elements of abstract harmonic analysis

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Preface

Abstract Harmonic Analysis is an active branch of modern analysis which is increasing in importance as a standard course for the beginning graduate student. Concepts like Banach algebras, Haar measure, locally compact Abelian groups, etc., appear in many current research papers. This book is intended to enable the student to approach the original literature more quickly by informing him of these concepts and the basic theorems involving them.

In order to give a reasonably complete and self-contained introduction to the subject, most of the proofs have been presented in great detail thereby making the development understandable to a very wide audience. Exercises have been supplied at the end of each chapter. Some of these are meant to extend the theory slightly while others should serve to test the reader's understanding of the material presented.

The first chapter and part of the second give a brief review of classical Fourier analysis and present concepts which will subsequently be generalized to a more abstract framework. The presentation of this material is not meant to be detailed but is given mainly to motivate the generalizations obtained later in the book. The next five chapters present an introduction to commutative Banach algebras, general topological spaces, and topological groups. We hope that Chapters 2–6 might serve as an adequate introduction for those students primarily interested in the theory of commutative Banach algebras as well as serving as needed prerequisite material for the abstract harmonic analysis. The remaining chapters contain some of the measure theoretic background, including the Haar integral, and an extension of the concepts of the first two chapters to Fourier analysis on locally compact topological abelian groups.

In an attempt to make the book as self-contained and as introductory as possible, it was felt advisable to start from scratch with many concepts—in particular with general topological spaces. However, within the space limitations, it was not possible to do this with certain other background material—notably some measure theory and a few facts from functional analysis. Nevertheless, the material needed from these areas has all been listed in various appendices to the chapters to which they are most relevant.

There are now a number of more advanced books on abstract harmonic analysis which go deeper into the subject. We cite in particular the references to Rudin, Loomis, and the recent book by Hewitt and Ross. Our treatment of the latter part of Chapter 12 follows to some extent the initial chapter of the book by Rudin, which would be an excellent continuation for the reader who wishes to pursue these matters further.

The present book is based on a one semester course in abstract harmonic analysis given at the Polytechnic Institute of Brooklyn during the summer of 1963, for which lecture notes were written by Lawrence Narici. A few revisions and expansions have been made.

I would like to express my sincere gratitude to Mr. Narici for his effort in writing the notes, improving many of the proofs, and for editing the entire manuscript. I would like finally to express also my appreciation to Melvin Maron for his help in the preparation of the manuscript.

George Bachman

Contents

Symbols Used in Text

$\{x \mid p\}$ The set of all x with property p

$f : X \to Y$, $x \to y$ The function f mapping the set X into the set Y where $x \in X$ is mapped into $y \in Y$. Further, if $E \subset X, f(E)$ will denote the image set of E under f; i.e., $f(E) = \{f(x) \mid x \in E\}$. If $M \subset Y, f^{-1}(M)$ will denote those $x \in X$ such that $f(x) \in M$. The notation $f|_E$, $E \subset X$, will denote the restriction of f to E.

CE The complement of the set E

$\emptyset$ The null set

C The complex numbers

In the list below the number immediately following the symbol will denote the page on which the symbol is defined. The symbols are listed according to the order in which they appear in the text.

R, 1
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CHAPTER 1

The Fourier Transform on the Real Line for Functions in L_1

Introduction

This section will be devoted to some Fourier analysis on the real line. The notion of convolution will be introduced and some relationships between functions and their transforms will be derived. Some ideas from real variables (e.g. passage to the limit under the Lebesgue integral and interchanging the order of integration in iterated integrals) will be heavily relied upon and a summary of certain theorems that will be used extensively in this section are included in the appendix at the end of this chapter.

Notation

Throughout, R will denote the real axis, $(-\infty, \infty)$, and f will denote a measurable function on R (f may be real or complex valued). We will denote by L_p the set of all measurable functions on R with the property†

$$\int_{-\infty}^{\infty} |f(x)|^p \, dx < \infty, \qquad 1 \leq p < \infty.$$

Also, since we will so frequently be integrating from minus infinity to plus infinity, we will denote

$$\int_{-\infty}^{\infty} \quad \text{by simply} \quad \int$$

where all integrations are taken in the Lebesgue sense.

† One often says that f is pth *power summable.*

Noting without proof that the space L_p is a linear space we can now define the *norm of f with respect to p*, $||f||_p$, where $f \in L_p$ as follows:

$$||f||_p = \left(\int |f(x)|^p \, dx\right)^{1/p}.$$

It is simple to verify that the following assertions are valid:

1. $||f||_p \geq 0$ and $||f||_p = 0$ if and only if $f \sim 0$.†
2. $||kf||_p = |k| \cdot ||f||_p$ where k is a real or complex number. Note that this immediately implies $||f||_p = ||-f||_p$.
3. $||f + g||_p \leq ||f||_p + ||g||_p$. To verify this all one needs is Minkowski's inequality for integrals which is listed in the appendix to this chapter (p. 15).

Thus it is seen that L_p is a *normed* linear space. One can go one step further however; although it will not be proven here, it turns out that L_p is actually a *complete* normed linear space or *Banach space*, or that any Cauchy sequence (in the norm with respect to p) of functions in L_p converges (in the norm with respect to p) to a function that is pth power summable.

The Fourier Transform

In this section we consider the Fourier transform of a function f in L_1 and note certain properties of the Fourier transform. Let $f \in L_1$ and consider as the Fourier transform of f

$$\hat{f}(x) = \int e^{ixt} f(t) \, dt.$$

† We will say that two functions, f and g, are equivalent, denoted by $f \sim g$, if $f = g$ almost everywhere. Thus L_p is actually the set of all equivalence classes of functions under this equivalence relation. If we did not do this then $|| \; ||_p$ would represent a *pseudonorm* because it would be possible for $||f||_p$ to equal zero even though f was *not* zero.

We first note that $\hat{f}(x)$ exists for, since $f \in L_1$,

$$|\hat{f}(x)| \le \int |f(t)|\,dt < \infty.$$

Further, since

$$\int |f(t)|\,dt = ||f||_1$$

we can say

$$|\hat{f}(x)| \le ||f||_1$$

for any x; or that the 1-norm of f is an upper bound for the Fourier transform of f. Since it is an upper bound it is certainly greater than or equal to the least upper bound or

$$\sup_{x\epsilon R} |\hat{f}(x)| \le ||f||_1.$$

It will now be shown that the Fourier transform of f is a continuous function of x. Consider the difference

$$\hat{f}(x + h_n) - \hat{f}(x) = \int e^{ixt}(e^{ih_nt} - 1)f(t)\,dt$$

where $h_n \in R$:

$$|\hat{f}(x + h_n) - \hat{f}(x)| \le \int |\exp(ih_nt) - 1|\,|f(t)|\,dt.$$

Since this is true for any h_n it is true in the limit as h_n approaches zero, or

$$\lim_{h_n \to 0} |\hat{f}(x + h_n) - \hat{f}(x)| \le \lim_{h_n \to 0} \int |e^{ih_nt} - 1|\,|f(t)|\,dt.$$

Our wish now is to take the limit operation under the integral sign, and Lebesgue's dominated convergence theorem will allow this manipulation (see appendix to Chapter 1) for certainly the integrand in the last expression is dominated by the summable function $2\,|f(t)|$. Taking the limit inside yields the desired result; namely, that

$$\lim_{h_n \to 0} \hat{f}(x + h_n) = \hat{f}(x)$$

or that the Fourier transform $\hat{f}$ of a function f in L_1 is a continuous function.

The following fact can also be demonstrated about $\hat{f}(x)$ and is usually referred to as the Riemann–Lebesgue lemma:

$$\lim_{x \to \pm\infty} \hat{f}(x) = 0. \tag{1}$$

We note in passing that there are continuous functions, $F(x)$, satisfying (1) but such that no $f(t)$ can be found that satisfies

$$F(x) = \int e^{ixt} f(t)\, dt.$$

This brings us to our next problem: knowing $\hat{f}(x)$, how can the function that it came from, $f(t)$, be found again?

Recovery

In elementary treatments one often sees the following inversion formula:

$$f(t) = \frac{1}{2\pi} \int e^{-ixt} \hat{f}(x)\, dx.$$

It will now be demonstrated, by a counterexample, that the above formula is *not* true in general.

Example. Consider the function

$$f(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\hat{f}(x) = \int_0^\infty e^{(ix-1)t}\, dt = \frac{-1}{ix - 1}.$$

It is now clearly impossible to recover $f(t)$ using the formula for

$$\int | e^{-ixt}\hat{f}(x) | \, dx = \int \frac{dx}{\sqrt{1 + x^2}}.$$

Since the last integrand behaves like $1/x$, for large x the integral will become infinite as $\log x$ and, therefore, we cannot recover $f(t)$ using that formula. (Note that a Lebesgue integral must be absolutely convergent in order to converge.)

Before proceeding further, two results from real variables will be needed:

Definition. t is said to be a *Lebesgue point of the function* f if

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} | f(t) - f(x) | \, dx = 0.$$

Theorem. If $f \in L_1$, then almost all of its points are Lebesgue points.

Theorem. Every point of continuity of a function is a Lebesgue point.

The following two theorems on inversion are lengthy and somewhat intricate and the reader is referred to Goldberg [2] for the proofs.

Theorem 1. Let $f, \hat{f} \in L_1$ and suppose f is continuous at t, then

$$f(t) = \frac{1}{2\pi} \int e^{-ixt}\hat{f}(x) \, dx.$$

Theorem 2. Let $f \in L_1$ and let t be a Lebesgue point for the function, f, then

$$f(t) = \lim_{\alpha\to\infty} \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \left(1 - \frac{| x |}{\alpha}\right) e^{-ixt}\hat{f}(x) \, dx.$$

(Note that this limiting process is analogous to $(C, 1)$ summability criterion for infinite series.)

Corollary 1. Suppose $f \in L_1$ and $\hat{f}(x) = 0$, for all x, then $f(t) = 0$, a.e.

Corollary 2. Suppose $f_1, f_2 \in L_1$. If $\hat{f}_1 = \hat{f}_2$, then $f_1(t) = f_2(t)$ a.e.

This result follows immediately from Corollary 1, for we have

$$\widehat{f_1 - f_2} = \hat{f}_1 - \hat{f}_2 = 0$$

therefore

$$f_1 - f_2 = 0, \qquad \text{a.e.}$$

Convolution. Let $f, g \in L_1$ and consider the function

$$h(x) = \int f(x - t)g(t)\,dt = (f * g)(x)$$

and called the *convolution of f with g*. It is now contended that $h(x)$ exists for almost all x and that $h(x)$ is summable.

Proof. It is easily shown that

$$\int f(x - t)\,dx = \int f(x)\,dx \tag{2}$$

by simply making a change of variable. Now consider

$$\int dt \int |f(x - t)g(t)|\,dx = \int |g(t)|\,dt \int |f(x - t)|\,dx$$

$$= \|g\|_1 \|f\|_1 < \infty$$

by using (2). Now, by the Tonelli–Hobson theorem (see appendix to Chapter 1), it follows that

$$\iint f(x - t)g(t)\,dt\,dx$$

is absolutely convergent. By the Fubini theorem it follows that $h(x)$ exists a.e. and is integrable.

It will now be shown that the operation of convolution is a commutative one.

Theorem. $f * g = g * f$ for $f, g \in L_1$.

Proof.

$$(f * g)(x) = \int f(x - t) g(t)\, dt.$$

Let $u = x - t$. Then

$$(f * g)(x) = \int_{\infty}^{-\infty} f(u) g(x - u)(-du) = (g * f)(x).$$

It also follows that the operation of convolution is associative or

$$f * (g * h) = (f * g) * h$$

where $f, g, h \in L_1$.

The proof of this result, although straightforward, is rather messy and will be omitted.

Theorem. Let $f, g \in L_1$. Then

$$\| f * g \|_1 \leq \| f \|_1 \| g \|_1.$$

Proof.

$$\| f * g \|_1 = \int dx \left| \int f(x - t) g(t)\, dt \right|$$

$$\leq \int dx \int | f(x - t) g(t) |\, dt. \tag{3}$$

We now note that

$$\int dt \int | f(x - t) g(t) |\, dx = \int | g(t) |\, dt \int | f(x - t) |\, dx$$

$$= \| g \|_1 \| f \|_1 < \infty \tag{4}$$

or that (3) converges absolutely. By the Tonelli-Hobson theorem,†

$$\int dx \int | f(x - t) g(t) |\, dt$$

converges absolutely and, by the same theorem must also

† See appendix to Chapter 1.