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Daniel Huybrechts

Complex Geometry

An Introduction

复几何导论

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Preface

Complex geometry is a highly attractive branch of modern mathematics that has witnessed many years of active and successful research and that has recently obtained new impetus from physicists' interest in questions related to mirror symmetry. Due to its interactions with various other fields (differential, algebraic, and arithmetic geometry, but also string theory and conformal field theory), it has become an area with many facets. Also, there are a number of challenging open problems which contribute to the subject's attraction. The most famous among them is the Hodge conjecture, one of the seven one-million dollar millennium problems of the Clay Mathematics Institute. So, it seems likely that this area will fascinate new generations for many years to come.

Complex geometry, as presented in this book, studies the geometry of (mostly compact) complex manifolds. A complex manifold is a differentiable manifold endowed with the additional datum of a complex structure which is much more rigid than the geometrical structures in differential geometry. Due to this rigidity, one is often able to describe the geometry of complex manifolds in very explicit terms. E.g. the important class of projective manifolds can, in principle, be described as zero sets of polynomials.

Yet, a complete classification of all compact complex manifolds is too much to be hoped for. Complex curves can be classified in some sense (involving moduli spaces etc.), but already the classification of complex surfaces is tremendously complicated and partly incomplete.

In this book we will concentrate on more restrictive types of complex manifolds for which a rather complete theory is in store and which are also relevant in the applications. A prominent example are Calabi–Yau manifolds, which play a central role in questions related to mirror symmetry. Often, interesting complex manifolds are distinguished by the presence of special Riemannian metrics. This will be one of the central themes throughout this text. The idea is to study cases where the Riemannian and complex geometry on a differentiable manifold are not totally unrelated. This inevitably leads to

Kähler manifolds, and a large part of the book is devoted to techniques suited for the investigation of this prominent type of complex manifolds.

The book is based on a two semester course taught in 2001/2002 at the university of Cologne. It assumes, besides the usual facts from the theory of holomorphic functions in one variable, the basic notions of differentiable manifolds and sheaf theory. For the convenience of the reader we have summarized those in the appendices A and B. The aim of the course was to introduce certain fundamental concepts, techniques, and results in the theory of compact complex manifolds, without being neither too basic nor too sketchy.

I tried to teach the subject in a way that would enable the students to follow recent developments in complex geometry and in particular some of the exciting aspects of the interplay between complex geometry and string theory. Thus, I hope that the book will be useful for both communities, those readers aiming at understanding and doing research in complex geometry and those using mathematics and especially complex geometry in mathematical physics.

Some of the material was intended rather as an outlook to more specialized topics, and I have added those as appendices to the corresponding chapters. They are not necessary for the understanding of the subsequent sections.

I am aware of several shortcomings of this book. As I found it difficult to teach the deeper aspects of complex analysis to third-year students, the book cannot serve as an introduction to the fascinating program initiated by Siu, Demailly, and others, that recently has lead to important results in complex and algebraic geometry. So, for the analysis I have to refer to Demailly's excellent forthcoming (?) text book [35]. I also had to leave out quite a number of important tools, like higher direct image sheaves, spectral sequences, intermediate Jacobians, and others. The hope was to create a streamlined approach to some central results and so I did not want to enter too many of the promising side-roads. Finally, although relevant examples have been included in the text as well as in the exercises, the book does not discuss in depth any difficult type of geometry, e.g. Calabi–Yau or hyperkähler manifolds. But I believe that with the book at hand, it should not be too difficult to understand more advanced texts on special complex manifolds.

Besides Demailly's book [35], there are a number of text books on complex geometry, Hodge theory, etc. The classic [59] and the more recent one by Voisin [113] are excellent sources for more advanced reading. I hope that this book may serve as a leisurely introduction to those.

In the following, we will give an idea of the content of the book. For more information, the reader may consult the introductions at the beginning of each chapter.

Chapter 1 provides the minimum of the local theory needed for the global description of complex manifolds. It may be read along with the later chapters or worked through before diving into the general theory of complex manifolds beginning with Chapter 2.

Section 1.1 shows a way from the theory of holomorphic functions of one variable to the general theory of complex functions. Eventually, it would lead to the local theory of complex spaces, but we restrict ourselves to those aspects strictly necessary for the understanding of the rest of the book. The reader interested in this attractive combination of complex analysis and commutative algebra may consult [35] or any of the classics, e.g. [57, 64].

Section 1.2 is a lesson in linear algebra and as such rather elementary. We start out with a real vector space endowed with a scalar product and the additional datum of an almost complex structure. We shall investigate what kind of structure is induced on the exterior algebra of the vector space. I tried to present the material with some care in order to make the reader feel comfortable when later, in the global context, the machinery is applied to compact Kähler manifolds.

Section 1.3 proves holomorphic versions of the Poincaré lemma and is supposed to accustom the reader to the yoga of complex differential forms on open sets of $\mathbb{C}^n$.

With **Chapter 2** the story begins. Sections 2.1 and 2.2 deal with complex manifolds and holomorphic vector bundles, both holomorphic analogues of the corresponding notions in real differential geometry. But a few striking differences between the real and the complex world will become apparent right away. The many concrete examples of complex manifolds are meant to motivate the discussion of the more advanced techniques in the subsequent chapters.

Section 2.3 illuminates the intimate relation between complex codimension one submanifolds (or, more generally, divisors) and holomorphic line bundles with their global sections. This builds the bridge to classical algebraic geometry, e.g. Veronese and Segre embedding are discussed. The section ends with a short discussion of the curve case. .

Section 2.4 is devoted to the complex projective space $\mathbb{P}^n$, a universal object in complex (algebraic) geometry comparable to spheres in the real world. We describe its tangent bundle by means of the Euler sequence and certain tautological line bundles. A discussion of the Riemannian structure of $\mathbb{P}^n$ (e.g. the Fubini–Study metric) is postponed until Section 3.1.

Section 2.5 provides an example of the universal use of the projective space. It explains a complex surgery, called blow-up, which modifies a given complex manifold along a distinguished complex submanifold, replacing a point by a projective space. Apart from its importance in the birational classification of complex manifolds, blow-ups will turn out to be of use in the proof of the Kodaira embedding theorem in Section 5.2.

Section 2.6 interprets complex manifolds as differentiable manifolds together with an additional linear datum (an almost complex structure) satisfying an integrability condition. Here, the linear algebra of Section 1.2 comes in handy. The crucial Newlander–Nierenberg theorem, asserting the equivalence of the two points of view, is formulated but not proved.

Chapter 3 is devoted to (mostly compact) Kähler manifolds. The existence of a Kähler metric on a compact complex manifold has far reaching consequences for its cohomology. Behind most of the results on Kähler manifolds one finds the so-called Kähler identities, a set of commutator relations for the various differential and linear operators. They are the topic of Section 3.1.

In Section 3.2, Hodge theory for compact manifolds is used to pass from arbitrary forms to harmonic forms and eventually to cohomology classes. This immediately yields central results, like Serre duality and, in Section 3.3, Lefschetz decomposition.

Section 3.3 also explains how to determine those classes in the second cohomology $H^2(X)$ of a compact Kähler manifold X that come from holomorphic line bundles. This is the Lefschetz theorem on $(1,1)$-classes. A short introduction to the hoped for generalization to higher degree cohomology classes, i.e. the Hodge conjecture, ends this section.

There are three appendices to Chapter 3. Appendix 3.A proves the formality of compact Kähler manifolds, a result that interprets the crucial $\partial\bar{\partial}$-lemma of Section 3.2 homologically. Appendix 3.B is a first introduction to some mathematical aspects of supersymmetry. The cohomological structures encountered in the bulk of the chapter are formalized by the notion of a Hodge structure. Appendix 3.C collects a few basic notions and explains how they fit in our context.

Chapter 4 provides indispensable tools for the study of complex manifolds: connections, curvature, and Chern classes. In contrast to previous sections, we will not just study complex manifolds and their tangent bundles but broaden our perspective by considering arbitrary holomorphic vector bundles. However, we will not be in the position to undertake an indepth analysis of all fundamental questions. E.g. the question whether there exist holomorphic vector bundles besides the obvious ones on a given manifold (or holomorphic structures on a given complex vector bundle) will not be addressed. This is partially due to the limitations of the book, but also to the state of the art. Only for curves and projective surfaces the situation is fairly well understood (see [70]).

In the appendices to Chapter 4 we discuss the interplay of complex and Riemannian geometry. Appendix 4.A tries to clarify the relation between the Levi-Civita connection and the Chern connection on a Kähler manifold. The concept of holonomy, well known in classical Riemannian geometry, allows to view certain features in complex geometry from a slightly different angle. Appendix 4.B outlines some basic results about Kähler–Einstein and Hermite–Einstein metrics. Before, the hermitian structure on a holomorphic vector bundle was used as an auxiliary in order to apply Hodge theory, etc. Now, we ask whether canonical hermitian structures, satisfying certain compatibility conditions, can be found.

In order to illustrate the power of cohomological methods, we present in **Chapter 5** three central results in complex algebraic geometry. Except for the

Hirzebruch–Riemann–Roch theorem, complete proofs are given, in particular for Kodaira's vanishing and embedding theorems. The latter one determines which compact complex manifolds can be embedded into a projective space. All three results are of fundamental importance in the global theory of complex manifolds.

Chapter 6 is relevant to the reader interested in Calabi–Yau manifolds and mirror symmetry. It is meant as a first encounter with deformation theory, a notoriously difficult and technical subject. In Section 6.1 we leave aside convergence questions and show how to study deformations by a power series expansion leading to the Maurer–Cartan equation. This approach can successfully be carried out for compact Kähler manifolds with trivial canonical bundle (Calabi–Yau manifolds) due to the Tian–Todorov lemma. Section 6.2 surveys the more abstract parts of deformation theory, without going into detail. The appendix to this chapter is very much in the spirit of appendix 3.A. Here, the content of Section 6.1 is put in the homological language of Batalin–Vilkovisky algebras, a notion that has turned out to be useful in the construction of Frobenius manifolds and in the formulation of mirror symmetry.

In general, all results are proved except for assertions presented as 'theorems', indicating that they are beyond the scope of this book, and a few rather sketchy points in the various appendices to the chapters. Certain arguments, though, are relegated to the exercises, not because I wanted to leave unpleasant bits to the reader, but because sometimes it is just more rewarding performing a computation on ones own.

Acknowledgement: I learned much of the material from the two classics [8, 59] and from my teacher H. Kurke. Later, the interplay of algebraic geometry and gauge theory as well as the various mathematical aspects of mirror symmetry have formed my way of thinking about complex geometry. The style of the presentation has been influenced by stimulating discussions with D. Kaledin, R. Thomas, and many others over the last few years.

I want to thank G. Hein, M. Nieper-Wißkirchen, D. Ploog, and A. Schmidt, who read preliminary versions of the text and came up with long lists of comments, corrections, and suggestions. Due to their effort, the text has considerably improved.

Paris, June 2004 *Daniel Huybrechts*

Contents

1

Local Theory

This chapter consists of three sections. Section 1.1 collects the principal facts from the theory of holomorphic functions of several variables. We assume that the reader has mastered the theory of holomorphic functions of one variable, but the main results shall be briefly recalled.

Section 1.2 is pure linear algebra. The reader may skip this part, or skim through it, and come back to it whenever feeling uncomfortable about certain points in the later chapters. I tried to present the material with great care. In particular, the interplay between the Hodge and Lefschetz operators is explained with all the details.

In Section 1.3 the techniques of the previous two sections are merged. The reader will be introduced to the theory of complex differential forms on open subsets of $\mathbb{C}^n$. This gives him the opportunity to do some explicit calculations before these notions will be reconsidered in the global context. The central result of this section is the $\bar{\partial}$-Poincaré lemma.

1.1 Holomorphic Functions of Several Variables

Let us first recall some basic facts and definitions from the theory of holomorphic functions of one variable. For proofs and further discussion the reader may consult any textbook on the subject, e.g. [98].

Let $U \subset \mathbb{C}$ be an open subset. A function $f : U \to \mathbb{C}$ is called *holomorphic* if for any point $z_0 \in U$ there exists a ball $B_\varepsilon(z_0) \subset U$ of radius $\varepsilon > 0$ around z_0 such that f on $B_\varepsilon(z_0)$ can be written as a convergent power series, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ for all } z \in B_\varepsilon(z_0). \tag{1.1}$$

There are equivalent definitions of holomorphicity. The most important one uses the *Cauchy–Riemann equations*. Let us denote the real and imaginary part of $z \in \mathbb{C}$ by x respectively y. Thus, f can be regarded as a complex function $f(x,y)$ of two real variables x and y. Furthermore, f can be written in the form $f(x,y) = u(x,y) + iv(x,y)$, where $u(x,y)$ and $v(x,y)$ denote the real and imaginary part of f, respectively. Then one shows that f is holomorphic if and only if u and v are continuously differentiable and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.2}$$

In other words, the derivative of f has to be complex linear. Let us introduce the differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{1.3}$$

The notation is motivated by the properties $\frac{\partial}{\partial z}(z) = \frac{\partial}{\partial \bar{z}}(\bar{z}) = 1$ and $\frac{\partial}{\partial z}(\bar{z}) = \frac{\partial}{\partial \bar{z}}(z) = 0$. Then, the Cauchy–Riemann equations (1.2) can be rewritten as $\frac{\partial f}{\partial \bar{z}} = 0$. This is easy if one uses $f = u + iv$. It might be instructive to do the same calculation for f written as the vector $\binom{u}{v}$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\frac{\partial}{\partial y})$.

As the transition from the real partial differentials $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ to the complex partial differentials $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ is a crucial point, let us discuss this a little further. Consider a differentiable map $f : U \subset \mathbb{C} = \mathbb{R}^2 \to \mathbb{C} = \mathbb{R}^2$. Its differential $df(z)$ at a point $z \in U$ is an $\mathbb{R}$-linear map between the tangent spaces $df(z) : T_z\mathbb{R}^2 \to T_{f(z)}\mathbb{R}^2$. Writing the complex coordinate on the left hand side as $z = x + iy$ and on the right hand side as $w = r + is$ the two tangent spaces can be given canonical bases $\langle \partial/\partial x, \partial/\partial y \rangle$ and $\langle \partial/\partial r, \partial/\partial s \rangle$, respectively. With respect to these the differential $df(z)$ is given by the real Jacobian

$$J_{\mathbb{R}}(f) = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix},$$

where $f = u + iv$ as before, i.e. $u = r \circ f$ and $v = s \circ f$.

After extending $df(z)$ to a $\mathbb{C}$-linear map $df(z)_{\mathbb{C}} : T_z\mathbb{R}^2 \otimes \mathbb{C} \to T_{f(z)}\mathbb{R}^2 \otimes \mathbb{C}$, we may choose different bases $\langle \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})\rangle$ and correspondingly for the right hand side. With respect to those $df(z)$ is given by the matrix

$$\begin{pmatrix} \dfrac{\partial f}{\partial z} & \dfrac{\partial f}{\partial \bar{z}} \\ \\ \dfrac{\partial \bar{f}}{\partial z} & \dfrac{\partial \bar{f}}{\partial \bar{z}} \end{pmatrix}.$$

E.g. the vector $\frac{\partial}{\partial z}$ is sent to the vector $\frac{\partial f}{\partial z}\cdot\frac{\partial}{\partial w}+\frac{\partial \bar{f}}{\partial z}\cdot\frac{\partial}{\partial \bar{w}}$. For the chain rule it would be more natural to change the order of $\frac{\partial}{\partial w}$ and $\frac{\partial f}{\partial z}$ (and of $\frac{\partial}{\partial \bar{w}}$ and $\frac{\partial \bar{f}}{\partial z}$). In the following, we will use that for any function f one has $\frac{\partial \bar{f}}{\partial \bar{z}}=\overline{\left(\frac{\partial f}{\partial z}\right)}$. If f is holomorphic, then $\frac{\partial f}{\partial \bar{z}}=\frac{\partial \bar{f}}{\partial z}=0$ and thus $df(z)$ in the new base is given by the diagonal matrix

$$\begin{pmatrix} \frac{\partial f}{\partial z} & 0 \\ 0 & \frac{\partial \bar{f}}{\partial \bar{z}} \end{pmatrix}.$$

Holomorphicity of f is also equivalent to the *Cauchy integral formula.* More precisely, a function $f: U \to \mathbb{C}$ is holomorphic if and only if f is continuously differentiable and for any $B_\varepsilon(z_0) \subset U$ the following formula holds true

$$f(z_0)=\frac{1}{2\pi i}\int_{\partial B_\varepsilon(z_0)}\frac{f(z)}{z-z_0}dz. \tag{1.4}$$

Actually, the formula holds true for any continuous function $f: \overline{B_\varepsilon(z_0)} \to \mathbb{C}$ which is holomorphic in the interior. Let us remind that the Cauchy integral formula is used to prove the existence of a power series expansion of any function satisfying the Cauchy–Riemann equations. (If f is just continuous, one only has $f(z_0)=(1/2\pi i)\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(z_0)} f(z)/(z-z_0)dz$.)

The following list collects a few well-known facts, which will be important for our purposes.

Maximum principle. Let $U \subset \mathbb{C}$ be open and connected. If $f: U \to \mathbb{C}$ is holomorphic and non-constant, then $|f|$ has no local maximum in U. If U is bounded and f can be extended to a continuous function $f: \overline{U} \to \mathbb{C}$, then $|f|$ takes its maximal values on the boundary ∂U.

Identity theorem. If $f, g: U \to \mathbb{C}$ are two holomorphic functions on a connected open subset $U \subset \mathbb{C}$ such that $f(z)=g(z)$ for all z in a non-empty open subset $V \subset U$, then $f=g$. There are stronger versions of the identity theorem, but in this form it immediately generalizes to higher dimensions.

Riemann extension theorem. Let $f: B_\varepsilon(z_0)\setminus\{z_0\} \to \mathbb{C}$ be a bounded holomorphic function. Then f can be extended to a holomorphic function $f: B_\varepsilon(0) \to \mathbb{C}$.

Riemann mapping theorem. Let $U \subset \mathbb{C}$ be a simply connected proper open subset. Then U is biholomorphic to the unit ball $B_1(0)$, i.e. there exists a bijective holomorphic map $f: U \to B_1(0)$ such that its inverse f^{-1} is also holomorphic.

Liouville. Every bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant. In particular, there is no biholomorphic map between $\mathbb{C}$ and a ball $B_\varepsilon(0)$ with $\varepsilon < \infty$. This is a striking difference to the real situation and will cause a different concept of locality for complex manifolds than the one we are used to from real differential geometry.

Residue theorem. Let $f : B_\varepsilon(0) \setminus \{0\} \to \mathbb{C}$ be a holomorphic function. Then f can be expanded in a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and the coefficient a_{-1} is given by the residue formula $a_{-1} = (1/2\pi i) \int_{|z|=\varepsilon/2} f(z)dz$. The residue theorem is usually applied to more general situations where the function f has several isolated singularities in a connected open subset and the integral is taken over a closed contractible path surrounding the singularities.

The notion of a holomorphic function of one variable can be extended in two different ways. Firstly, one can consider functions of several variables $\mathbb{C}^n \to \mathbb{C}$ and, secondly, functions that take values in $\mathbb{C}^n$. As a basis for the topology in higher dimensions we will usually take the *polydiscs* $B_\varepsilon(w) = \{z \mid |z_i - w_i| < \varepsilon_i\}$, where $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_n)$.

Definition 1.1.1 Let $U \subset \mathbb{C}^n$ be an open subset and let $f : U \to \mathbb{C}$ be a continuously differentiable function. Then f is said to be *holomorphic* if the Cauchy–Riemann equations (1.2) holds for all coordinates $z_i = x_i + iy_i$, i.e.

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}, \quad \frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i}, \quad i = 1, \ldots, n. \tag{1.5}$$

(It should be clear that i appears with two different meanings here, as an index and as $\sqrt{-1}$. This is a bit unfortunate, but it will always be clear which one is meant.)

By definition, a continuous(ly differentiable) function f is holomorphic if the induced functions

$$U \cap \{(z_1, \ldots, z_{i-1}, z, z_{i+1}, \ldots, z_n) \mid z \in \mathbb{C}\} \to \mathbb{C}$$

are holomorphic for all choices of i and fixed $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n \in \mathbb{C}$.

Introducing

$$\frac{\partial}{\partial z_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right),$$

(1.5) can be rewritten as

$$\frac{\partial f}{\partial \bar{z}_i} = 0 \text{ for } i = 1, \ldots, n. \tag{1.6}$$

Sometimes all these equations together are written as $\bar{\partial} f = 0$. Later in Section 1.3, a precise meaning will be given to this equation.

The comparison between real and complex Jacobian can be carried over to several variables. This will be discussed shortly.

But before, let us discuss the Cauchy integral formula for functions of several variables and a few central results.

Proposition 1.1.2 *Let* $f : \overline{B_\varepsilon(w)} \to \mathbb{C}$ *be a continuous function such that* f *is holomorphic with respect to every single component* z_i *in any point of* $B_\varepsilon(w)$. *Then for any* $z \in B_\varepsilon(w)$ *the following formula holds true*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_i - w_i| = \varepsilon_i} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1) \ldots (\xi_n - z_n)} d\xi_1 \ldots d\xi_n. \tag{1.7}$$

Proof. Repeated application of the Cauchy integral formula in one variable yields

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_1 - w_1| = \varepsilon_1} \cdots \int_{|\xi_n - w_n| = \varepsilon_n} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1) \ldots (\xi_n - z_n)} d\xi_n \ldots d\xi_1.$$

Since the integrand is continuous on the boundary of $B_\varepsilon(w)$, the iterated integral can be replaced by the multiple integral. This proves the assertion. □

The proposition can easily be applied to show that any continuous(!) function on an open subset $U \subset \mathbb{C}^n$ with the property that the function is holomorphic with respect to any single coordinate is holomorphic itself (Osgood's Lemma, cf. [64]). Clearly the integrand in the above integral is holomorphic as a function of $\xi = (\xi_1, \ldots, \xi_n)$.

As in the one-dimensional case, the integral formula (1.7) can be used to write down a power series expansion of any holomorphic function $f : U \to \mathbb{C}$. More precisely, for any $w \in U$ there exists a polydisc $B_\varepsilon(w) \subset U \subset \mathbb{C}^n$ such that the restriction of f to $B_\varepsilon(w)$ is given by a power series

$$\sum_{i_1, \ldots, i_n = 0}^{\infty} a_{i_1 \ldots i_n} (z_1 - w_1)^{i_1} \ldots (z_n - w_n)^{i_n},$$

with

$$a_{i_1 \ldots i_n} = \frac{1}{i_1! \ldots i_n!} \cdot \frac{\partial^{i_1 + \ldots + i_n} f}{\partial z_1^{i_1} \ldots \partial z_n^{i_n}}.$$

From the above list the maximum principle, the identity theorem, and the Liouville theorem generalize easily to the higher dimensional situation. A version of the Riemann extension theorem holds true, although the proof needs some work. The Riemann mapping theorem definitely fails (see Exercise 1.1.16). There are also some new unexpected features in dimension > 1, e.g. Hartogs' theorem (see Proposition 1.1.4).

Often the holomorphicity of a function of several variables is shown by representing the function as an integral, using residue theorem or Cauchy integral formula, of a function which is known to be holomorphic. For later use we state this principle as a separate lemma.

Lemma 1.1.3 *Let $U \subset \mathbb{C}^n$ be an open subset and let $V \subset \mathbb{C}$ be an open neighbourhood of the boundary of $B_\varepsilon(0) \subset \mathbb{C}$. Assume that $f : V \times U \to \mathbb{C}$ is a holomorphic function. Then*

$$g(z) := g(z_1, \ldots, z_n) := \int_{|\xi|=\varepsilon} f(\xi, z_1, \ldots, z_n) d\xi$$

is a holomorphic function on U.

Proof. Let $z \in U$. If $|\xi| = \varepsilon$ then there exists a polydisc $B_{\delta(\xi)}(\xi) \times B_{\delta'(\xi)}(z) \subset V \times U$ on which f has a power series expansion.

Since $\partial B_\varepsilon(0)$ is compact, we can find a finite number of points $\xi_1, \ldots, \xi_k \in \partial B_\varepsilon(0)$ and positive real numbers $\delta(\xi_1), \ldots, \delta(\xi_k)$ such that

$$\bigcup \left(\partial B_\varepsilon(0) \cap B_{\delta(\xi_i)/2}(\xi_i)\right) \quad \text{is a disjoint union}$$

and

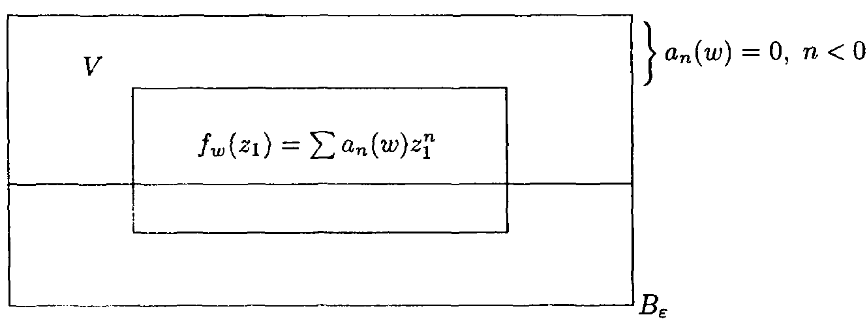
$$\partial B_\varepsilon(0) = \bigcup \left(\partial B_\varepsilon(0) \cap \overline{B_{\delta(\xi_i)/2}(\xi_i)}\right).$$

Hence, $g(z) = \int_{|\xi|=\varepsilon} f(\xi, z_1, \ldots, z_n) d\xi = \sum_{i=1}^k \int_{|\xi|=\varepsilon, |\xi_i - \xi| < \delta(\xi_i)/2} f d\xi$. Each summand is holomorphic, as the power series expansion of f converges uniformly on $\overline{B_{\delta(\xi_i)/2}(\xi_i)}$ and thus commutes with the integral. □

The next result is only valid in dimension at least two.

Proposition 1.1.4 (Hartogs' theorem) *Suppose $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ and $\varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_n)$ are given such that for all i one has $\varepsilon'_i < \varepsilon_i$. If $n > 1$ then any holomorphic map $f : B_\varepsilon(0) \setminus \overline{B_{\varepsilon'}(0)} \to \mathbb{C}$ be can be uniquely extended to a holomorphic map $f : B_\varepsilon(0) \to \mathbb{C}$.*

Proof. We may assume that $\varepsilon = (1, \ldots, 1)$. Moreover, there exists $\delta > 0$ such that the open subset $V := \{z \mid 1 - \delta < |z_1| < 1, |z_{i \neq 1}| < 1\} \cup \{z \mid 1 - \delta < |z_2| < 1, |z_{i \neq 2}| < 1\}$ is contained in the complement of $B_{\varepsilon'}(0)$.



In particular, f is holomorphic on V. Thus, for any $w := (z_2, \ldots, z_n)$ with $|z_i| < 1$ this yields a holomorphic function $f_w(z_1) := f(z_1, z_2, \ldots, z_n)$ on the