

# Optimization Methods

*in operations research  
and systems analysis*

K V Mital

0224  
M2

8164711

# Optimization Methods

in operations research and  
systems analysis



E8164711

**K. V. MITAL**  
*University of Roorkee*



**WILEY EASTERN LIMITED**  
New Delhi Bangalore Bombay Calcutta

Copyright © 1976, Wiley Eastern Limited  
Reprint, 1978, 1979

This book or any part thereof may not be  
reproduced in any form without the written  
permission of the publisher

This book is not to be sold outside  
the country to which it is consigned by  
Wiley Eastern Limited

10 9 8 7 6 5 4 3

ISBN 0 85226 582 4

Published by Ravi Acharya for Wiley Eastern Limited, 4835/24  
Ansari Road, Daryaganj, New Delhi 110002 and printed by  
Pramodh Kapur at Raj Bandhu Industrial Company, C 61  
Maya Puri II, New Delhi 110064. Printed in India.

# OPTIMIZATION METHODS



**To Father  
whose memory lingers**

## Preface

The subject which started as operations research during the last World War in the early forties has been growing theoretically and also in its applications to a variety of problems in diverse fields, such as, engineering, management and economics. In its more comprehensive sense which includes survey and data collection, mathematical modelling, solutions of large mathematical problems and improvements through feedback of results, the subject has come to be known as systems analysis. The mathematical contents of this subject concerned with optimization of objectives may be grouped more expressively under optimization methods which form the subject matter of this volume.

This book is an elementary mathematical introduction to linear and nonlinear programming, dynamic programming, geometric programming, direct search methods and theory of games. It has grown out of lectures given to M.Sc. and M.E. classes and to short-term courses under the Refresher Courses Department and the Quality Improvement Programme at the University of Roorkee during the last six years. Only deterministic problems have been dealt with. Stochastic problems have not been touched. The book is intended to serve as a suitable text for students of mathematics, operations research, engineering, economics or management whose courses of study include some or all of the topics treated here. Most of the chapters can be studied independently of each other. A knowledge of algebra (including matrices), calculus and geometry as is usually given in the B.Sc. and the B.E. courses in India is assumed. Chapters I and II provide the additional necessary topics in mathematics. Convex sets have been treated in some detail as they are not included in the usual mathematics courses given to our students but are fundamental to the theory of mathematical programming.

Bibliography at the end lists a number of books, mostly recent, on the various topics discussed in this volume. A short bibliographical note at the end of each chapter is meant to guide the reader to a few standard books which he may profitably consult either

along with the present book or subsequently for more advanced study. No references are made to research papers, as it is seldom profitable for students to go direct to them without acquiring a working knowledge of the established concepts. A short historical note at the end of some of the chapters is meant to acquaint the student with the pioneer workers in the field. For original research papers and credits, books listed in the bibliography can be usefully consulted.

In their application to real life, problems in systems analysis and operations research usually involve large number of variables, parameters, equations and constraints. The problems generally involve too much numerical work which can be handled only by the digital computer. For this reason the methods of solution are computer oriented. The criterion of suitability of a method is often the economy and efficiency with which it can be programmed on the computer. In this book we are not concerned with computer programming. The illustrative examples in the text and also the problems at the end of each chapter are small enough to be solved by hand and may not apparently justify the methods recommended to solve them. But the student should not lose sight of the fact that the problems are only illustrative and the methods are really designed for large problems of the same type.

It is my pleasant duty to acknowledge gratefully the generous help I have received from many colleagues and friends in the preparation of this book. I am particularly grateful to Professor C. Prasad, Dr. O.P. Varshney and Dr. A.P. Gupta for their assistance in chapters I, II and VI; to Dr. U.S. Gupta for chapter III; to Dr. C. Mohan for chapters VIII and X; to Dr. R.K. Gupta for chapter IX; and to Dr. Bal Krishna for critically reading through chapter I. It is a truism that a teacher learns through his students. I am thankful to all my students of M.Sc. and M.E. classes and all those participants of special short-term courses who have attended my lectures over the last many years. Without their knowing it, they taught me a great deal and have contributed in some measure to the writing of this book.

K. V. MITAL

Roorkee  
June 26, 1976

# Contents

## PREFACE

vii

## I MATHEMATICAL PRELIMINARIES

<b>Euclidean Space</b>	<b>1</b>
1 Vectors and vector spaces	1
2 Linear dependence	3
3 Dimensions of a vector space, basis	4
4 Euclidean space	4
5 Norm of a vector	6
<b>Linear Algebraic Equations</b>	<b>8</b>
6 General form	8
7 Particular case: when $A$ is nonsingular square matrix	9
8 Consistent system of equations	10
9 Linearly independent consistent equations	11
10 Homogeneous equations	13
<b>Convex Sets</b>	<b>15</b>
11 Open and closed sets in $E_n$	15
12 Convex linear combinations, convex sets	17
13 Intersection of convex sets, convex hull of a set	20
14 Vertices or extreme points of a convex set	22
15 Convex polyhedron	23
16 Hyperplanes, half-spaces and polytopes	24
17 Separating and supporting hyperplanes	26
18 Vertices of a closed bounded convex set	28
19 Summary	31
<b>Quadratic Forms</b>	<b>31</b>
20 Quadratic forms	31
<i>Bibliographical note</i>	34
<i>Problems I</i>	34



**II EXTREMA OF FUNCTIONS**

1	Real-valued function	40
2	Partial derivatives, gradient vector	40
3	Taylor series	41
4	Directional derivative, direction of steepest descent	41
5	Local and global extrema	42
6	Limitations of the method of differential calculus	43
7	Unconstrained extrema of differentiable functions	44
8	Constrained extrema	46
9	Implicit function theorem	47
10	Method of Lagrange multipliers	48
11	Convex functions	50
12	General problem of mathematical programming	54
	<i>Bibliographical note</i>	55
	<i>Problems II</i>	56

**III LINEAR PROGRAMMING**

1	Introduction	58
2	LP in two-dimensional space	58
3	General LP problem	61
4	Feasible solutions	62
5	Basic solutions	63
6	Basic feasible solutions	64
7	Optimal solutions	66
8	Summary	68
9	Simplex method	69
10	Canonical form of equations	70
11	Simplex method (numerical example)	71
12	Simplex tableau	74
13	Finding the first b.f.s., artificial variables	75
14	Degeneracy	78
15	Simplex multipliers	79
16	Revised simplex method	80
17	Duality in LP problems	84
18	Duality theorems	86
19	Sensitivity analysis and parametric LP	89
20	Integer programming	92
21	Further developments in LP	93
	<i>Historical note</i>	93
	<i>Bibliographical note</i>	94

*Problems III*

94

**IV TRANSPORTATION PROBLEM**

1	Introduction	97
2	Transportation problem	97
3	<b>Triangular basis</b>	98
4	Finding a basic feasible solution	100
5	Testing for optimality	102
6	Changing the basis	104
7	Degeneracy	105
8	Unbalanced problem	106
9	Transportation with transshipment	106
10	Caterer problem	109
11	Assignment problem	112
12	Generalized transportation problem	115
	<i>Historical note</i>	115
	<i>Bibliographical note</i>	116
	<i>Problems IV</i>	116

**V FLOW AND POTENTIAL IN NETWORKS**

1	Introduction	121
2	Graphs: definitions and notation	121
3	Minimum path problem	125
4	Spanning tree of minimum length	131
5	Problem of minimum potential difference	135
6	Scheduling of sequential activities	137
7	Maximum flow problem	140
8	Duality in the maximum flow problem	146
9	Generalized problem of maximum flow	147
	<i>Historical note</i>	150
	<i>Bibliographical note</i>	150
	<i>Problems V</i>	150

**VI NONLINEAR CONVEX PROGRAMMING**

1	Introduction	154
2	Lagrangian function; saddle point	154
3	Kuhn-Tucker theory	155
4	Quadratic programming	161
	<i>Historical note</i>	167

<i>Bibliographical note</i>	167
<i>Problems VI</i>	167

## VII DYNAMIC PROGRAMMING

1 Introduction	169
2 Problem I: a minimum path problem	169
3 Problem II: single additive constraint, additively separable return	173
4 Problem III: single multiplicative constraint, additively separable return	177
5 Problem IV: single additive constraint, multiplicatively separable return	179
6 Computational economy in DP	180
7 Serial multistage model	181
8 Examples of failure	183
9 Decomposition	184
10 Backward and forward recursion	186
11 Systems with more than one constraint	189
12 Application of DP to continuous systems	191
<i>Bibliographical note</i>	192
<i>Problems VII</i>	193

## VIII GEOMETRIC PROGRAMMING

1 Introduction	196
2 Illustrative examples	197
3 General method	203
<i>Historical note</i>	204
<i>Bibliographical note</i>	205
<i>Problems VIII</i>	205

## IX THEORY OF GAMES

1 Introduction	206
2 Matrix (or rectangular) games	207
3 Problem of game theory	208
4 Minimax theorem, saddle point	209
5 Strategies and pay off	213
6 Theorems of matrix games	214
7 Graphical solution	219
8 Notion of dominance	222

9	Rectangular game as an LP problem	222
	<i>Historical note</i>	224
	<i>Bibliographical note</i>	224
	<i>Problems IX</i>	224

## X DIRECT SEARCH AND GRADIENT METHODS

1	Introduction	226
	<b>One-dimensional Search</b>	228
2	Unimodal functions	228
3	Search plans (one variable)	228
4	Fibonacci search plan	230
5	Golden section plan	233
6	Rosenbrock method	233
7	Methods requiring $f(x)$ to be differentiable	234
	<b>Multi-dimensional Search</b>	236
8	$n$ -dimensional problem	236
9	The basic step—search along a line	237
10	Basic methods of choosing a direction	239
11	Conjugate directions	241
12	Conjugate gradient method	243
13	PARTAN method	244
14	Scaling	245
15	Constrained problem, gradient projection	245
	<i>Historical note</i>	249
	<i>Bibliographical note</i>	249
	<i>Problems X</i>	249

<b>BIBLIOGRAPHY</b>	251
---------------------	-----

<b>INDEX</b>	255
--------------	-----

# I / Mathematical Preliminaries

## EUCLIDEAN SPACE

### 1 Vectors and vector spaces

A mathematical model of a system may contain  $n$  variables  $x_1, x_2, \dots, x_n$ , each of which may vary within a subset of real numbers  $R$ . A collection of  $n$  real numbers, taken in order, such that the first number is the value of  $x_1$ , the second of  $x_2$ , and so on, is called an ordered  $n$ -tuple of real numbers. We may denote an ordered  $n$ -tuple by a single symbol  $\mathbf{X}$ , so that

$$\mathbf{X} = (x_1, x_2, \dots, x_n),$$

and a set of such  $n$ -tuples by  $R_n$ , so that

$$R_n = \{\mathbf{X} \mid \mathbf{X} = (x_1, x_2, \dots, x_n)\}.$$

In order to deal with such sets it is convenient to establish an analogy with geometrical concepts which are easy to visualize. We therefore assume that  $\mathbf{X}$  and  $R_n$  satisfy certain postulates which are generalizations of notions familiar in two- and three-dimensional geometry, and then call  $\mathbf{X}$  a vector and  $R_n$  a vector space. We start with general definitions of vector and vector space.

**DEFINITION 1.** Let  $V$  be a set such that if  $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in V$  and  $a, b \in R$ , then the following postulates (defining the binary operation of sum and the operation of product with a real number) hold.

*Sum:*

- (i)  $\mathbf{X} + \mathbf{Y} \in V$ ;
- (ii)  $\mathbf{X} + \mathbf{Y} = \mathbf{Y} + \mathbf{X}$ ;
- (iii)  $(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = \mathbf{X} + (\mathbf{Y} + \mathbf{Z})$ ;
- (iv) There exists an element  $\mathbf{0} \in V$ , called the null or zero vector, such that  $\mathbf{X} + \mathbf{0} = \mathbf{X}$ ;
- (v) There exists an element  $-\mathbf{X} \in V$ , called the additive inverse of  $\mathbf{X}$ , such that  $\mathbf{X} + (-\mathbf{X}) = \mathbf{0}$ ;

*Product:*

- (vi)  $a\mathbf{X} \in V$ ;

- (vii)  $a(\mathbf{X} + \mathbf{Y}) = a\mathbf{X} + a\mathbf{Y}$ ;
- (viii)  $(a + b)\mathbf{X} = a\mathbf{X} + b\mathbf{X}$ ;
- (ix)  $(ab)\mathbf{X} = a(b\mathbf{X})$ ;
- (x)  $1\mathbf{X} = \mathbf{X}$ .

Then  $V$  is called a vector space and its elements are called vectors. Throughout this chapter  $V$  shall denote a vector space.

**Example:** Let  $V$  be a set of all polynomials in  $x$  of degree  $n$  or less.

$$V = \{f_1(x), f_2(x), \dots, f_i(x), \dots\},$$

where 
$$f_i(x) = \sum_{j=1}^n a_{ij} x^j, a_{ij} \in R.$$

If the two operations be the usual operations of sum and product by a real number, then it can be verified that the postulates (i) to (x) hold, and so  $V$  is a vector space. The additive inverse of  $f_i(x)$  and the zero vector can be easily identified.

**Example:** Let  $\mathbf{X} = (x_1, x_2, \dots, x_n)$  be an ordered  $n$ -tuple of real numbers and  $R_n$  be the set of all such  $n$ -tuples. If we define the sum of two  $n$ -tuples as

$$\begin{aligned} \mathbf{X} + \mathbf{Y} &= (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) \\ &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n), \end{aligned}$$

and the product with a real number  $a$  as

$$a\mathbf{X} = (ax_1, ax_2, \dots, ax_n),$$

then it can be verified that  $R_n$  is a vector space. The zero vector of the space is  $(0, 0, \dots, 0)$ .

In matrix operations it is convenient to regard  $\mathbf{X}$  as a column vector:

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \ x_2 \ \dots \ x_n]'$$

Its transpose  $\mathbf{X}'$  is a row vector. To indicate that  $\mathbf{X}$  has  $n$  components it is customary to call  $\mathbf{X}$  an  $n$ -vector.

**DEFINITION 2.** A subset  $W$  of  $V$  is called a subspace of the vector space  $V$  if  $W$  is itself a vector space with respect to the operations of sum and product defined in  $V$ .

**Example:** Let  $W \subseteq R_n$  (of last example) such that  $W = \{\mathbf{X} | \mathbf{X} = (x_1, 0, x_3, \dots, x_n)\}$ . Then  $W$  is a subspace of  $R_n$ . For, if  $\mathbf{X}, \mathbf{Y} \in W$ ,

$$\mathbf{X} + \mathbf{Y} = (x_1 + y_1, 0, x_3 + y_3, \dots, x_n + y_n) \in W,$$

$$a\mathbf{X} = (ax_1, 0, ax_3, \dots, ax_n) \in W, a \in R,$$

and similarly all other postulates can be seen to hold.

## 2 Linear dependence

**DEFINITION 3.** Let  $X_i, 1 \leq i \leq m$ , be vectors of  $V$ . Then  $X$  is called a linear combination of the vectors  $X_i$  if

$$X = \sum_{i=1}^m a_i X_i, a_i \in R.$$

**THEOREM 1.** The set  $W$  of all linear combinations of  $X_i, 1 \leq i \leq m$ , of vector space  $V$  is a subspace of  $V$ .

*Proof.* Let  $X = \sum_{i=1}^m a_i X_i, Y = \sum_{i=1}^m b_i X_i$ , so that  $X, Y \in W$ . Then

$$X + Y = \sum_{i=1}^m (a_i + b_i) X_i \in W,$$

$$\lambda X = \sum_{i=1}^m (\lambda a_i) X_i \in W, \lambda \in R.$$

It follows (see problem 1) that  $W$  is a vector space and hence a subspace of  $V$ . Proved.

$W$  is said to be spanned by (or a span of)  $X_i$ .

**DEFINITION 4.** The vectors  $X_i, 1 \leq i \leq m$ , of  $V$  are said to be linearly dependent if there exist real numbers  $a_i$ , not all zero, such that

$$\sum_{i=1}^m a_i X_i = 0.$$

If, however, this is so only if  $a_i = 0$  for all  $i$ , then the vectors are said to be linearly independent.

**Example:** Let  $X_1 = [2 \ -1 \ 3 \ 2]', X_2 = [1 \ 2 \ 2 \ -4]', X_3 = [4 \ 3 \ 7 \ -6]'$ . Since  $X_1 + 2X_2 - X_3 = 0$ , the vectors are linearly dependent. But  $X_1$  and  $X_2$  are linearly independent. So are  $X_2, X_3$ .

To test whether the  $n$ -vectors  $X_i, 1 \leq i \leq m$ , are linearly independent or not, one has to examine the equations

$$\sum_{i=1}^m a_i X_i = 0,$$

or putting  $X_i$  as  $[x_{1i} \ x_{2i} \ \dots \ x_{ni}]'$ ,

$$a_1 x_{11} + a_2 x_{12} + \dots + a_m x_{1m} = 0,$$

$$a_1 x_{21} + a_2 x_{22} + \dots + a_m x_{2m} = 0,$$

$$\dots \dots \dots$$

$$a_1 x_{n1} + a_2 x_{n2} + \dots + a_m x_{nm} = 0,$$

and investigate whether values of  $a_i$ , not all zero, exist which satisfy these equations, or in other words, whether the  $n$  equations in  $m$  unknowns  $a_i$  have a nontrivial solution. (A solution  $a_i = 0$  for all  $i$  is

called a trivial solution.) We shall discuss the solution of such equations later in this chapter.

### 3 Dimension of a vector space, basis

**DEFINITION 5.**  *$V$  is said to be of dimension  $m$  if there exists at least one set of  $m$  linearly independent vectors in  $V$ , while every set of  $m+1$  vectors in  $V$  is linearly dependent. The linearly independent set is called a basis of  $V$ .*

**THEOREM 2.** *A set of  $m$  linearly independent vectors in a vector space  $V$  of dimension  $m$  spans  $V$ .*

*Proof.* Let  $\mathbf{Y}_i$ ,  $1 \leq i \leq m$ , be  $m$  linearly independent vectors in  $V$ , and let  $\mathbf{X}$  be any vector in  $V$ . Since  $V$  is of dimension  $m$ , the  $m+1$  vectors  $\mathbf{X}, \mathbf{Y}_i$  must be linearly dependent. Hence

$$a_0 \mathbf{X} + \sum_{i=1}^m a_i \mathbf{Y}_i = \mathbf{0},$$

where  $a_0 \neq 0$ . For,  $a_0 = 0$  will imply that  $\mathbf{Y}_i$ ,  $1 \leq i \leq m$ , are linearly dependent vectors which, by hypothesis, they are not. It follows that

$$\mathbf{X} = - \sum_{i=1}^m (a_i/a_0) \mathbf{Y}_i$$

which means that  $\mathbf{X}$  is a linear combination of  $\mathbf{Y}_i$ . Since  $\mathbf{X}$  is any vector in  $V$  the theorem is proved.

It can be seen that the set of linearly independent vectors spanning a vector space is not unique. Consequently the basis of a vector space is also not unique. But once the basis is chosen every vector of the vector space has a unique linear combination expression in terms of the chosen basis.

### 4 Euclidean space

**DEFINITION 6.** *The inner product  $\langle \mathbf{X}, \mathbf{Y} \rangle$  of any two vectors  $\mathbf{X}$  and  $\mathbf{Y}$  of  $V$  is a real number satisfying the following properties.*

- (i)  $\langle \mathbf{X}, \mathbf{Y} \rangle = \langle \mathbf{Y}, \mathbf{X} \rangle$ ;
- (ii)  $\langle \mathbf{X} + \mathbf{Z}, \mathbf{Y} \rangle = \langle \mathbf{X}, \mathbf{Y} \rangle + \langle \mathbf{Z}, \mathbf{Y} \rangle$ ,  $\mathbf{Z} \in V$ ;
- (iii)  $\langle a\mathbf{X}, \mathbf{Y} \rangle = a\langle \mathbf{X}, \mathbf{Y} \rangle$ ,  $a \in R$ ;
- (iv)  $\langle \mathbf{X}, \mathbf{X} \rangle > 0$  if  $\mathbf{X} \neq \mathbf{0}$ ,  $\langle \mathbf{X}, \mathbf{X} \rangle = 0$  if  $\mathbf{X} = \mathbf{0}$ .

Two nonzero vectors are said to be *orthogonal* if their inner product is zero.



**DEFINITION 7.** A vector space with an inner product defined on it is called an Euclidean space.

For vectors of the vector space  $R_n$ , the expression

$$\mathbf{X}'\mathbf{Y} = \sum_{i=1}^n x_i y_i$$

satisfies the definition of inner product. With this definition  $R_n$  becomes a Euclidean space. This Euclidean space, if of dimension  $n$ , shall be denoted by  $E_n$ . On account of its importance in the present work we give afresh the definition of  $E_n$  which may be understood without reference to general definitions of vectors and vector spaces given above.

**DEFINITION 8.** Let  $R_n$  be a set of ordered  $n$ -tuples of real numbers. For every pair of  $n$ -tuples  $\mathbf{X}, \mathbf{Y} \in R_n$ , let

- (i) **Sum:**  $\mathbf{X} + \mathbf{Y} = \mathbf{Y} + \mathbf{X} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in R_n$ ;
  - (ii) **Product:**  $a\mathbf{X} = (ax_1, ax_2, \dots, ax_n) \in R_n, a \in R$ ;
  - (iii) **Inner product:**  $\mathbf{X}'\mathbf{Y} = \mathbf{Y}'\mathbf{X} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n \in R$ ;
- be defined. Then the  $n$ -tuples are called vectors and  $R_n$  is called a Euclidean space. Also let
- (iv) There be at least one set of  $n$  linearly independent vectors in  $R_n$ . Then  $R_n$  is a Euclidean space of dimension  $n$  which we shall denote as  $E_n$ .

It should be noticed that the additional condition 'every set of  $n+1$  vectors in  $R_n$  is linearly dependent' which was included in definition 5 has been dropped in (iv) above. The reason is that in this case it is implied and its explicit statement will be superfluous (see problem 10).

**Example:** The set of column vectors  $[100]', [010]', [001]'$  is a basis of  $R_3$ . For, these vectors are linearly independent, and any vector  $[x_1 \ x_2 \ x_3]'$  of  $R_3$  can be expressed as a linear combination of these vectors as follows.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This basis is called the canonical or the natural basis of  $R_3$ . Another basis of  $R_3$  is  $[100]', [110]', [111]'$ . For,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where  $c = x_3, b = x_2 - x_3, a = x_1 - x_2$ .