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Problem-Solving Through Problems

With 104 Illustrations

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To Elizabeth

Preface

The purpose of this book is to isolate and draw attention to the most important problem-solving techniques typically encountered in undergraduate mathematics and to illustrate their use by interesting examples and problems not easily found in other sources. Each section features a single idea, the power and versatility of which is demonstrated in the examples and reinforced in the problems. The book serves as an introduction and guide to the problems literature (e.g., as found in the problems sections of undergraduate mathematics journals) and as an easily accessed reference of essential knowledge for students and teachers of mathematics.

The book is both an anthology of problems and a manual of instruction. It contains over 700 problems, over one-third of which are worked in detail. Each problem is chosen for its natural appeal and beauty, but primarily for its unique challenge. Each is included to provide the context for illustrating a given problem-solving method. The aim throughout is to show how a basic set of simple techniques can be applied in diverse ways to solve an enormous variety of problems. Whenever possible, problems within sections are chosen to cut across expected course boundaries and to thereby strengthen the evidence that a single intuition is capable of broad application. Each section concludes with "Additional Examples" that point to other contexts where the technique is appropriate.

The book is written at the upper undergraduate level. It assumes a rudimentary knowledge of combinatorics, number theory, algebra, analysis, and geometry. Much of the content is accessible to students with only a year of calculus, and a sizable proportion does not even require this. However, most of the problems are at a level slightly beyond the usual contents of textbooks. Thus, the material is especially appropriate for students preparing for mathematical competitions.

The methods and problems featured in this book are drawn from my experience of solving problems at this level. Each new issue of *The American Mathematical Monthly* (and other undergraduate journals) contains material that would be just right for inclusion. Because these ideas continue to find new expression, the reader should regard this collection as a starter set and should be encouraged to create a personal file of problems and solutions to extend this beginning in both breadth and depth. Obviously, we can never hope to develop a "system" for problem-solving; however, the acquiring of ideas is a valuable experience at all stages of development.

Many of the problems in this book are old and proper referencing is very difficult. I have given sources for those problems that have appeared more recently in the literature, citing contests whenever possible. I would appreciate receiving exact references for those I have not mentioned.

I wish to take this opportunity to express my thanks to colleagues and students who have shared many hours of enjoyment working on these problems. In this regard I am particularly grateful to O. E. Stanaitis, Professor Emeritus of St. Olaf College. Thanks to St. Olaf College and the Mellon Foundation for providing two summer grants to help support the writing of this manuscript. Finally, thanks to all individuals who contributed by posing problems and sharing solutions. Special acknowledgement goes to Murray S. Klamkin who for over a quarter of a century has stood as a giant in the area of problem-solving and from whose problems and solutions I have learned a great deal.

March 21, 1983 LOREN C. LARSON

Contents

Chapter 1. Heuristics

Strategy or tactics in problem-solving is called *heuristics*. In this chapter we will be concerned with the heuristics of solving mathematical problems. Those who have thought about heuristics have described a number of basic ideas that are typically useful. Among these, we shall focus on the following:

(1) Search for a pattern.
(2) Draw a figure.
(3) Formulate an equivalent problem.
(4) Modify the problem.
(5) Choose effective notation.
(6) Exploit symmetry.
(7) Divide into cases.
(8) Work backward.
(9) Argue by contradiction.
(10) Pursue parity.
(11) Consider extreme cases.
(12) Generalize.

Our interest in this list of problem-solving ideas is not in their description but in their implementation. By looking at examples of how others have used these simple but powerful ideas, we can expect to improve our problem-solving skills.

Before beginning, a word of advice about the problems at the end of the sections: Do not be overly concerned about using the heuristic treated in that section. Although the problems are chosen to give practice in the use of the heuristic, a narrow focus may be psychologically debilitating. A single problem usually admits several solutions, often employing quite different heuristics. Therefore, it is best to approach each problem with an open mind rather than with a preconceived notion about how a particular heuristic should be applied. In working on a problem, solving it is what matters. It is the accumulated experience of all the ideas working together that will result in a heightened awareness of the possibilities in a problem.

1.1. Search for a Pattern

Virtually all problem solvers begin their analysis by getting a feel for the problem, by convincing themselves of the plausibility of the result. This is best done by examining the most immediate special cases; when this exploration is undertaken in a systematic way, patterns may emerge that will suggest ideas for proceeding with the problem.

1.1.1. Prove that a set of n (different) elements has exactly 2^n (different) subsets.

When the problem is set in this imperative form, a beginner may panic and not know how to proceed. Suppose, however, that the problem were cast as a query, such as

(i) How many subsets can be formed from a set of n objects?
(ii) Prove or disprove: A set with n elements has 2^n subsets.

In either of these forms there is already the implicit suggestion that one should begin by checking out a few special cases. This is how each problem should be approached: remain skeptical of the result until convinced.

Solution 1. We begin by examining what happens when the set contains 0, 1, 2, 3 elements; the results are shown in the following table:

n	Elements of S	Subsets of S	Number of subsets of S
0	none	$\emptyset$	1
1	x_1	$\emptyset, \{x_1\}$	2
2	x_1, x_2	$\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}$	4
3	x_1, x_2, x_3	$\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}$ $\{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}$	8

Our purpose in constructing this table is not only to verify the result, but also to look for patterns that might suggest how to proceed in the general case. Thus, we aim to be as systematic as possible. In this case, notice when $n = 3$, we have listed first the subsets of $\{x_1, x_2\}$ and then, in the second line, each of these subsets augmented by the element x_3. This is the key idea that allows us to proceed to higher values of n. For example, when $n = 4$, the subsets of $S = \{x_1, x_2, x_3, x_4\}$ are the eight subsets of $\{x_1, x_2, x_3\}$ (shown in the table) together with the eight formed by adjoining x_4 to each of these. These sixteen subsets constitute the entire collection of possibilities; thus, a set with 4 elements has 2^4 $(= 16)$ subsets.

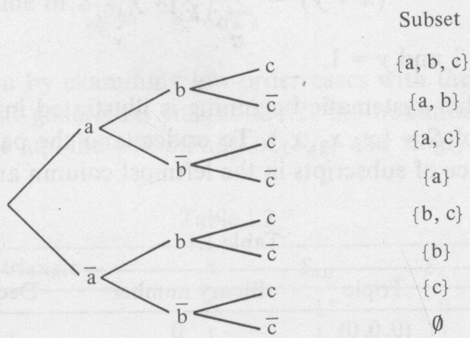
A proof based on this idea is an easy application of mathematical induction (see Section 2.1).

Solution 2. Another way to present the idea of the last solution is to argue as follows. For each n, let A_n denote the number of (different) subsets of a set with n (different) elements. Let S be a set with $n + 1$ elements, and designate one of its elements by x. There is a one-to-one correspondence between those subsets of S which do not contain x and those subsets that do contain x (namely, a subset T of the former type corresponds to $T \cup \{x\}$). The former types are all subsets of $S - \{x\}$, a set with n elements, and therefore, it must be the case that

$$A_{n+1} = 2A_n .$$

This recurrence relation, true for $n = 0, 1, 2, 3, \ldots$, combined with the fact that $A_0 = 1$, implies that $A_n = 2^n$. $(A_n = 2A_{n-1} = 2^2A_{n-2} = \cdots = 2^nA_0 = 2^n.)$

Solution 3. Another systematic enumeration of subsets can be carried out by constructing a "tree". For the case $n = 3$ and $S = \{a, b, c\}$, the tree is as shown below:



Each branch of the tree corresponds to a distinct subset of S (the bar over the name of the element means that it is not included in the set corresponding to that branch). The tree is constructed in three stages, corresponding to the three elements of S. Each element of S leads to two possibilities: either it is in the subset or it is not, and these choices are represented by two branches. As each element is considered, the number of branches doubles.

Thus, for a three-element set, the number of branches is $2 \times 2 \times 2 = 8$. For an n-element set the number of branches is

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n} = 2^n;$$

thus, a set with n elements has 2^n subsets.

Solution 4. Suppose we enumerate subsets according to their size. For example, when $S = \{a, b, c, d\}$, the subsets are

Number of elements		Number of subsets
0	$\varnothing$	1
1	$\{a\}, \{b\}, \{c\}, \{d\}$	4
2	$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$	6
3	$\{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}$	4
4	$\{a,b,c,d\}$	1

This beginning could prompt the following argument. Let S be a set with n elements. Then

$$\text{No. of subsets of } S = \sum_{k=0}^{n} (\text{No. of subsets of } S \text{ with } k \text{ elements})$$

$$= \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

The final step in this chain of equalities follows from the binomial theorem,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

upon setting $x = 1$ and $y = 1$.

Solution 5. Another systematic beginning is illustrated in Table 1.1, which lists the subsets of $S = \{x_1, x_2, x_3\}$. To understand the pattern here, notice the correspondence of subscripts in the leftmost column and the occurrence

Table 1.1

Subset	Triple	Binary number	Decimal number
$\varnothing$	(0, 0, 0)	0	0
$\{x_3\}$	(0, 0, 1)	1	1
$\{x_2\}$	(0, 1, 0)	10	2
$\{x_2, x_3\}$	(0, 1, 1)	11	3
$\{x_1\}$	(1, 0, 0)	100	4
$\{x_1, x_3\}$	(1, 0, 1)	101	5
$\{x_1, x_2\}$	(1, 1, 0)	110	6
$\{x_1, x_2, x_3\}$	(1, 1, 1)	111	7

of 1's in the second column of triples. Specifically, if A is a subset of $S = \{x_1, x_2, \ldots, x_n\}$, define a_i, for $i = 1, 2, \ldots, n$, by

$$a_i = \begin{cases} 1 & \text{if } a_i \in A, \\ 0 & \text{if } a_i \notin A. \end{cases}$$

It is clear that we can now identify a subset A of S with $(a_1, a_2, \ldots, a_n)$, an n-tuple of 0's and 1's. Conversely, each such n-tuple will correspond to a unique subset of S. Thus, the number of subsets of S is equal to the number of n-tuples of 0's and 1's. This latter set is obviously in one-to-one correspondence with the set of nonnegative binary numbers less than 2^n. Thus, each nonnegative integer less than 2^n corresponds to exactly one subset of S, and conversely. Therefore, it must be the case that S has 2^n subsets.

Normally, we will give only one solution to each example—a solution which serves to illustrate the heuristic under consideration. In this first example, however, we simply wanted to reiterate the earlier claim that a single problem can usually be worked in a variety of ways. The lesson to be learned is that one should remain flexible in the beginning stages of problem exploration. If an approach doesn't seem to lead anywhere, don't despair, but search for a new idea. Don't get fixated on a single idea until you've had a chance to think broadly about a variety of alternative approaches.

1.1.2. Let $S_{n,0}$, $S_{n,1}$, and $S_{n,2}$ denote the sum of every third element in the nth row of Pascal's Triangle, beginning on the left with the first element, the second element, and the third element respectively. Make a conjecture concerning the value of $S_{100,1}$.

Solution. We begin by examining low-order cases with the hope of finding patterns that might generalize. In Table 1.2, the nonunderlined terms are those which make up the summands of $S_{n,0}$; the singly underlined and

Table 1.2

Pascal's triangle	n	$S_{n,0}$	$S_{n,1}$	$S_{n,2}$
1	0	1^+	0	0
$1\ \underline{1}$	1	1	1	0^-
$1\ \underline{2}\ \underline{\underline{1}}$	2	1	2^+	1
$1\ \underline{3}\ \underline{\underline{3}}\ 1$	3	2^-	3	3
$1\ \underline{4}\ \underline{\underline{6}}\ 4\ \underline{1}$	4	5	5	6^+
$1\ \underline{5}\ \underline{\underline{10}}\ 10\ \underline{5}\ \underline{\underline{1}}$	5	11	10^-	11
$1\ \underline{6}\ \underline{\underline{15}}\ 20\ \underline{15}\ \underline{\underline{6}}\ 1$	6	22^+	21	21
$1\ \underline{7}\ \underline{\underline{21}}\ 35\ \underline{35}\ \underline{\underline{21}}\ 7\ \underline{1}$	7	43	43	42^-

doubly underlined terms are those of $S_{n,1}$ and $S_{n,2}$, respectively. The three columns on the right show that, in each case, two of the sums are equal, whereas the third is either one larger (indicated by a superscript $+$) or one smaller (indicated by a superscript $-$). It also appears that the unequal term in this sequence changes within a cycle of six. Thus, from the pattern established in the first rows, we expect the anomaly for $n = 8$ to occur in the middle column and it will be one less than the other two.

We know that $S_{n,0} + S_{n,1} + S_{n,2} = 2^n$ (see 1.1.1). Since $100 = 6 \times 16 + 4$, we expect the unequal term to occur in the third column ($S_{100,2}$) and to be one more than the other two. Thus $S_{100,0} = S_{100,1} = S_{100,2} - 1$, and $S_{100,1} + S_{100,1} + S_{100,1} + 1 = 2^{100}$. From these equations we are led to conjecture that

$$S_{100,1} = \frac{2^{100} - 1}{3}.$$

A formal proof of this conjecture is a straightforward application of mathematical induction (see Chapter 2).

1.1.3. Let $x_1, x_2, x_3, \ldots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}, \quad n = 3, 4, 5, \ldots$$

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

Solution. To get a feel for the sequence, we will compute the first few terms, expressing them in terms of x_1 and x_2. We have (omitting the algebra)

$$x_3 = \frac{x_1 x_2}{2x_1 - x_2},$$

$$x_4 = \frac{x_1 x_2}{3x_1 - 2x_2},$$

$$x_5 = \frac{x_1 x_2}{4x_1 - 3x_2}.$$

We are fortunate in this particular instance that the computations are manageable and a pattern emerges. An easy induction argument establishes that

$$x_n = \frac{x_1 x_2}{(n-1)x_1 - (n-2)x_2},$$

which, on isolating the coefficient of n, takes the form

$$x_n = \frac{x_1 x_2}{(x_1 - x_2)n + (2x_2 - x_1)}.$$

In this form, we see that if $x_1 \neq x_2$, the denominator will eventually exceed the numerator in magnitude, so x_n then will not be an integer. However, if $x_1 = x_2$, all the terms of the sequence are equal. Thus, x_n is an integer for infinitely many values of n if and only if $x_1 = x_2$.

1.1.4. Find positive numbers n and $a_1, a_2, \ldots, a_n$ such that $a_1 + \cdots + a_n = 1000$ and the product $a_1 a_2 \cdots a_n$ is as large as possible.

Solution. When a problem involves a parameter which makes the analysis complicated, it is often helpful in the discovery stage to replace it temporarily with something more manageable. In this problem, we might begin by examining a sequence of special cases obtained by replacing 1000 in turn with $2, 3, 4, 5, 6, 7, 8, 9, \ldots$. In this way we are led to discover that in a maximum product

(i) no a_i will be greater than 4,
(ii) no a_i will equal 1,
(iii) all a_i's can be taken to be 2 or 3 (because $4 = 2 \times 2$ and $4 = 2 + 2$),
(iv) at most two a_i's will equal 2 (because $2 \times 2 \times 2 < 3 \times 3$ and $2 + 2 + 2 = 3 + 3$).

Each of these is easy to establish. Thus, when the parameter is 1000 as in the problem at hand, the maximum product must be $3^{332} \times 2^2$.

1.1.5. Let S be a set and $*$ be binary operation on S satisfying the two laws

$$x * x = x \qquad \text{for all } x \text{ in } S,$$

$$(x * y) * z = (y * z) * x \qquad \text{for all } x, y, z \text{ in } S.$$

Show that $x * y = y * x$ for all x, y in S.

Solution. The solution, which appears so neatly below, is actually the end result of considerable scratch work; the procedure can only be described as a search for pattern (the principle pattern is the cyclic nature of the factors in the second condition). We have, for all x, y in S, $x * y = (x * y) * (x * y) = [y * (x * y)] * x = [(x * y) * x] * y = [(y * x) * x] * y = [(x * x) * y] * y = [(y * y)] * (x * x) = y * x$.

Problems

Develop a feel for the following problems by searching for patterns. Make appropriate conjectures, and think about how the proofs might be carried out.

1.1.6. Beginning with 2 and 7, the sequence $2, 7, 1, 4, 7, 4, 2, 8, \ldots$ is constructed by multiplying successive pairs of its members and adjoining the result as the next one or two members of the sequence, depending on whether the product is a one- or a two-digit number. Prove that the digit 6 appears an infinite number of times in the sequence.

1.1.7. Let S_1 denote the sequence of positive integers $1, 2, 3, 4, 5, 6, \ldots$, and define the sequence S_{n+1} in terms of S_n by adding 1 to those integers in S_n which are divisible by n. Thus, for example, S_2 is $2, 3, 4, 5, 6, 7, \ldots$, S_3 is $3, 3, 5, 5, 7, 7, \ldots$. Determine those integers n with the property that the first $n-1$ integers in S_n are n.

1.1.8. Prove that a list can be made of all the subsets of a finite set in such a way that

(i) the empty set is first in the list,
(ii) each subset occurs exactly once, and
(iii) each subset in the list is obtained either by adding one element to the preceding subset or by deleting one element of the preceding subset.

1.1.9. Determine the number of odd binomial coefficients in the expansion of $(x+y)^{1000}$. (See 4.3.5.)

1.1.10. A well-known theorem asserts that a prime $p > 2$ can be written as a sum of two perfect squares ($p = m^2 + n^2$, with m and n integers) if and only if p is one more than a multiple of 4. Make a conjecture concerning which primes $p > 2$ can be written in each of the following forms, using (not necessarily positive) integers x and y: (a) $x^2 + 16y^2$, (b) $4x^2 + 4xy + 5y^2$. (See 1.5.10.)

1.1.11. If $\langle a_n \rangle$ is a sequence such that for $n \geqslant 1$, $(2 - a_n)a_{n+1} = 1$, what happens to a_n as n tends toward infinity? (See 7.6.4.)

1.1.12. Let S be a set, and let $*$ be a binary operation on S satisfying the laws

$$x * (x * y) = y \quad \text{for all } x, y \text{ in } S,$$

$$(y * x) * x = y \quad \text{for all } x, y \text{ in } S.$$

Show that $x * y = y * x$ for all x, y in S.

Additional Examples

Most induction problems are based on the discovery of a pattern. Thus, the problems in Sections 2.1, 2.2, 2.3, 2.4 offer additional practice in this heuristic. Also see 1.7.2, 1.7.7, 1.7.8, 2.5.6, 3.1.1, 3.4.6, 4.3.1, 4.4.1, 4.4.3, 4.4.15, 4.4.16, 4.4.17.

1.2. Draw a Figure

Whenever possible it is helpful to describe a problem pictorially, by means of a figure, a diagram, or a graph. A diagrammatic representation usually makes it easier to assimilate the relevant data and to notice relationships and dependences.

1.2.1. A chord of constant length slides around in a semicircle. The midpoint of the chord and the projections of its ends upon the base form the vertices of a triangle. Prove that the triangle is isosceles and never changes its shape.

Solution. Let AB denote the base of the semicircle, let XY be the chord, M the midpoint of XY, C and D the projections of X and Y on AB (Figure 1.1). Let the projection of M onto AB be denoted by N. Then N is the midpoint of CD and it follows that $\triangle CMD$ is isosceles.

To show that the shape of the triangle is independent of the position of the chord, it suffices to show that $\angle MCD$ remains unchanged, or equivalently, that $\angle XCM$ is constant, for all positions of XY. To see that this is the case, extend XC to cut the completed circle at Z (Figure 1.2). Then CM is parallel to ZY (C and M are the midpoints of XZ and XY, respectively),

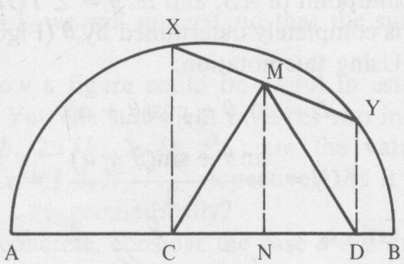


Figure 1.1.

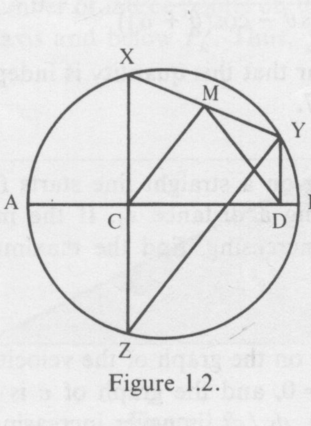


Figure 1.2.