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Hans Triebel

Theory of Function Spaces III

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Preface

This book may be considered as the continuation of the monographs [Triβ] and [Triγ] with the same title. It deals with the theory of function spaces of type B^s_{pq} and F^s_{pq} as it stands at the beginning of this century. These two scales of spaces cover many well-known spaces of functions and distributions such as Hölder-Zygmund spaces, (fractional and classical) Sobolev spaces, Besov spaces and Hardy spaces.

On the one hand this book is essentially self-contained. On the other hand we concentrate principally on those developments in recent times which are related to the nowadays numerous applications of function spaces to some neighboring areas such as numerics, signal processing and fractal analysis, to mention only a few of them.

Chapter 1 in [Triγ] is a self-contained historically-oriented survey of the function spaces considered and their roots up to the beginning of the 1990s entitled

How to measure smoothness.

Chapter 1 of the present book has the same heading indicating continuity. As far as the history is concerned we will now be very brief, restricting ourselves to the essentials needed to make this book self-contained and readable. We complement [Triγ], Chapter 1, by a few points omitted there. But otherwise we jump to the 1990s, describing more recent developments. Some of them will be treated later on in detail. In other words, [Triγ], Chapter 1, and Chapter 1 of the present book complement each other, providing a sufficiently comprehensive picture of the theory of the spaces B^s_{pq} and F^s_{pq} and their roots from the beginning up to our time. But quite obviously as far as very recent topics are concerned we are somewhat selective, emphasizing those developments which are near to our own interests.

This book has 9 chapters. Chapter 1 is the indicated self-contained survey.

Chapters 2 and 3 deal with building blocks in (isotropic) spaces of type B^s_{pq} and F^s_{pq} in $\mathbb{R}^n$, especially with (non-smooth) atoms (Chapter 2) and with wavelet bases and wavelet frames (Chapter 3). We discuss some consequences: pointwise multiplier assertions, positivity properties and local smoothness problems.

In recent times there is a growing interest in function spaces in (bounded) Lipschitz domains in $\mathbb{R}^n$. Here we split our presentation, collecting some old and a few new results in the introductory Section 1.11 and returning to this subject in greater detail in Chapter 4.

Wavelet representations of anisotropic function spaces and of weighted function spaces on $\mathbb{R}^n$ will be treated in Chapters 5 and 6, respectively.

Chapter 7 might be considered as the direct continuation of our studies in [Triδ] and [Triϵ] about fractal quantities of measures and spectral assertions of fractal elliptic operators.

Finally in Chapters 8 and 9 we develop a new theory for function spaces on quasi-metric spaces and on sets.

Formulas are numbered within the nine chapters. Furthermore, within each of these chapters all definitions, theorems, propositions, corollaries, remarks and examples are jointly and consecutively numbered. Chapter n is divided in subsections $n.k$, which occasionally are subdivided in subsubsections $n.k.l$. But when quoted we refer simply to Section $n.k$ or Section $n.k.l$ instead of Subsection $n.k$ or Subsubsection $n.k.l$, respectively.

It is a pleasure to acknowledge the great help I have received from my colleagues and friends round the world who made valuable suggestions which have been incorporated in the text. This applies in particular to Chapter 1 of this book. I am especially indebted to Dorothee D. Haroske for her remarks and for producing all the figures. Last, but not least, I wish to thank my friend David Edmunds in Brighton who looked through the whole manuscript and offered many comments.

Jena, Spring 2006 Hans Triebel

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Chapter 1

How to Measure Smoothness

1.1 Introduction

This chapter has the same title as the historically-oriented survey in [Triγ], Chapter 1. But our aim now is somewhat different. As far as the background is concerned we will be very brief, restricting ourselves to the bare minimum and referring to [Triγ], Chapter 1, for more details. We are now mainly interested in a description of the theory of function spaces from the 1990s up to our time. Quite obviously we are somewhat selective, emphasizing topics of our own interest. Furthermore we prepare to some extent what follows in the subsequent chapters.

The function spaces B^s_{pq} and F^s_{pq} on $\mathbb{R}^n$ and on domains with respect to the full range of the parameters

$$s \in \mathbb{R}, \quad 0 < p \leq \infty, \quad 0 < q \leq \infty, \tag{1.1}$$

were introduced between 1959 and 1975. They cover many well-known classical concrete function spaces having their own history. In Section 1.2 we give a corresponding short list. These two scales of spaces and their special cases attracted a lot of attention and have been treated systematically with numerous applications given. We mention in particular the following books, reflecting also the development of this theory: [Sob50], [Nik77] (first edition 1969), [Ste70], [BIN75], [Pee76], [Triα] (1978), [Triβ] (1983) and [Triγ] (1992). Special aspects but related to our intentions have been studied in [AdF03] (Sobolev spaces; first edition 1975), [Maz85] (Sobolev spaces), [Zie89] (Sobolev spaces) and [ST87] (periodic spaces, anisotropic spaces and spaces with dominating mixed smoothness). The two surveys [BKLN88] and [KuN88] cover in particular the Russian literature. More recent developments of the spaces B^s_{pq} and F^s_{pq} in the last decade may be found in [ET96], [RuS96], [AdH96], [Triδ] (1997), [Triϵ] (2001), [HeN04] and [Har06]. More detailed references especially to the original papers may be found in [Triγ], Chapter 1.

The recent theory of the above function spaces is characterised by the extensive use of building blocks such as atoms, quarks, and wavelets. Hence it seems to be appropriate to complement the above literature by some more specific references. Atomic decompositions of the spaces B^s_{pq} and F^s_{pq} go back to [FrJ85] and [FrJ90]. Descriptions are also given in [FJW91], [Tor91], [Triγ], [ET96], and [Triδ], Section 13. The theory of subatomic or quarkonial decompositions has been developed in [Triδ] and, in greater detail, in [Triϵ]. Wavelet expansions (bases or frames) are a fashionable subject, preferably with respect to L_2-spaces or L_p-spaces where $1 < p < \infty$. Other types of function spaces such as classical Sobolev or Besov spaces are also treated but not as a major topic. We refer to [Mey92], [Dau92] and [Woj97]. In this book the theory of diverse building blocks such as atoms, quarks, wavelet bases and wavelet frames, and its applications to some problems of the spaces B^s_{pq} and F^s_{pq} play a central role, both in this introductory survey and in the subsequent chapters.

1.2 Concrete spaces

The systematic study in this book begins with Chapter 2. Then we collect the notation needed in the sequel in detail. On this somewhat preliminary basis we list a few special cases of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ without further comments. In particular we postpone the (Fourier-analytical) definition of these spaces to the following Section 1.3. Our aim is twofold. First we wish to substantiate what has been said in Section 1.1. Secondly, as far as the classical function spaces are concerned we fix our notation. Of course, $\mathbb{R}^n$ is Euclidean n-space and $L_p(\mathbb{R}^n)$ is the usual complex Lebesgue space with respect to Lebesgue measure. Otherwise we use standard notation. In case of doubt one might consult the list of symbols at the end of the book and the references given there. We will be brief. More details may be found in [Triβ], especially Section 2.2.2, pp. 35–38, and [Triγ], especially Chapter 1.

(i) Let $1 < p < \infty$. Then

$$F^0_{p2}(\mathbb{R}^n) = L_p(\mathbb{R}^n). \tag{1.2}$$

This is a Paley-Littlewood theorem, see [Triβ], Section 2.5.6, pp. 87–88.

(ii) Let $1 < p < \infty$ and $s \in \mathbb{N}_0$. Then

$$F^s_{p2}(\mathbb{R}^n) = W^s_p(\mathbb{R}^n) \tag{1.3}$$

are the *classical Sobolev spaces*, usually normed by

$$\|f\,|W^s_p(\mathbb{R}^n)\| = \left(\sum_{|\alpha|\le s} \|D^\alpha f\,|L_p(\mathbb{R}^n)\|^p\right)^{\frac{1}{p}}. \tag{1.4}$$

This generalises assertion (i). We refer again to [Triβ], Section 2.5.6, pp. 87–88.

(iii) Let $\sigma \in \mathbb{R}$. Then

$$I_\sigma : \quad f \mapsto \left(\langle \xi \rangle^\sigma \widehat{f} \right)^\vee , \tag{1.5}$$

with $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, is a one-to-one map of the Schwartz space $S(\mathbb{R}^n)$ onto itself and of the space of tempered distributions $S'(\mathbb{R}^n)$ onto itself. Here $\widehat{f}$ and $f^\vee$ are the Fourier transform of f and its inverse, respectively. Then I_σ is a lift for the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ for the F-scale), $0 < q \le \infty$:

$$I_\sigma B^s_{pq}(\mathbb{R}^n) = B^{s-\sigma}_{pq}(\mathbb{R}^n) \quad \text{and} \quad I_\sigma F^s_{pq}(\mathbb{R}^n) = F^{s-\sigma}_{pq}(\mathbb{R}^n) \tag{1.6}$$

(equivalent quasi-norms), [Triβ], Section 2.3.8, pp. 58–59. In particular, let

$$H^s_p(\mathbb{R}^n) = I_{-s} L_p(\mathbb{R}^n), \qquad s \in \mathbb{R}, \quad 1 < p < \infty. \tag{1.7}$$

Then one gets by (1.2), (1.3), and (1.6),

$$H^s_p(\mathbb{R}^n) = F^s_{p2}(\mathbb{R}^n), \qquad s \in \mathbb{R}, \quad 1 < p < \infty, \tag{1.8}$$

and

$$H^s_p(\mathbb{R}^n) = W^s_p(\mathbb{R}^n) \quad \text{if} \quad s \in \mathbb{N}_0 \quad \text{and} \quad 1 < p < \infty. \tag{1.9}$$

We call $H^s_p(\mathbb{R}^n)$ *Sobolev spaces* (sometimes denoted as fractional Sobolev spaces or Bessel potential spaces) and its special cases (1.9) with (1.4) *classical Sobolev spaces.*

(iv) We denote

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n), \qquad s \in \mathbb{R}, \tag{1.10}$$

as *Hölder-Zygmund spaces.* Let

$$\left(\Delta^1_h f\right)(x) = f(x+h) - f(x), \quad \left(\Delta^{l+1}_h f\right)(x) = \Delta^1_h \left(\Delta^l_h f\right)(x), \tag{1.11}$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $l \in \mathbb{N}$, be the iterated differences in $\mathbb{R}^n$. Let $0 < s < m \in \mathbb{N}$. Then

$$\|f \,|\mathcal{C}^s(\mathbb{R}^n)\|_m = \sup_{x \in \mathbb{R}^n} |f(x)| + \sup |h|^{-s} \left|\Delta^m_h f(x)\right| \tag{1.12}$$

where the second supremum is taken over all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$ with $0 < |h| \le 1$, are equivalent norms in $\mathcal{C}^s(\mathbb{R}^n)$. For more details we refer again to [Triβ], Sections 2.2.2, 2.5.12. Hence if $s > 0$ then $\mathcal{C}^s(\mathbb{R}^n)$ are the well-known *Hölder-Zygmund spaces.* We extend this notation to all $s \in \mathbb{R}$.

(v) Assertion (iv) can be generalised as follows. Once more let $0 < s < m \in \mathbb{N}$ and $1 \le p \le \infty$, $1 \le q \le \infty$. Then

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|_m = \|f \,|L_p(\mathbb{R}^n)\| + \left(\int\limits_{|h| \le 1} |h|^{-sq} \, \|\Delta^m_h f \,|L_p(\mathbb{R}^n)\|^q \, \frac{dh}{|h|^n} \right)^{1/q} \tag{1.13}$$

(with the usual modification if $q = \infty$) are equivalent norms in $B^s_{pq}(\mathbb{R}^n)$. As for details we refer to [Triβ], Sections 2.2.2, 2.5.12. These are the *classical Besov spaces.*

Remark 1.1. There are further concrete spaces which fit in the scales $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. For example, the (inhomogeneous) *Hardy spaces* $h_p(\mathbb{R}^n)$ with $0 < p < \infty$ can be identified with $F^0_{p2}(\mathbb{R}^n)$. Furthermore for all of the above spaces one has numerous equivalent norms and characterisations. We refer to the literature in Section 1.1 and in particular to [Triα], [Triβ], [Triγ].

1.3 The Fourier-analytical approach

Recall that this introductory first chapter should be seen in continuation of Chapter 1 in [Triγ] with the same title. We do not repeat the history presented there. Just on the contrary, we restrict ourselves to those ingredients needed later on and which are the basis of the theory of the spaces B^s_{pq} and F^s_{pq} up to recent times. In particular from now onwards we incorporate immediately distinguished results of the last decade.

We use now standard notation which will be detailed later on beginning with Chapter 2. In case of doubt one may consult the list of symbols at the end of the book and the references given there. In particular, $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ are the Schwartz space of all complex-valued rapidly decreasing C^∞ functions on $\mathbb{R}^n$, and the dual space of all tempered distributions. The Fourier transform of $\varphi \in S(\mathbb{R}^n)$ is denoted by $\widehat{\varphi}$ or $F\varphi$. As usual, $\varphi^\vee$ and $F^{-1}\varphi$ stand for the inverse Fourier transform. Both F and F^{-1} are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(y) = 0 \text{ if } |y| \ge 3/2, \tag{1.14}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \qquad x \in \mathbb{R}^n, \quad k \in \mathbb{N}. \tag{1.15}$$

Then, since

$$1 = \sum_{j=0}^{\infty} \varphi_j(x) \qquad \text{for all} \quad x \in \mathbb{R}^n, \tag{1.16}$$

the φ_j form a dyadic resolution of unity in $\mathbb{R}^n$. Recall that $(\varphi_j \widehat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$. In particular, $(\varphi_j \widehat{f})^\vee(x)$ makes sense pointwise.

Definition 1.2. *Let* $\varphi = \{\varphi_j\}_{j=0}^\infty$ *be the dyadic resolution of unity according to* (1.14)–(1.16).

(i) *Let* $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, *and*

$$\|f\,|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq}\,\|(\varphi_j \widehat{f})^\vee\,|L_p(\mathbb{R}^n)\|^q\right)^{1/q} \tag{1.17}$$

(with the usual modification if $q = \infty$ *). Then*

$$B^s_{pq}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \;\; \|f\,|B^s_{pq}(\mathbb{R}^n)\|_\varphi < \infty \right\}. \tag{1.18}$$

(ii) *Let* $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, *and*

$$\|f\,|F^s_{pq}(\mathbb{R}^n)\|_\varphi = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq}|(\varphi_j \widehat{f})^\vee(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\right\| \tag{1.19}$$

(with the usual modification if $q = \infty$*). Then*

$$F^s_{pq}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \;\; \|f\,|F^s_{pq}(\mathbb{R}^n)\|_\varphi < \infty \right\}. \tag{1.20}$$

Remark 1.3. The history of these definitions may be found in [Triγ], Section 1.5, especially on p. 29, which will not be repeated here. Some distinguished special cases have been listed in the preceding Section 1.2. The huge corresponding literature, mostly books, may be found in Section 1.1. A systematic study of these spaces in the above generality has been given in [Triβ], [Triγ], and more recently in [Triδ], [Triϵ], including many references.

It is convenient to complement these definitions by some maximal functions. Again let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the above resolution of unity. Then we introduce the maximal functions

$$\left(\varphi^*_{j,a} f\right)(x) = \sup_{y \in \mathbb{R}^n} \frac{|(\varphi_j \widehat{f})^\vee(x-y)|}{1 + |2^j y|^a}, \qquad f \in S'(\mathbb{R}^n), \quad a > 0. \tag{1.21}$$

Maximal functions play a crucial role in diverse types of function spaces as demonstrated in [Ste93]. The above version and its use in connection with the spaces introduced in the definition goes back to J. Peetre, [Pee75], [Pee76]. But otherwise we refer to [Triβ], [Triγ] for history and literature. Recall that (1.21) always makes sense, accepting that the right-hand side might be infinite. More precisely: Let $\varphi \in S(\mathbb{R}^n)$ and $f \in S'(\mathbb{R}^n)$. Then for $x \in \mathbb{R}^n$,

$$\begin{aligned}(f * \varphi)(x) &= f\left(\varphi(x - \cdot)\right) \\ &= \int_{\mathbb{R}^n} \varphi(x-y)\, f(y)\, \mathrm{d}y = \int_{\mathbb{R}^n} \varphi(y)\, f(x-y)\, \mathrm{d}y \end{aligned} \tag{1.22}$$