned a great deal about since the 1950s, when the advent of computers made possible research on artificial intelligence and computer simulation of human problem solving. I have tried to organize the discussion of these ideas in a simple, logical way that will help you understand, remember, and apply them.

You should be warned, however, that the theory of problem solving is far from being precise enough at present to provide simple cookbook instructions for solving most problems. Partly for this reason and partly for reasons of intrinsic merit, *teaching by example* is the primary approach used in this book. First, a problem-solving method will be discussed theoretically, then it will be applied to a variety of problems, so that you may see how to use the method in actual practice.

To master these methods, it is essential to work through the examples of their application to a variety of problems. Thus, much of the book is devoted to analyzing problems that exemplify the use of different methods. You should pay careful attention to these problems and should not be discouraged if you do not perfectly understand the theoretical discussions. The theory of problem solving will undoubtably help those students with sufficient mathematical background to understand it, but students who lack such a background can compensate by spending greater time on the examples.

SCOPE OF THE BOOK

This book is primarily a practical guide to how to solve a certain class of problems, specifically, what I call *formal problems* or just "problems" (with the adjective *formal* being understood in later contexts). Formal problems include all mathematical problems of either the "to find" or the "to prove" character but do not include problems of defining "mathematically interesting" axiom systems. A student taking mathematics courses will hardly be aware of the practical significance of this exclusion, since defining interesting axiom systems is a problem not typically encountered except in certain areas of basic research in mathematics. Similarly, the problem of constructing a new mathematical theory in any field of science is not a formal problem, as I use the term, and I will not discuss it in this book. However, any other mathematical problem that comes up in any field of science, engineering, or mathematics is a formal problem in the sense of this book.

Problems such as what you should eat for breakfast, whether you should marry x or y, whether you should drop out of school, or how can you get yourself to spend more time studying are not formal problems. These problems are virtually impossible at the present time to turn into formal problems because we have no good ways of restricting our thinking to a specified set of given information and operations (courses of action we might take), nor do we often even know how to specify precisely what our goals are in solving these problems. Understanding formal problems can undoubtedly make some contributions to your thinking in regard to these poorly specified personal problems, but the scope of the present book does not include such problems. Even if it did, it would be extremely difficult to specify any precise methods for solving them.

However, formal problems include a large class of practical problems that people might encounter in the real world, although they usually encounter them as games or puzzles presented by friends or appearing in magazines. A practical problem such as how to build a bridge across a river is a formal problem if, in solving the problem, one is limited to some specified set of materials (givens), operations, and, of course, the goal of getting the bridge built.

In actuality, you might limit yourself in this way for a while and, if no solution emerged, decide to consider the use of some additional materials, if possible. Expanding the set of given materials (by means other than the use of acceptable operations) is not a part of formal problem solving, but often the situation presents certain givens in sufficiently disguised or implicit form that recognition of all the givens is an important part of skill in formal problem solving. That skill will be discussed later.

Practical problems or puzzles of the type we will consider differ from problems in mathematics, science, or engineering in that to pose them requires less background information and training. Thus, puzzle problems are especially suitable as examples of problem-solving methods in this book, because they communicate the workings of the methods most easily to the widest range of readers. For this reason, puzzle problems will constitute a large proportion of the examples used in this book—at least prior to the last chapter.

In principle, it might seem that most important problem-solving methods would be unique to each specialized area of mathematics, science, or engineering, but this is probably not the case. There are many extremely general problem-solving methods, though, to be sure, there are also special methods that can be of use in only a limited range of fields.

It may be quite difficult to learn the special methods and knowledge required in a particular field, but at least such methods and knowledge are the specific object of instruction in courses. By contrast, general problem-solving methods are rarely, if ever, taught, though they are quite helpful in solving problems in every field of mathematics, science, and engineering.

GENERAL VERSUS SPECIAL METHODS

The relation between specific knowledge and methods, on the one hand, and general problem-solving methods, on the other hand, appears to be as follows. When you understand the relevant material and specific methods quite well and already have considerable experience in applying this knowledge to similar problems, then in solving a new problem you use the same specific methods you used before. Considering the methods used in similar problems is a general problem-solving technique. However, in cases where it is obvious that a particular problem is a member of a class of problems you have solved before, you do not need to make explicit, conscious use of the method: simply go ahead and solve the problem, using methods that you have learned to apply to this class. Once you have this level of understanding of the relevant material, general problem-solving methods are of little value in solving the vast majority of homework and examination problems for mathematics, science, and engineering courses.

When problems are more complicated, in the sense of involving more component steps, and are not highly similar to previously solved problems, the use of general problem-solving methods can be a substantial aid in solution. However, such complex problems will be encountered only rarely by the beginning mathematics, science, and engineering students taking courses in high school and college. More important to the immediate needs of such students is the role of general problem-solving methods in simple homework and examination problems where one does *not* completely understand the relevant material and does *not* have considerable experience in solving the relevant class of problems. In such cases, general problem-solving methods serve to guide the student to recognize what relevant background information needs to be understood. For example, when one understands the general problem-solving method of setting subgoals, one can often set particular subgoals that directly indicate what types of specific information are being tested (and thereby taught) by a particular problem. One then knows what sections of the textbook to reread in order to understand the relevant material.

If, however, the book is not available, as in many examination situations, general problem-solving methods provide one with powerful general methods for retrieving from memory the relevant background information. For example, the use of general problem-solving methods can indicate for which quantities one needs a formula and can provide a basis for choosing among different alternative formulas. Frequently, a student may know all the definitions, formulas, and so on, but not have strong associations to this knowledge from the cues present in each type of problem to which this knowledge is relevant.

With experience in solving a variety of problems to which the knowledge is relevant, one will develop strong direct associations between the cues in such problems and this relevant knowledge. However, in the early stages of learning the material, a student will lack such direct associations and will need to use general problem-solving methods to indicate where in one's memory to retrieve relevant information or where in the book to look it up. Assuming this idea is true (and this book aims to convince you it is), mastering general problem-solving methods is important to you both so you can use problems as a learning device and so you can achieve the maximum range of applicability of the knowledge you have stored in mind—on an examination, on a job, or whatever.

The goal of this book is to teach as many of these general problem-solving methods as I know about, so that if you spend the time to master these methods you can more effectively learn the subject matter of your courses. Also, since the ability to use the information given in most mathematics, science, and engineering courses is often primarily the ability to solve problems in these fields, the book aims to increase this ability to use knowledge.

RELATION TO ARTIFICIAL INTELLIGENCE

It should be emphasized that this text is primarily a practical how-to-do-it book in a field where the level of precise (mathematical) formulation is far below what I am sure it will be in the future, perhaps even the near future. Artificial intelligence and computer simulation of human problem solving are currently very active fields of research, and results from some of this work have heavily influenced this book. However, theoretical formulations of problem solving superior to those we currently have will eventually make the present formulation outdated. Nevertheless, the methods described in the present book, however imperfectly, can be of substantial benefit to any student who masters them. When someone has a beautiful mathematical theory of problems and problem solving sometime in the future, then clearer and more effective how-to-do-it books can be written. Meanwhile, it is my hope that this book will help many people to solve problems better than they did before.

APPLYING METHODS TO PROBLEMS

As discussed previously, to master the problem-solving methods described in this book, it is necessary to study the example problems illustrating their use. The problems and solutions analyzed in Chapters 3 to 10 illustrate the use of the methods discussed in the particular chapter. Chapter 11 considers a variety of homework and examination problems for mathematics, science, and engineering courses. Of course, you probably have lots of your own problems to solve in school or work, and you should begin using the methods on these problems immediately. Merely reading this book provides only the beginning concepts necessary to mastering general problem-solving methods. *Practice* in using the methods is essential to achieving a high level of skill.

Everyone who solves problems uses many or all of the methods described in this book, but if you are not an extremely good problem solver, you may be using the methods less effectively or more haphazardly than you could be by more explicit training in the methods. At first, the application of such explicitly taught problem-solving methods involves a rather slow, conscious analysis of each problem.

There is no particular reason to engage in this careful, conscious analysis of a problem when you can immediately get some good ideas on how to solve it. Just go ahead and solve the problem "naturally." However, after you solve it or, even better, while you are solving it, analyze what you are doing. It will greatly deepen your understanding of problem-solving methods, and you might discover new methods or a new application of an old method.

As you get extensive practice in using these problem-solving methods you should become so skilled in their use that the process becomes less conscious and more automatic or natural. This is the way of all skill learning, whether driving a car, playing tennis, or solving mathematical problems.