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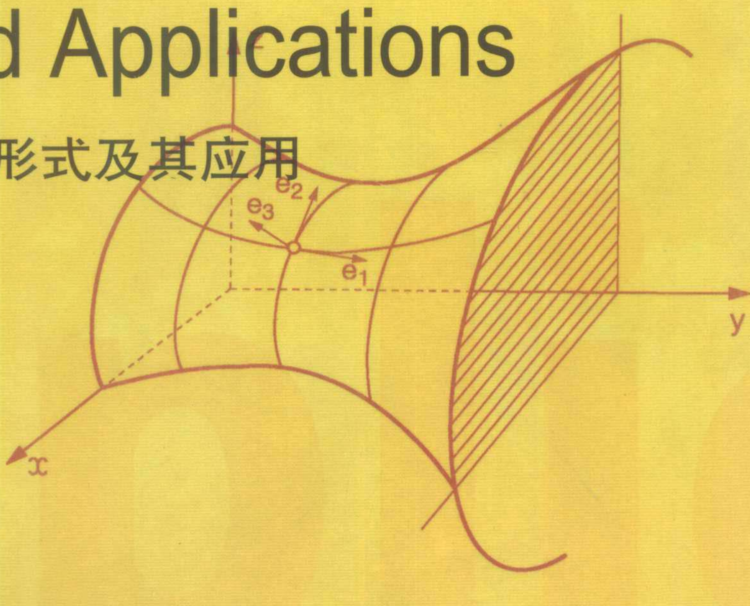
Manfredo P. do Carmo

Differential Forms and Applications

微分形式及其应用



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Manfredo P. do Carmo

Differential Forms and Applications

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Preface

This is a free translation of a set of notes published originally in Portuguese in 1971. They were translated for a course in the College of Differential Geometry, ICTP, Trieste, 1989. In the English translation we omitted a chapter on the Frobenius theorem and an appendix on the nonexistence of a complete hyperbolic plane in euclidean 3-space (Hilbert's theorem). For the present edition, we introduced a chapter on line integrals.

In Chapter 1 we introduce the differential forms in R^n. We only assume an elementary knowledge of calculus, and the chapter can be used as a basis for a course on differential forms for "users" of Mathematics.

In Chapter 2 we start integrating differential forms of degree one along curves in R^n. This already allows some applications of the ideas of Chapter 1. This material is not used in the rest of the book.

In Chapter 3 we present the basic notions of differentiable manifolds. It is useful (but not essential) that the reader be familiar with the notion of a regular surface in R^3.

In Chapter 4 we introduce the notion of manifold with boundary and prove Stokes theorem and Poincare's lemma.

Starting from this basic material, we could follow any of the possible routes for applications: Topology, Differential Geometry, Mechanics, Lie Groups, etc. We have chosen Differential Geometry. For simplicity, we restricted ourselves to surfaces.

Thus in Chapter 5 we develop the method of moving frames of Elie Cartan for surfaces. We first treat immersed surfaces and next the intrinsic geometry of surfaces

Finally, in Chapter 6, we prove the Gauss-Bonnet theorem for compact orientable surfaces. The proof we present here is essentially due to S.S.Chern. We also prove a relation, due to M. Morse, between the Euler characteristic of such a surface and the critical points of a certain class of differentiable functions on the surface.

As most authors, I am indebted to so many sources that it is hardly possible to acknowledge them all. Let me at least mention that the first four chapters were strongly influenced by the writings of my friend and colleague Elon Lima and the last two chapters bear the imprint of my teacher and friend S.S. Chern.

For the present version I am indebted to my colleagues M. Dajczer, L. Rodríguez and W. Santos for reading critically the manuscript and offering a number of useful suggestions. Special thanks are due to Lucio Rodríguez for his care in the camera ready presentation of the final text.

Rio de Janeiro, February 1994. Manfredo Perdigão do Carmo

Table of Contents

1. Differential Forms in $\mathbf{R}^n$

The goal of this chapter is to define in $\mathbf{R}^n$ "fields of alternate forms" that will be used later to obtain geometric results.

In order to fix the ideas, we will work initially with the three-dimensional space $\mathbf{R}^3$.

Let p be a point of $\mathbf{R}^3$. The set of vectors $q-p$, $q \in \mathbf{R}^3$ (that have origin at p) will be called the *tangent space of* $\mathbf{R}^3$ *at* p and will be denoted by $\mathbf{R}^3_p$. The vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ of the canonical basis of $\mathbf{R}^3_0$ will be identified with their translates $(e_1)_p, (e_2)_p, (e_3)_p$ at the point p.

A *vector field* in $\mathbf{R}^3$ is a map v that associates to each point $p \in \mathbf{R}^3$ a vector $v(p) \in \mathbf{R}^3_p$. We can write v as

$$v(p) = a_1(p)e_1 + a_2(p)e_2 + a_3(p)e_3,$$

thereby defining three functions $a_i \colon \mathbf{R}^3 \to \mathbf{R}$, $i = 1,2,3$, that characterize the vector field v. We say that v is *differentiable* if the functions a_i are differentiable.

To each tangent space $\mathbf{R}^3_p$ we can associate its *dual space* $(\mathbf{R}^3_p)^*$ which is the set of linear maps $\varphi \colon \mathbf{R}^3_p \to \mathbf{R}$. A basis for $(\mathbf{R}^3_p)^*$ is obtained by taking $(dx_i)_p$, $i = 1,2,3$, where $x_i \colon \mathbf{R}^3 \to \mathbf{R}$ is the map which assigns to each point its i^{th} -coordinate. The set

$$\{(dx_i)_p;\ i = 1,2,3\}$$

is in fact the dual basis of $\{(e_i)_p\}$ since

$$(dx_i)_p(e_j) = \frac{\partial x_i}{\partial x_j} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

Definition 1. A *field of linear forms* (or an *exterior form of degree* 1) in $\mathbf{R}^3$ is a map ω that associates to each $p \in \mathbf{R}^3$ an element $\omega(p) \in (\mathbf{R}^3_p)^*$; ω can be written as

$$\omega(p) = a_1(p)(dx_1)_p + a_2(p)(dx_2)_p + a_3(p)(dx_3)_p$$

or

$$\omega = \sum_{i=1}^{3} a_i \, dx_i,$$

where a_i are real functions in $\mathbf{R}^3$. If the functions a_i are differentiable, ω is called a *differential form of degree* 1.

Now let $\Lambda^2(\mathbf{R}^3_p)^*$ be the set of maps $\varphi: \mathbf{R}^3_p \times \mathbf{R}^3_p \to \mathbf{R}$ that are bilinear (i.e., φ is linear in each variable) and alternate (i.e., $\varphi(v_1, v_2) = -\varphi(v_2, v_1)$). With the usual operations of functions, the set $\Lambda^2(\mathbf{R}^3_p)^*$ becomes a vector space.

When φ_1 and φ_2 belong to $(\mathbf{R}^3_p)^*$, we can obtain an element $\varphi_1 \wedge \varphi_2 \in \Lambda^2(\mathbf{R}^3_p)^*$ by setting

$$(\varphi_1 \wedge \varphi_2)(v_1, v_2) = \det(\varphi_i(v_j))$$

The element $(dx_i)_p \wedge (dx_j)_p \in \Lambda^2(\mathbf{R}^3_p)^*$ will be denoted by $(dx_i \wedge dx_j)_p$. It is easy to see that the set $\{(dx_i \wedge dx_j)_p, \, i < j\}$ is a basis for $\Lambda^2(\mathbf{R}^3_p)^*$ (this will be proved in a more general setting in Proposition 1 below). Furthermore,

$$(dx_i \wedge dx_j)_p = -(dx_j \wedge dx_i)_p, \qquad i \neq j,$$

and

$$(dx_i \wedge dx_i)_p = 0.$$

Definition 2. A *field of bilinear alternating forms* or *an exterior form of degree 2* in $\mathbf{R}^3$ is a correspondence w that associates to each $p \in \mathbf{R}^3$ an element $\omega(p) \in \Lambda^2(\mathbf{R}^3_p)^*$; ω can be written in the form

$$\omega(p) = a_{12}(p)(dx_1 \wedge dx_2)_p + a_{13}(p)(dx_1 \wedge dx_3)_p + a_{23}(p)(dx_2 \wedge dx_3)_p$$

or

$$\omega = \sum_{i<j} a_{ij} dx_i \wedge dx_j, \qquad i, j = 1, 2, 3,$$

where a_{ij} are real functions in $\mathbf{R}^3$. When the functions a_{ij} are differentiable, ω is a *differential form of degree* 2.

We will now generalize the notion of differential form to $\mathbf{R}^n$. Let $p \in \mathbf{R}^n$, $\mathbf{R}^n_p$ the tangent space of $\mathbf{R}^n$ at p and $(\mathbf{R}^n_p)^*$ its dual space. Let $\Lambda^k(\mathbf{R}^n_p)^*$ be the set of all k-linear alternating maps

$$\varphi: \underbrace{\mathbf{R}^n_p \times \ldots \times \mathbf{R}^n_p}_{k \text{ times}} \to \mathbf{R}$$

(alternating means that φ changes signs with the interchange of two consecutive arguments). With the usual operations, $\Lambda^k(\mathbf{R}^n_p)^*$ is a vector space. Given $\varphi_1, \ldots, \varphi_k \in (\mathbf{R}^n_p)^*$, we can obtain an element $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_k$ of $\Lambda^k(\mathbf{R}^n_p)^*$ by setting

$$(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_k)(v_1, v_2, \ldots, v_k) = \det(\varphi_i(v_j)), \ \ i,j = 1, \ldots, k.$$

It follows from the properties of determinants that $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_k$ is in fact k-linear and alternate. In particular $(dx_{i_1})_p \wedge (dx_{i_2})_p \wedge \ldots \wedge (dx_{i_k}) \in \Lambda^k(\mathbf{R}^n_p)^*$, $i_1, i_2, \ldots, i_k = 1, \ldots, n$. We will denote this element by $(dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k})_p$.

Proposition 1. *The set*

$$\{(dx_{i_1} \wedge \ldots \wedge dx_{i_k})_p, \quad i_1 < i_2 < \ldots < i_k, \quad i_j \in \{1, \ldots, n\}\}$$

is a basis for $\Lambda^k(\mathbf{R}^n_p)^*$.

Proof. The elements of the set are linearly independent. For, if

$$\sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} \ dx_{i_1} \wedge \ldots \wedge dx_{i_k} = 0,$$

is applied to

$$(e_{j_1}, \ldots, e_{j_k}), \ j_1 < \ldots < j_k, \ j_\ell \in \{1, \ldots, n\},$$

we obtain (Exercise 2)

$$\sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} \quad dx_{i_1} \wedge \ldots \wedge dx_{i_k} \quad (e_{j_1}, \ldots, e_{j_k}) = a_{j_1 \ldots j_k} = 0.$$

We now show that if $f \in \Lambda^k(\mathbf{R}^n_p)^*$, then f is a linear combination of the form

$$f = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} \quad dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

For that, set

$$g = \sum_{i_1 < \ldots < i_k} f(e_{i_1}, \ldots, e_{i_k})\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

Notice that $g \in \Lambda^k(\mathbf{R}^n_p)^*$ and that

$$g(e_{i_1}, \ldots, e_{i_k}) = f(e_{i_1}, \ldots, e_{i_k}),$$

for all $i_1, \ldots, i_k$. It follows that $f = g$. Setting $f(e_{i_1}, \ldots, e_{i_k}) = a_{i_1 \ldots i_k}$, we obtain the above expression for f. □

Definition 3. An *exterior k-form* in $\mathbf{R}^n$ is a map ω that associates to each $p \in \mathbf{R}^n$ an element $\omega(p) \in \Lambda^k(\mathbf{R}^n_p)^*$; by Proposition 1, ω can be written as

$$\omega(p) = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k}(p)(dx_{i_1} \wedge \ldots \wedge dx_{i_k})_p, \quad i_j \in \{1, \ldots, n\},$$

where $a_{i_1 \dots i_k}$ are real functions in $\mathbf{R}^n$. When the $a_{i_1 \dots i_k}$ are differentiable functions, ω is called a *differential k-form*.

For notational convenience, we will denote by I the k-upla $(i_1, \dots, i_k)$, $i_1 < \dots < i_k$, $i_j \in \{1, \dots, n\}$, and will use the following notation for ω:

$$\omega = \sum_I a_I dx_I.$$

We also set the convention that a differential 0-form is a differentiable function $f: \mathbf{R}^n \to \mathbf{R}$.

Example 1. In $\mathbf{R}^4$ we have the following types of exterior forms (where a_i, a_{ij}, etc., are real functions in $\mathbf{R}^4$):

0-forms, functions in $\mathbf{R}^4$,

1-forms, $a_1 dx_1 + a_2 dx_2 + a_3 dx_3 + a_4 dx_4$,

2-forms, $a_{12} dx_1 \wedge dx_2 + a_{13} dx_1 \wedge dx_3 + a_{14} dx_1 \wedge dx_4 + a_{23} dx_2 \wedge dx_3 + a_{24} dx_2 \wedge dx_4 + a_{34} dx_3 \wedge dx_4$,

3-forms, $a_{123} dx_1 \wedge dx_2 \wedge dx_3 + a_{124} dx_1 \wedge dx_2 \wedge dx_4 + a_{134} dx_1 \wedge dx_3 \wedge dx_4 + a_{234} dx_2 \wedge dx_3 \wedge dx_4$,

4-forms, $a_{1234} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$.

From now on, we will restrict ourselves to differential k-forms and we will call them simply k-forms.

We are going to define some operations on k-forms in $\mathbf{R}^n$.

First, if ω and φ are two k-forms:

$$\omega = \sum_I a_I dx_I, \qquad \varphi = \sum_I b_I dx_I,$$

we can define their *sum*

$$\omega + \varphi = \sum_I (a_I + b_I) dx_I.$$

Next, if ω is a k-form and φ is an s-form, we can define their *exterior product* $\omega \wedge \varphi$, which is an $(s+k)$-form, as follows.

Definition 4. Let

$$\omega = \sum a_I dx_I, \qquad I = (i_1, \dots, i_k), \qquad i_1 < \dots < i_k,$$

$$\varphi = \sum b_J dx_J, \qquad J = (j_1, \dots, j_s), \qquad j_1 < \dots < j_s.$$

By definition,

$$\omega \wedge \varphi = \sum_{IJ} a_I b_J dx_I \wedge dx_J.$$

Example 2. Let $\omega = x_1 dx_1 + x_2 dx_2 + x_3 dx_3$ be a 1-form in $\mathbf{R}^3$ and $\varphi = x_1 dx_1 \wedge dx_2 + dx_1 \wedge dx_3$ be a 2-form in $\mathbf{R}^3$. Then, since $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$, $i \neq j$, we obtain

$$\begin{aligned} \omega \wedge \varphi &= x_2 dx_2 \wedge dx_1 \wedge dx_3 + x_3 x_1 dx_3 \wedge dx_1 \wedge dx_2 \\ &= (x_1 x_3 - x_2) dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

Remark 1. The definition of exterior product is made in such a way that if $\varphi_1, \ldots, \varphi_k$ are 1-forms, then the exterior product $\varphi_1 \wedge \ldots \wedge \varphi_k$ agrees with the k-form previously defined by

$$\varphi_1 \wedge \ldots \wedge \varphi_k(v_1, \ldots, v_k) = \det(\varphi_i(v_j)).$$

This follows immediately from the definition and will be left as an exercise (Exercise 3).

The exterior product of forms in $\mathbf{R}^n$ has the following properties.

Proposition 2. *Let ω be a k-form, φ be an s-form and θ be an r-form. Then:*

a) $(\omega \wedge \varphi) \wedge \theta = \omega \wedge (\varphi \wedge \theta)$,
b) $(\omega \wedge \varphi) = (-1)^{ks}(\varphi \wedge \omega)$,
c) $\omega \wedge (\varphi + \theta) = \omega \wedge \varphi + \omega \wedge \theta$, *if* $r = s$.

Proof. (a) and (c) are straightforward. To prove (b), we write

$$\omega = \sum a_I dx_I, \qquad I = (i_1, \ldots, i_k), \qquad i_1 < \ldots < i_k,$$

$$\varphi = \sum b_J dx_J, \qquad J = (j_1, \ldots, j_s), \qquad j_1 < \ldots < j_s.$$

Then

$$\begin{aligned} \omega \wedge \varphi &= \sum_{IJ} a_I b_J dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_s} \\ &= \sum_{IJ} b_J a_I (-1) dx_{i_1} \wedge \ldots \wedge dx_{i_{k-1}} \wedge dx_{j_1} \wedge dx_{i_k} \wedge \ldots \wedge dx_{j_s} \\ &= \sum_{IJ} b_J a_I (-1)^k dx_{j_1} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_s}. \end{aligned}$$

Since J has s elements, we obtain, by repeating the above argument for each $dx_{j_\ell}, j_\ell \in J$,

$$\begin{aligned} \omega \wedge \varphi &= \sum_{JI} b_J a_I (-1)^{ks} dx_{j_1} \wedge \ldots \wedge dx_{j_s} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\ &= (-1)^{ks} \varphi \wedge \omega. \end{aligned}$$

□

Remark 2. Although $dx_i \wedge dx_i = 0$, it is not true that for any form $\omega \wedge \omega = 0$. For instance, if

$$\omega = x_1 dx_1 \wedge dx_2 + x_2 dx_3 \wedge dx_4,$$

then

$$\omega \wedge \omega = 2x_1 x_2 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

See however Exercise 4.

One of the most important features of differential forms is the way they behave under differentiable maps. Let $f: \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map. Then f induces a map f^* that takes k-forms in $\mathbf{R}^m$ into k-forms in $\mathbf{R}^n$ and is defined as follows. Let ω be a k-form in $\mathbf{R}^m$. By definition, $f^*\omega$ is the k-form in $\mathbf{R}^n$ given by

$$(f^*\omega)(p)(v_1, \dots, v_k) = \omega(f(p))(df_p(v_1), \dots, df_p(v_k)).$$

Here $p \in \mathbf{R}^n$, $v_1, \dots, v_k \in \mathbf{R}^n_p$, and $df_p: \mathbf{R}^n_p \to \mathbf{R}^m_{f(p)}$ is the differential of the map f at p. We set the convention that if g is a 0-form,

$$f^*(g) = g \circ f.$$

We are going to show that the operation f^* on forms is equivalent to "substitution of variables". Before that, we need some properties of f^*.

Proposition 3. *Let $f: \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map, ω and φ be k-forms on $\mathbf{R}^m$ and $g: \mathbf{R}^m \to \mathbf{R}$ be a 0-form on $\mathbf{R}^m$. Then:*

a) $f^*(\omega + \varphi) = f^*\omega + f^*\varphi$,

b) $f^*(g\omega) = f^*(g) f^*(\omega)$,

c) If $\varphi_1, \dots, \varphi_k$ are 1-forms in $\mathbf{R}^m$, $f^*(\varphi_1 \wedge \dots \wedge \varphi_k) = f^*(\varphi_1) \wedge \dots \wedge f^*(\varphi_k)$.

Proof. The proofs are very simple. Let $p \in \mathbf{R}^n$ and let $v_1, \dots, v_k \in \mathbf{R}^n_p$. Then

(a) $f^*(\omega + \varphi)(p)(v_1, \dots, v_k) = (\omega + \varphi)(f(p))(df_p(v_1), \dots, df_p(v_k)) = (f^*\omega)(p)(v_1, \dots, v_k) + (f^*\varphi)(p)(v_1, \dots, v_k) = (f^*w + f^*\varphi)(p)(v_1, \dots, v_k)$.

(b) $f^*(g\omega)(p)(v_1, \dots, v_k) = (g\omega)(f(p))(df_p(v_1), \dots, df_p(v_k)) = (g \circ f)(p) \cdot f^*\omega(p)(v_1, \dots, v_k) = f^*g(p) \cdot f^*\omega(p)(v_1, \dots, v_k)$.

(c) By omitting the indication of the point p, we obtain

$$\begin{aligned} f^*(\varphi_1 \wedge \dots \wedge \varphi_k)(v_1, \dots, v_k) &= (\varphi_1 \wedge \dots \wedge \varphi_k)(df(v_1), \dots, df(v_k)) \\ &= \det(\varphi_i(df(v_j)) = \det(f^*\varphi_i(v_j)) \\ &= (f^*\varphi_1 \wedge \dots \wedge f^*\varphi_k)(v_1, \dots, v_k). \end{aligned}$$

Remark 3. We will show below (See Proposition 4) that (c) holds not only for 1-forms but for k-forms as well.

We can now present the promised interpretation of f^*. Let $(x_1, \ldots, x_n)$ be coordinates in $\mathbf{R}^n$, $(y_1, \ldots, y_m)$ be coordinates in $\mathbf{R}^m$ and let $f: \mathbf{R}^n \to \mathbf{R}^m$ be written as

$$y_1 = f_1(x_1, \ldots, x_n), \ldots, y_m = f_m(x_1, \ldots, x_n). \qquad (*)$$

Let $\omega = \sum_I a_I dy_I$ be a k-form in $\mathbf{R}^m$. By using the above properties of f^*, we obtain

$$f^*\omega = \sum_I f^*(a_I)(f^*dy_{i_1}) \wedge \ldots \wedge (f^*dy_{i_k}).$$

Since

$$f^*(dy_i)(v) = dy_i(df(v)) = d(y_i \circ f)(v) = df_i(v),$$

we have

$$f^*\omega = \sum_I a_I(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) df_{i_1} \wedge \ldots \wedge df_{i_k},$$

where f_i and df_i are functions of x_j. Thus to apply f^* to ω is equivalent to "*substitute*" in ω the variables y_i and their differentials by the functions of x_k and dx_k obtained from $(*)$.

Remark 4. In various situations, it is convenient to use differential forms defined only on some open set $U \subset \mathbf{R}^n$ and not on the entire $\mathbf{R}^n$. It is clear that everything done so far extends trivially to this situation.

Example. (Polar coordinates). Let ω be the 1-form in $\mathbf{R}^2 - \{0, 0\}$ by

$$\omega = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

Let U be the set in the plane (r, θ) given by

$$U = \{r > 0; 0 < \theta < 2\pi\}$$

and let $f: U \to \mathbf{R}^2$ be the map

$$f(r, \theta) = \begin{cases} x = r \cos\theta \\ y = r \operatorname{sen}\theta \end{cases}$$

Let us compute $f^*\omega$. Since

$$dx = \cos\theta dr - r \operatorname{sen}\theta d\theta,$$

$$dy = \operatorname{sen}\theta dr + r \cos\theta d\theta,$$

we obtain

$$\begin{aligned} f^*\omega &= -\frac{r \operatorname{sen}\theta}{r^2}(\cos\theta dr - r \operatorname{sen}\theta d\theta) + \frac{r \cos\theta}{r^2}(\operatorname{sen}\theta dr + r \cos\theta d\theta) \\ &= d\theta. \end{aligned}$$

Notice that (a) of Proposition 3 states that the addition of differential forms commutes with the "substitution of variables". We will now show that the same holds for the exterior product.

Proposition 4. *Let $f\colon \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map. Then*

(a) $f^*(\omega \wedge \varphi) = (f^*\omega) \wedge (f^*\varphi)$, where ω and φ any two forms in $\mathbf{R}^m$.

(b) $(f \circ g)^*\omega = g^*(f^*\omega)$, where $g\colon \mathbf{R}^p \to \mathbf{R}^n$ is a differentiable map.

Proof. By setting $(y_1, \dots, y_m) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) \in \mathbf{R}^m$, $(x_1, \dots, x_n) \in \mathbf{R}^n$, $\omega = \sum_I a_I dy_I$, $\varphi = \sum_J b_J dy_J$, we obtain

$$\begin{aligned} f^*(\omega \wedge \varphi) &= f^*(\sum_{IJ} a_I b_J dy_I \wedge dy_J) \\ &= \sum_{IJ} a_I(f_1, \dots, f_m) b_J(f_1, \dots, f_m) df_I \wedge df_J \\ &= \sum_I a_I(f_1, \dots, f_m) df_I \wedge \sum_J b_J(f_1, \dots, f_m) df_J \\ &= f^*\omega \wedge f^*\varphi. \end{aligned}$$

b) $(f \circ g)^*\omega = \sum_I a_I((f \circ g)_1, \dots, (f \circ g)_m) d(f \circ g)_I$
$= \sum_I a_I(f_1(g_1, \dots, g_n), \dots, f_m(g_1, \dots, g_n)) df_I(dg_1, \dots, dg_n)$
$= g^*(f^*(\omega))$. □

We are now going to define an operation on differential form that generalizes the differentiation of functions. Let $g\colon \mathbf{R}^n \to \mathbf{R}$ be a 0-form (i.e., a differentiable function). Then the differential

$$dg = \sum_{i=1}^n \frac{\partial g}{\partial x_i} dx_i$$

is a 1-form. We want to generalize this process by defining an operation that takes k-forms into $(k+1)$-forms.

Definition 5. Let $\omega = \sum a_I dx_I$ be a k-form in $\mathbf{R}^n$. The *exterior differential* $d\omega$ of ω is defined by

$$d\omega = \sum_I da_I \wedge dx_I.$$

Example 4. Let $\omega = xyzdx + yzdy + (x+z)dz$ and let us compute $d\omega$:

$$\begin{aligned} d\omega &= d(xyz) \wedge dx + d(yz) \wedge dy + d(x+z) \wedge dz \\ &= (yzdx + xzdy + xydz) \wedge dx + (zdy + ydz) \wedge dy + (dx + dz) \wedge dz \\ &= -xzdx \wedge dy + (1 - xy)dx \wedge dz - ydy \wedge dz. \end{aligned}$$

We now present some properties of exterior differentiation. Item (c) is probably the most important one and item (d) means that the operation d commutes with substitution of variables.

Proposition 5.

a) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$, *where* ω_1 *and* ω_2 *are* k*-forms*

b) $d(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^k \omega \wedge d\varphi$, *where* ω *is a* k*-form and* φ *is an* s*-form*

c) $d(d\omega) = d^2\omega = 0$.

d) $d(f^*\omega) = f^*(d\omega)$, *where* ω *is a* k*-form in* $\mathbf{R}^m$ *and* $f: \mathbf{R}^n \to \mathbf{R}^m$ *is a differentiable map.*

Proof.

(a) is straightforward.

(b) Let $\omega = \sum_I a_I dx_I$, $\varphi = \sum_J b_J dx_J$. Then

$$
\begin{aligned}
&= \sum_{IJ} d(a_I b_J) \wedge dx_I \wedge dx_J \\
&= \sum_{IJ} b_J da_I \wedge dx_I \wedge dx_J + \sum_{IJ} a_I db_J \wedge dx_I \wedge dx_J \\
&= d\omega \wedge \varphi + (-1)^k \sum_{IJ} a_I dx_I \wedge db_J \wedge dx_J \\
&= d\omega \wedge \varphi + (-1)^k \omega \wedge d\varphi.
\end{aligned}
$$

(c) Let us first assume that ω is a 0-form, i.e., ω is a function $f: \mathbf{R}^n \to \mathbf{R}$ that associates to each $(x_1, \ldots, x_n) \in \mathbf{R}^n$ the value $f(x_1, \ldots, x_n) \in \mathbf{R}$. Then

$$
\begin{aligned}
d(df) &= d\left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j\right) = \sum_{j=1}^n d\left(\frac{\partial f}{\partial x_j}\right) \wedge dx_j \\
&= \sum_{j=1}^n \left(\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j\right).
\end{aligned}
$$

Since $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$, $i \neq j$, we obtain that

$$
d(df) = \sum_{i<j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j = 0.
$$

Now let $w = \sum a_I dx_I$. By (a), we can restrict ourselves to the case $w = a_I dx_I$ with $a_I \neq 0$. By (b), we have that

$$
dw = da_I \wedge dx_I + a_I d(dx_I).
$$

But $d(dx_I) = d(1) \wedge dx_I = 0$. Therefore,

$$d(d\omega) = d(da_I \wedge dx_I) = d(da_I) \wedge dx_I + da_I \wedge d(dx_I) = 0,$$

since $d(da_I) = 0$ and $d(dx_I) = 0$, which proves (*c*).
(d) We will first prove the result for a 0-form. Let $g: \mathbf{R}^m \to \mathbf{R}$ be a differentiable function that associates to each $(y_1, \ldots, y_m) \in \mathbf{R}^m$ the value $g(y_1, \ldots, y_m)$. Then

$$\begin{aligned} f^*(dg) &= f^*\left(\sum_i \frac{\partial g}{\partial y_i} dy_i\right) = \sum_{ij} \frac{\partial g}{\partial y_i}\frac{\partial f_i}{\partial x_j} dx_j \\ &= \sum_j \frac{\partial (g \circ f)}{\partial x_j} dx_j = d(g \circ f) = d(f^* g). \end{aligned}$$

Now, let $\varphi = \sum_I a_I dx_I$ be a k-form. By using the above, and the fact that f^* commutes with the exterior product, we obtain

$$\begin{aligned} d(f^*\varphi) &= d(\sum_I f^*(a_I) f^*(dx_I)) \\ &= \sum_I d(f^*(a_I)) \wedge f^*(dx_I)) = \sum_I f^*(da_I) \wedge f^*(dx_I) \\ &= f^*(\sum_I da_I \wedge dx_I) = f^*(d\varphi) \end{aligned}$$

which proves (*d*). □

In the exercises that follow we will often use the canonical isomorphism between $\mathbf{R}^n_p$ and its dual $(\mathbf{R}^n_p)^*$ that is established by the natural inner product $\langle\, ,\, \rangle$ of $\mathbf{R}^n$. We recall that if $\{e_i\}$ is the canonical basis of $\mathbf{R}^n$ and $v_1 = \sum a_i e_i$, $v_2 = \sum b_i e_i$ belong to $(\mathbf{R}^n)_p$, then $\langle v_1, v_2 \rangle = \sum a_i b_i$. The above canonical isomorphism takes a vector $v \in \mathbf{R}^n_p$ to an element $\omega \in (\mathbf{R}^n_p)^*$ given by $\omega(u) = \langle v, u \rangle$, for all $u \in \mathbf{R}^n_p$. If we let the point p vary, this establishes a one-to-one correspondence between vector fields in $\mathbf{R}^n$ and exterior 1-forms in $\mathbf{R}^n$; it is easily seen that this correspondence takes differentiable vector fields into differential 1-forms and conversely.

EXERCISES

1) Prove that a bilinear form $\varphi: \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ is alternate if and only if $\varphi(v, v) = 0$, for all $v \in \mathbf{R}^3$.
2) Prove that if $i_1 < i_2 < \ldots < i_k$ and $j_1 < j_2 < \ldots < j_k$, then

$$(dx_{i_1} \wedge \ldots \wedge dx_{i_k})(e_{j_1}, \ldots, e_{j_k}) = \begin{cases} 1, & \text{if } i_1 = j_1, \ldots, i_k = j_k, \\ 0, & \text{otherwise.} \end{cases}$$