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Singular Electromagnetic Fields and Sources

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PREFACE

The present monograph is meant to complement the several excellent texts on electromagnetic theory which are available today. The contents should be of interest to graduate students in electrical engineering and physics, as well as to practising electromagneticists in industrial and academic laboratories. The overall purpose of the book is to discuss, in more detail than would a typical general treatise, the various 'infinities' which occur in electromagnetic fields and sources. To achieve this goal, the text has been divided into three parts.

The first discusses the 'distributional' representation of strongly concentrated charges and currents. It is well-known that a point charge at r_0 can be represented by a volume density $\rho = q\delta|\boldsymbol{r} - \boldsymbol{r}_0|$, i.e. the first term in a *multipole* expansion for ρ. More general sources—scalar or vectorial—require additional terms. These involve derivatives of δ-functions, and are discussed at length in Chapter 2. The analysis there is based on an elementary presentation of Schwartz' Theory of Distributions, given in Chapter 1. In that chapter, as in the rest of the book, the approach is unashamedly that of the 'applied mathematician'.

Multipole expansions can be written for magnetic currents $\boldsymbol{K}$ as well as electric currents $\boldsymbol{J}$. It is well-known that electric currents can be replaced by equivalent magnetic currents, and vice versa. Several sections in Chapter 2 are devoted to an extensive study of these equivalences, particularly with respect to sources which are concentrated on a *surface*.

The second part of the book analyses the *fields* associated with concentrated sources. In the case of a static point charge, potential and electric field have singularities of the order of $1/|\boldsymbol{r} - \boldsymbol{r}_0|$ and $1/|\boldsymbol{r} - \boldsymbol{r}_0|^2$, respectively. When the source is time harmonic, however, stronger singularities occur. For the electric Green's dyadic, for example, they are of the order of $1/|\boldsymbol{r} - \boldsymbol{r}_0|^3$. The way to handle such singularities has generated an abundant—and often controversial—literature. Chapter 3 surveys the various possible approaches, but ultimately puts the accent on the distributional and modal aspects of the theory.

The third part of the book is devoted to an analysis of field behaviour near *geometrical* singularities such as sharp edges, tips of cones, and vertices of sectors. The mathematics involved are quite elementary, and the emphasis is laid on the presentation of numerical data useful to the practising electromagneticist.

The scope of the monograph is seen to be quite modest. It is clear that additional topics could have been included, e.g. a treatment of Green's dyadics in non-homogeneous media (particularly layered ones), or a discussion of field behaviour near foci and caustics. These topics were deliberately left aside to safeguard the compact character of the book.

Many authors mentioned in the text took the trouble to read the paragraphs in which their work was quoted, and to suggest improvements and additions. These distinguished colleagues cannot be thanked individually, given their number. An exception must be made for Professor J. Boersma, whose extensive and critical remarks considerably increased the mathematical accuracy of many a section, particularly in Chapters 3 and 5.

Any formal qualities present in the text should be credited to the author's daughter Viveca, who applied her literary talents to a thorough criticism of the style of the original manuscript.

Finally, the author wishes to acknowledge the support given by his colleague and friend Professor P. E. Lagasse, and the competence with which two devoted secretaries, Mrs Buysse and Mrs Naessens, struggled with figures and equations.

J.V.B.

Ghent
October 1990

LIST OF SYMBOLS

General notation

Standard notations are used for:
—the electromagnetic fields $\boldsymbol{e}$, $\boldsymbol{h}$, $\boldsymbol{d}$, $\boldsymbol{b}$;
—coordinates such as polar coordinates ($\boldsymbol{r}$, φ, z) and spherical coordinates (R, φ, θ);
—special functions, such as Bessel and Hankel functions, and Legendre, Gegenbauer, and Chebychev polynomials;

The imaginary symbol is j, and the harmonic time factor is $e^{j\omega t}$.
Capitals are used to represent complex phasors (e.g. $\boldsymbol{E}$ for $\boldsymbol{e}$).
The Naperian logarithm is denoted by $\log_e$.
A_i denotes an incident quantity
A_{sc} denotes a scattered quantity
$o(1/x)$ and $O(1/x)$: a function $f(x)$ is $o(1/x)$ or $O(1/x)$ depending on whether $\lim_{x\to\infty} x f(x)$ is zero or different from zero (but finite).
$\doteqdot$ means proportional to;
$\approx$ means almost equal to;

$\underset{x}{\sim}$ means asymptotically equal to, for $x \to \infty$.

Symbols

$\boldsymbol{a}$ = magnetic vector potential (see Section 2.6)
$\boldsymbol{c}$ = electric vector potential (see Section 2.6)
$\boldsymbol{c}_s$ = strength of a double layer of current (see Section 1.12)
$\boldsymbol{e}_c$ = "cavity" electric field (see Section 3.18)
$\boldsymbol{e}_m, \boldsymbol{f}_m, \boldsymbol{g}_m, \boldsymbol{h}_m$ = cavity eigenvectors (see Appendix C)
f = frequency
h_i = metrical coefficient (see Section 1.5)
$\boldsymbol{j}$ = volume density of electric current
$\boldsymbol{j}_s$ = surface density of electric current
$k = \omega/c = 2\pi/\lambda$ = wavenumber in vacuum
$\boldsymbol{k}$ = volume density of magnetic current
$\boldsymbol{k}_s$ = surface density of magnetic current
p = acoustic pressure
$\boldsymbol{p}_e$ = electric dipole moment (see Section 2.1)
$\boldsymbol{p}_m$ = magnetic dipole moment (see Section 2.3)
q = electric charge

$\mathbf{q}_e$ = electric quadrupole dyadic (see Section 2.1)
$\mathbf{q}_m$ = magnetic quadrupole dyadic (see Section 2.5)
$\boldsymbol{r} = x\boldsymbol{u}_x + y\boldsymbol{u}_y + z\boldsymbol{u}_z$ = radius vector from the origin
$\boldsymbol{u}_a$ = unit vector in the direction in which parameter a is measured
$\boldsymbol{v}$ = velocity
$G(\boldsymbol{r}|\boldsymbol{r}')$ = a Green's function
$\mathbf{G}(\boldsymbol{r}|\boldsymbol{r}')$ = a Green's dyadic (see Appendix B)
$\mathbf{G}_e(\boldsymbol{r}|\boldsymbol{r}')$ = electric Green's dyadic (see Section 3.14)
$\mathbf{G}_m(\boldsymbol{r}|\boldsymbol{r}')$ = mixed Green's dyadic (see Section 3.12)
$\mathbf{G}_h(\boldsymbol{r}|\boldsymbol{r}')$ = magnetic Green's dyadic (see Section 3.19)
$\mathbf{I}_{xy} = \boldsymbol{u}_x\boldsymbol{u}_x + \boldsymbol{u}_y\boldsymbol{u}_y$ = identity dyadic in the (x, y) plane
$\mathbf{L}_{V^*}$ = depolarization dyadic relative to a volume V^* (see Section 3.10)
$R_c = \sqrt{(\mu_0/\varepsilon_0)}$ = characteristic resistance of vacuum
R_i = principal radius of curvature (see Section 1.5)
tr = trace of a dyadic (see Appendix B)
$\mathrm{Y}(x)$ = unit, step, or Heaviside function (see Section 1.3)
$Y_s(\boldsymbol{r})$ = three-dimensional Heaviside function (see Section 1.10)
γ_m, γ_n = propagation (or damping) constant (see Section 3.24)
$\delta(x)$ = one-dimensional delta-function (see Section 1.1)
$\delta^{(m)}(x) = m^{\text{th}}$ derivative of the delta-function (see Section 1.8)
$\delta(\boldsymbol{r})$ = three-dimensional delta-function (see Section 1.4)
δ_s = Dirac function on a surface (see Section 1.5)
δ_c = Dirac function on a curve (see Section 1.5)
δ_{ik} = Kronecker's delta ($\delta_{ii} = 1$; $\delta_{ik} = 0$ for $i \neq k$)
$\boldsymbol{\delta}_l$ = longitudinal Dirac dyadic (see Section 3.20)
$\boldsymbol{\delta}_t$ = transverse Dirac dyadic (see Section 3.20)
ε_r = dielectric constant (dimensionless)
$\varepsilon_0 = 4\pi\ 10^{-7}$ $(\mathrm{F\,m^{-1}})$
$\varepsilon = \varepsilon_0\varepsilon_r$
μ_r = magnetic permeability (dimensionless)
$\mu_0 = 10^{-9}/36\pi$ $(\mathrm{H\,m^{-1}})$
$\mu = \mu_r\mu_0$
$\omega = 2\pi f$ = angular frequency
π = a Hertz potential (see Section 3.18)
ρ = volume density of electric charge (in $\mathrm{C\,m^{-3}}$)
ρ_s = surface density of electric charge (in $\mathrm{C\,m^{-2}}$)
ρ_c = electric charge density on a curve C (in $\mathrm{C\,m^{-1}}$)
ρ_L = electric charge density on a straight line (in $\mathrm{C\,m^{-1}}$)
$\boldsymbol{\rho} = x\boldsymbol{u}_x + y\boldsymbol{u}_y$ = radius vector from the origin, in the (x, y) plane
σ = conductivity

τ = dipole-layer density on a surface (see Section 1.12)
ϕ_m = Dirichlet eigenfunction (see Appendices C and D)
ψ_n = Neumann eigenfunction (see Appendices C and D)
Ω = a solid angle
$\mathscr{D}$ = space of test functions (see Section 1.2)
$\mathscr{D}'$ = space of distributions (see Section 1.3)
$\mathscr{E}$ = an energy per unit length (see Section 4.2)

Operators

$\text{div}_s \boldsymbol{j}_s$ = surface divergence of a tangential vector (see Section 1.15)
$\text{grad}_s A$ = surface gradient of a scalar function (see Section 2.13)

CONTENTS

1

DELTA-FUNCTIONS AND DISTRIBUTIONS

The delta-function and its derivatives are frequently encountered in the technical literature. The function was first conceived as a tool which, if properly handled, could lead to useful results in a particularly concise way. Its popularity is now justified by solid mathematical arguments, developed over the years by authors such as Sobolev, Bochner, Mikusinski, and Schwartz. In the following pages we give the essentials of the Schwartz approach (distribution theory). The level of treatment is purely utilitarian. Rigorous exposés, together with descriptions of the historical evolution of the theory, may be found in the numerous texts quoted in the bibliography.

1.1 The δ-function

The idea of the δ-function is quite old, and dates back at least to the times of Kirchoff and Heaviside (van der Pol *et al.* 1951). In the early days of quantum mechanics, Dirac put the accent on the following properties of the function:

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1, \qquad \delta(x) = 0 \quad \text{for } x \neq 0. \tag{1.1}$$

The notation $\delta(x)$ was inspired by δ_{ik}, the Kronecker delta, equal to 0 for $i \neq k$, and to 1 for $i = k$. Clearly, $\delta(x)$ must be 'infinite' at $x = 0$ if the integral in (1.1) is to be unity. Dirac recognized from the start that $\delta(x)$ was not a function of x in the usual mathematical sense, but something more general which he called an 'improper' function. Its use, therefore, had to be confined to certain simple expressions, and subjected to careful codification. One of the expressions put forward by Dirac was the 'sifting' property

$$\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0). \tag{1.2}$$

This relationship can serve to define the delta function, not by its value at each point of the x axis, but by the ensemble of its scalar products with suitably chosen 'test' functions $f(x)$.

It is clear that the infinitely-peaked delta function can be interpreted intuitively as a strongly concentrated forcing function. The function may represent, for example, the force density produced by a unit force acting on a one-dimensional mechanical structure, e.g. a flexible string. This point of view leads to the concept of $\delta(x)$ being the limit of a function which becomes more and more concentrated in the vicinity of $x = 0$, whereas its integral from $-\infty$ to $+\infty$ remains equal to one. Some of the limit functions which behave in that manner are shown in Fig. 1.1. The first one is the rectangular pulse, which becomes 'needle-like' at high values of n (Fig. 1.1a). The other ones are (de Jager 1969; Bass 1971)

$$\lim_{n\to\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \qquad \text{(shown in Fig. 1.1b),} \qquad (1.3)$$

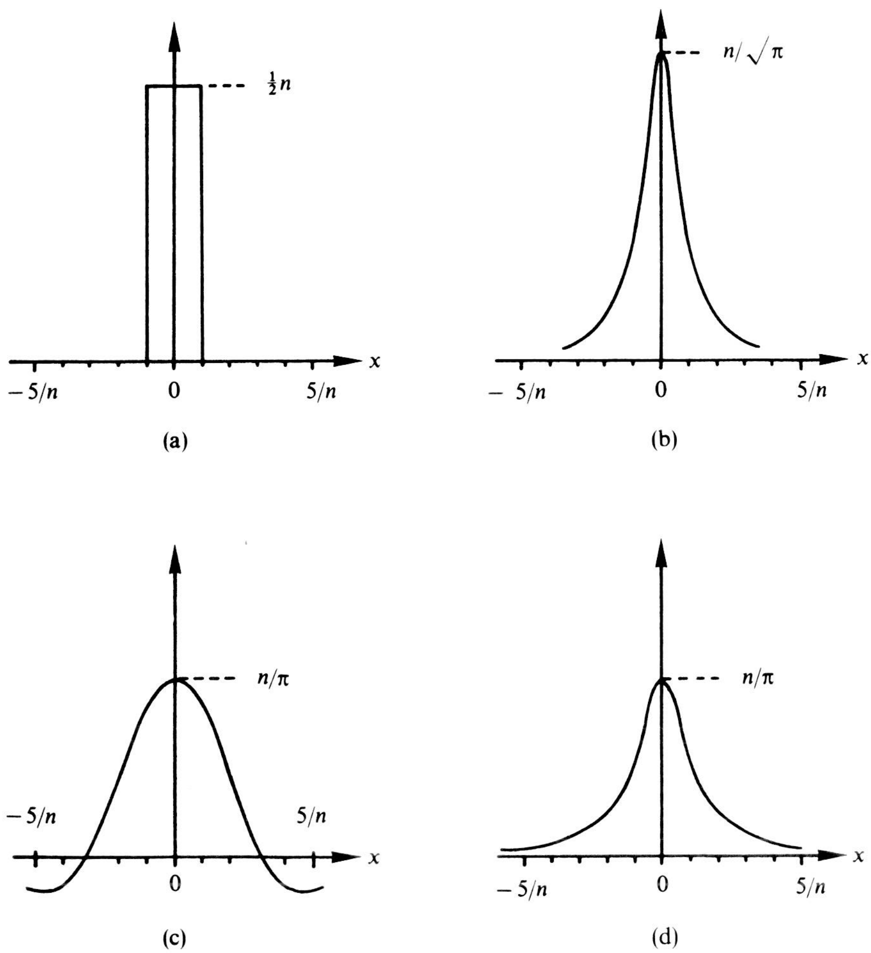

Fig. 1.1. Functions which represent Dirac's function in the limit $n \to \infty$.

$$\lim_{n\to\infty} \frac{\sin nx}{\pi x} \qquad \text{(shown in Fig. 1.1c)}, \tag{1.4}$$

$$\lim_{n\to\infty} \frac{n}{\pi(1+n^2x^2)} = \lim_{n\to\infty}\left(-\frac{n}{\pi}\operatorname{Im}\frac{1}{nx+\mathrm{j}}\right) \qquad \text{(shown in Fig. 1.1d)}. \tag{1.5}$$

1.2 Test functions and distributions

The notion of distribution is obtained by generalizing the idea embodied in (1.2), namely that a function is defined by the totality of its scalar products with reference functions termed *test functions*. The test functions used in the Schwartz theory are complex continuous functions $\phi(\boldsymbol{r})$ endowed with continuous derivatives of all orders. Such functions are often termed 'infinitely smooth'. They must vanish outside some finite domain, which may be different for each ϕ. They form a space $\mathscr{D}$. The smallest closed set which contains the set of points for which $\phi(\boldsymbol{r}) \neq 0$ is the *support* of ϕ. A typical one-dimensional test function is

$$\phi(x) = \begin{cases} \exp \dfrac{|ab|}{(x-a)(x-b)} & \text{for } x \text{ in } (a, b), \\ 0 & \text{for } x \text{ outside } (a, b). \end{cases} \tag{1.6}$$

The support of this function is the interval $[a, b]$. At the points $x = a$ and $x = b$, all derivatives vanish, and the graph of the function has a contact of infinite order with the x axis. A particular case of (1.6) is

$$\phi(x) = \begin{cases} \exp \dfrac{-1}{1-x^2} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geqslant 1. \end{cases} \tag{1.7}$$

In n dimensions, with $R^2 = x_1^2 + \cdots + x_n^2$, we have

$$\phi(\boldsymbol{r}) = \begin{cases} \exp \dfrac{-1}{1-R^2} & \text{for } |\boldsymbol{r}| < 1, \\ 0 & \text{for } |\boldsymbol{r}| \geqslant 1. \end{cases} \tag{1.8}$$

A few counterexamples are worth mentioning: $\phi(x) = x^2$ (for all x) is *not* a test function because its support is not bounded. The same is true of $\phi(x) = \sin|x|$, which furthermore has no continuous derivative at the origin.

To introduce the concept of 'distribution', it is necessary to first define *convergence* in $\mathscr{D}$ (Schwartz 1965). A sequence of functions $\phi_m(x)$ belonging to $\mathscr{D}$ is said to converge to $\phi(x)$ for $m \to \infty$ if

(1) the supports of the ϕ_m are contained in the same closed domain, independently of m;
(2) the ϕ_m and their derivatives of all orders converge uniformly to ϕ and its corresponding derivatives.

The next step is to define a *linear functional* on $\mathscr{D}$. This is an operation which associates a complex number $t(\phi)$ with every ϕ belonging to $\mathscr{D}$, in such a way that

$$t(\phi_1 + \phi_2) = t(\phi_1) + t(\phi_2), \qquad t(\lambda\phi) = \lambda t(\phi), \tag{1.9}$$

where λ is a complex constant. The complex number $t(\phi)$ is often written in the form

$$t(\phi) = \langle t, \phi \rangle \tag{1.10}$$

The functional is *continuous* if, when ϕ_m converges to ϕ for $m \to \infty$, the complex numbers $t(\phi_m)$ converge to $t(\phi)$. *Distributions are continuous linear functionals* on $\mathscr{D}$. They form a vector space $\mathscr{D}'$. To clarify these concepts, assume that $\tau(x)$ is a locally integrable function (i.e. a function which is integrable over any compact set). Such a function generates a distribution by the operation (Schwartz 1965)

$$\tau(\phi) = \langle \tau, \phi \rangle \overset{\text{def}}{=} \int_{-\infty}^{\infty} \tau(x)\phi(x)\,\mathrm{d}x. \tag{1.11}$$

Many distributions cannot be written as an integral of that form, except in a formal way. For such cases the 'generating function' $\tau(x)$ becomes a symbolic function, and (1.11) only means that the integral, whenever it is encountered in an analytical development, may be replaced by the value $\tau(\phi)$. It should be noted, in this respect, that experiments do not yield instantaneous, punctual values of quantities such as a force or an electric field. Instead, they generate *integrated* outputs, i.e. averages over some non-vanishing intervals of time and space. The description of a quantity by scalar products of the form (1.11) is therefore quite acceptable from a physical point of view.

1.3 Simple examples

A first simple example is the integral of ϕ from 0 to ∞. This integral is a distribution, which may be written as

$$\langle \mathrm{Y}, \phi \rangle \overset{\text{def}}{=} \int_{0}^{\infty} \phi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{Y}(x)\phi(x)\,\mathrm{d}x. \tag{1.12}$$