CONTEMPORARY MATHEMATICS

Nonlinear Partial Differential Equations

VOLUME 17

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Nonlinear Partial Differential Equations

Joel A. Smoller, Editor

AMERICAN MATHEMATICAL SOCIETY

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PREFACE

This is the volume of the proceedings of a conference on Nonlinear Partial Differential Equations, which took place from June 20 - 26, 1982, at the University of New Hampshire in Durham, New Hampshire. The conference was sponsored by the American Mathematical Society, and was funded by the National Science Foundation.

There were 67 participants, of which 16 people gave 1 hour talks, 17 people gave 1/2 hour talks and there were 11 informal 1/2 hour evening talks. In addition there was an informal session on computational and numerical aspects of shock waves.

The theme of the conference was on time-dependent nonlinear partial differential equations; in particular, the majority of the speakers lectured either on shock waves or reaction-diffusion equations and related areas. The first day speakers were asked to give an overview of their field: to describe the main results, and the open problems.

Perhaps the most interesting feature of this conference was the constant interplay between analysis, topology and computational methods.

I would like to thank the members of the organizing committee, consisting of C. Conley, P. Fife, T. P. Liu for their help and constant encouragement.

I am extremely grateful to Carole Kohanski for doing a superb job of making the arrangements, and for being so helpful (and cheerful!) throughout the week.

Joel Smoller

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THEORETICAL PROBLEMS AND NUMERICAL RESULTS FOR NONLINEAR CONSERVATION LAWS

James Glimm[1]

Adaptive methods of numerical computation are most important for singular problems, where ordinary methods give slow convergence and poor performance on reasonable grid sizes. The singularities and the adaptation can refer both to the geometrical space x, y, z, t of independent variables and to the state space of dependent variables (pressure, velocity, temperature, . . .). The simplest and most common singularities are jump discontinuities. Examples are contact or material discontinuities, shock waves, chemical reaction fronts, flame fronts and moving phase boundaries such as melting and boiling fronts. In the simplest approximation, the governing equations are often nonlinear conservation laws. These are systems of hyperbolic equations of the form

$$u_t + \Delta F(u) = 0 \tag{1}$$

and express conservation of the components u_i of the vector

$$u = \begin{pmatrix} u_1 \\ \cdot \; \cdot \; \cdot \\ u_n \end{pmatrix} \in R^n$$

(mass, momentum, etc.) Associated with such systems, there is usually a second order diffusion equation, for example

$$u_t + \nabla F(u) = \varepsilon \Delta u. \tag{2}$$

A jump discontinuity is locally one dimensional in the appropriate (normal and tangential) coordinates, and is analyzed through the study of a Riemann problem. A Riemann problem is a Cauchy problem for (1) in one space dimension with data

$$\begin{aligned} u(x,t=0) &= u_{left} = \text{const}, \quad x < 0 \\ u(x,t=0) &= u_{right} = \text{const}, \; x > 0. \end{aligned} \tag{3}$$

consisting of a single arbitrary jump discontinuity.

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[1]Supported in part by the National Science Foundation, PHY80-09179, the ARO, contract DAAG29-79-C-1079 and the DOE, contract DEA-CO2-76ER-03077.

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Because both the equation (1) and the data (2) are invariant under the scale transformations

$$u \to u^{(s)} = u(sx,st),$$

we anticipate that solutions of the Riemann problem will be functions of $\xi = x/t$ alone. However much more is true. It has been known since the fundamental paper of Lax [8] that the solution will consist of coherent waves, generally either shock or rarefaction waves. For an $n \times n$ system, there will generally be n waves, separated by wedges in which u takes on a constant value. Thus in figure 1, we have drawn two waves $(n = 2)$, three wedges and one

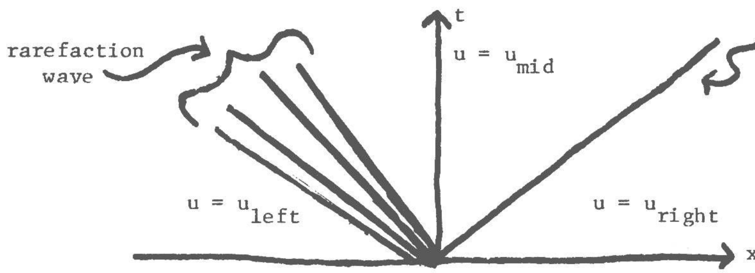

Figure 1

new constant state u_{mid}. The allowed elementary, or coherent waves, that is the shock, rarefaction and contact waves, are defined by solutions of *ordinary* differential equations or algebraic or functional equations in the state space R^n.

For a linear problem, that is $F(u) = Au$, with A an $n \times n$ matrix, the coherent waves result from the expansion of the jump discontinuity

$$u_{right} - u_{left} = \sum \alpha_i \, e_i$$

as a sum of right eigenvectors of the matrix u. In fact if the eigenvectors e_i are numbered in the order of increasing eigenvalues λ_i, then in the notation of figure 1,

$$u_{mid} - u_{left} = \alpha_1 \, e_1$$

$$u_{right} - u_{mid} = \alpha_2 \, e_2$$

Thus we see that the solution of the Riemann problem is equivalent to the expansion into normal modes, and that this expansion continues to have meaning in the nonlinear case. Having fixed the general ideas (see also [3,9,14] for more details), we now turn to specific problems.

Riemann Problems in the large. The picture sketched above is too simple, and we mention some of the complications which may arise. The research problem is partly to study Riemann problems associated with specific equations (chemical reactors, magnetohydrodynamics, ...) and to determine which further complications arise and it is partly to find properties of the nonlinear flux function F which characterize the phenomena in the solution of the Riemann problem.

Nonconvexity of F occurs naturally in chemistry, elasticity and oil reservoir applications. It gives rise to composite waves, that is waves associated with a single nonlinear mode which are rarefaction waves with embedded shocks. The phenomena is fully understood in the large only for single equations. In fact the meaning of convexity for $F: R^n \to R^n$, $n > 1$ is not clear.

Degenerate wave speeds $\lambda_i = \lambda_{i+1}$ occur naturally (the gas dynamics vacuum is an example). The degeneracy may occur on an open subset of R^n (this is typical of surfactant based tertiary recovery petroleum applications) or on a set of codimension one, two, ... in the state space R^n. In the case of degenerate wave speeds, the waves in the Riemann problem can be discontinuous as a function of the data, (u_{left}, u_{right}), however the solution $u = u(x,t)$, or its time slice $u(x,t = t_0)$ remains continuous in some norm, as a function of the same data. The degeneracy set $\lambda_i = \lambda_{i+1}$ can be a boundary between hyperbolic behavior (all λ_i real) and elliptic behavior (some λ_i occur in complex conjugate pairs). The transition between elastic and plastic behavior is an example of this phenomena.

Existence of an entropy decreasing solution of the Riemann problem in the large should be expected for "reasonable" flux functions F, including all F which occur in equations describing physical phenomena. One expects the solution to be composed of distinct coherent waves. When is uniqueness also expected?

Riemann Problems with source terms. Source terms may be introduced into the right hand side of (1) due to curvature of a wavefront in two dimensions, where x represents the normal direction to the wavefront. Also curvature of a boundary (flow in a duct of variable cross section) or coordinate system gives rise to source terms, when the flow is represented as a one dimensional flow. The source terms in general produce extra waves, as well as a curvature of the wave path. In order to retain scale invariance, we consider a delta function source in (1) concentrated at $x = 0$. (We thank G. Marshall and D. Marchesin for calling this suggestion to our attention.) For the case of scalar equations, or isothermal or polytropic gas, the resulting Riemann problem has been solved [10, 11]. The phenomena includes nonunique solutions and solutions discontinuous in the data. Here the discontinuity of the solution is with respect to any usual norm for $u(x,t)$, as a function of x, t.

Riemann Problems with second order data. See [5] for a discussion of this problem.

Riemann Problems in two and three dimensions. In two and three space dimensions, discontinuities move and bend, remaining locally one dimensional. However they also cross and form cusps, and become intrinsically two or three dimensional in their local behavior. A coherent wave or diffraction pattern is collection of one dimensional waves, meeting at a point $(d = 2)$ or on a line $(d = 3)$. To idealize this problem, we suppose the waves are planar or centered rarefaction waves and the intermediate states are constant. However to be called coherent, such a configuration should be dynamically stable. In general such a configuration is not dynamically stable, but bifurcates into two or more patterns which are stable. (These patterns could include rarefaction waves as well as shocks.) The higher dimensional Riemann problem is to describe the coherent, or dynamically stable diffraction patterns and to describe the nature of the bifurcation process whereby an arbitrary such pattern evolves into several stable diffraction wave patterns. For a single equation, work on this problem has been given by Wagner [17], based on different motivations.

The numerical results I will discuss were obtained in collaboration with O. McBryan and others. The general method has been a tracking method, to follow discontinuity surfaces in two dimensions explicitly, and to propagate them dynamically in time using the wave speeds obtained from solution of Riemann problems. See [3, 4] for a discussion of these methods. We have applied these methods to three areas: gas dynamics, oil reservoirs and the Rayleigh-Taylor problem. The latter is a gravity driven fingering instability of the interface between two fluids of different densities (say air and water). For gas dynamics, we present sample runs of three types. I-Liang Chern, Brad Plohr, and O. McBryan participated in this work.

Circular waves. This is a required test problem because the answer can be obtained by an elementary and accurate one dimensional calculation in polar coordinates. In Figure 2, we show the result of an expanding circular wave, on a 15×15 grid. The pressure ratio between the inside and outside is 5:1. Both the shock and the contact wave are tracked.

Diffraction by a wedge. Good experimental data [1] makes this a good test problem. An incident plane wave collides with a wedge or ramp, producing a reflected wave. See Figure 3.

Results of a 10×10 grid calculation. The results agree with a 40×40 grid calculation.

Vortex rollup (Helmholtz instability). We present results from early and later stages of the calculation in Figure 4 and Figure 5. The grid is 20×20.

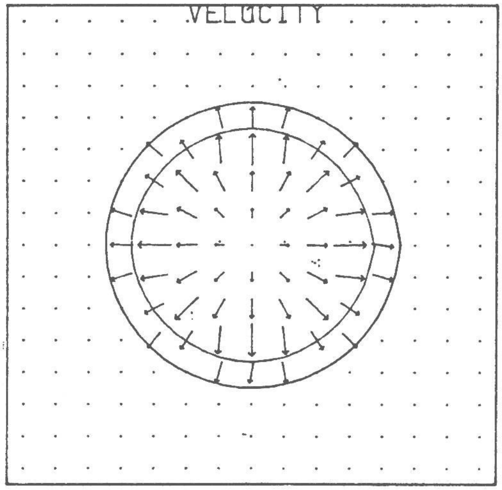

Figure 2

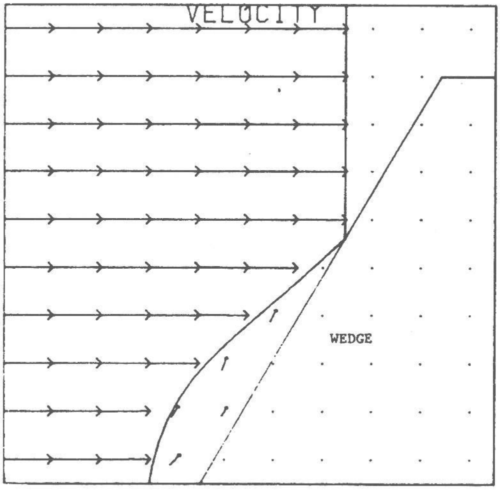

Figure 3

A statistical analysis of fingers in the Taylor-Saffman instability. For miscible displacement, we have studied the competing effects of an expanding circular geometry (i.e. the neighborhood of a single injection well) which gives stability and an adverse mobility ratio $M > 1$, which gives instability. The result of this competition is that the fingers stabilize at a finite size which is proportional to the distance from the source at the center, and thus have a fixed length on a logarithmic scale. This can be seen in Figure 6 where a number of runs with very distinct initial conditions give rise to the same (logarithmic) finger length. This work was in collaboration with E. Isaacson and O. McBryan.

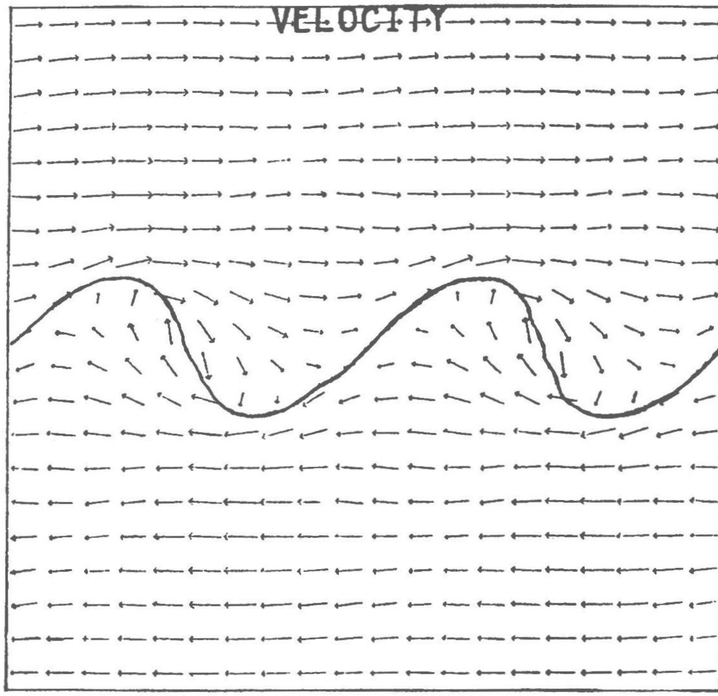

Figure 4

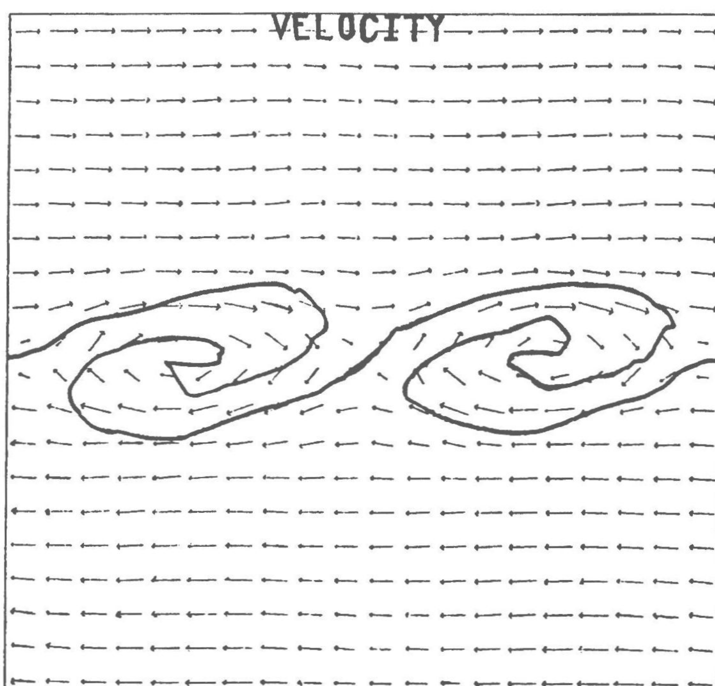

Figure 5

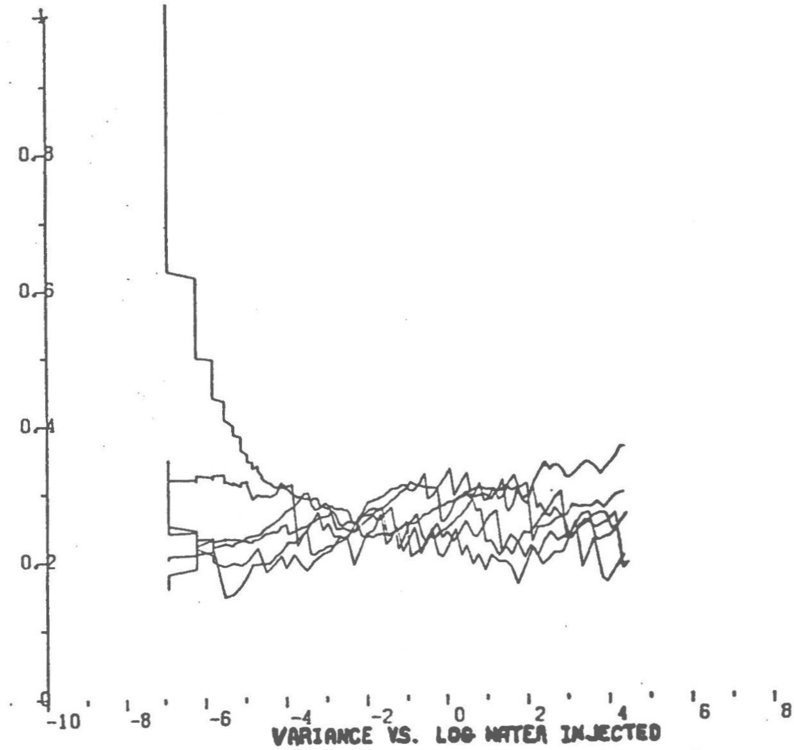

Figure 6

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