Infinite Abelian Groups

Revised Edition

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Ann Arbor
The University of Michigan Press

Second printing 1971

ISBN 0-472-08500-X
Library of Congress Catalog Card No. 69-15839
Published in the United States of America by
The University of Michigan Press and simultaneously
in Don Mills, Canada, by Longman Canada Limited
Manufactured in the United States of America

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PREFACE TO THE REVISED EDITION

In the second edition a number of misprints and errors have been corrected and Sections 6, 16, 17 (test problems, complete modules, algebraic compactness) have been extensively revised.

The new bibliography is limited to items to which there is an actual reference. The bibliography in the first edition had 145 entries and was intended to be complete up to about 1952. It would take over 400 additional entries to bring it up to 1968. It is doubtful whether the space occupied by such a large bibliography would be well spent.

The guide to the literature is omitted, but relevant parts have been incorporated into the new section of notes. For some things, e.g. my views on duality, the reader should consult the first edition.

I urge the reader to have Fuchs's definitive treatise at hand. My feeling is that there is nevertheless still room for a slim volume, not so imposing, gentle, and talkative (at least in the beginning).

I take the opportunity to disagree mildly with Professor Fuchs about the role of modules. In the applications of algebra (notably to topology), very general rings and modules over them are increasingly important. I stand by the compromise in *Infinite Abelian Groups:* start with familiar plain old abelian groups and switch completely to modules over principal ideal rings at about the half-way point.

In the appended notes there is indeed a good deal of exploration of modules, combined with remarks appropriate to a second look at the subject. These require from the reader some familiarity with the rudiments of commutative ring theory and homological algebra.

I am indebted to Peter Crawley, Alfred Hales, Charles Megibben, and Joseph Rotman for spirited comments on a draft of the second edition. Professor Fuchs kindly took time out from the preparation of his own second edition to send me valuable suggestions. In addition to the Office of Naval Research, whose aid was acknowledged in the first edition, I am happy to thank the Army Research Office, the Air Force Office of Scientific Research, and the National Science Foundation for their support over the years. Thanks also to Joyce Bolden for a splendid job of typing.

Chicago, Ill. May 1968

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INFINITE ABELIAN GROUPS

1. INTRODUCTION

In the early days of group theory attention was confined almost entirely to finite groups. But recently, and above all in the last two decades, the infinite group has come into its own. The results obtained on infinite *abelian* groups have been particularly penetrating. This monograph has been written with two objectives in mind: first, to make the theory of infinite abelian groups available in a convenient form to the mathematical public; second, to help students acquire some of the techniques used in modern infinite algebra.

For this second purpose infinite abelian groups serve admirably. No extensive background is required for their study, the rudiments of group theory being sufficient. There is a good variety in the transfinite tools employed, with Zorn's lemma being applied in several different ways. The traditional style of transfinite induction is not completely ignored either, for there is a theorem whose very formulation uses transfinite ordinals. The peculiar role sometimes played by a countability hypothesis makes a challenging appearance.

It is furthermore helpful that finite abelian groups are completely known. In other subjects, such as rings or nonabelian groups, there are distracting difficulties which occur even in the finite case. Here, however, our attention is concentrated on the problems arising from the fact that the groups may be infinite.

With a student audience in mind, I have given details and included remarks that would ordinarily be suppressed in print. However, as the discussion proceeds it becomes somewhat more concise. A serious effort has been made to furnish, in brief space, a reasonably complete account of the subject. In order to do this, I have relegated many results of some interest to the role of exercises, and a large part of the literature is merely surveyed in the guide to it provided in § 20.

This material is adapted from a course which I gave at the University of Chicago in the fall of 1950. I should like to record my indebtedness to the many able members of that class, particularly to George Backus, Arlen Brown, and Roger Farrell. Thanks are expressed to Isidore Fleischer for the ideas in §16 (the torsion-free case of Theorem 22 was discovered by him and appears in his doctoral dissertation); to Robert Heyneman and George Kolettis, who read a preliminary version of this work and made many valuable suggestions; to Tulane University and the University of Michigan, where I had the opportunity to lecture on abelian groups; and to the Office of Naval Research.

A special acknowledgment goes to Professor Reinhold Baer. It was from his papers that I learned much of the theory of abelian groups. Furthermore, when this monograph was nearly complete, I had the privilege of reading an unpublished manuscript (of book length) on abelian groups which he prepared in the late 1940's.

2. EXAMPLES OF ABELIAN GROUPS

Before beginning to develop the theory, it is desirable to have at hand a small collection of examples of abelian groups.

To avoid endless repetition, let it be agreed that "group" will always mean "abelian group."

(*a*) *Cyclic groups.* A group G is cyclic if it can be generated by a single element. If that element has infinite order, G is isomorphic to the additive group of integers, and is called an infinite cyclic group; if it has finite order n, G is cyclic of order n and is isomorphic to the additive group of integers mod n. We shall use the notation Z and Z_n respectively for these two groups.

(*b*) *External direct sums.* Let $\{G_i\}$ be any set of groups, where the subscript i runs over an index set I, which may be finite or infinite. We define the *direct sum* of the groups G_i. We take "vectors" $\{a_i\}$; that is, arrays indexed by $i \in I$ with a_i in G_i. Moreover, we impose the restriction that all but a finite number of the a_i's are to be 0 (we are writing 0 indifferently for the identity element of any G_i). Addition of vectors is defined by adding components. This gives an abelian group, called the direct sum of $\{G_i\}$.

If there is any danger of ambiguity, the object just defined may be referred to as the "weak," or "discrete," direct sum, as opposed to the "complete" direct sum, where the vectors are unrestricted. In pure algebra it is the weak direct sum which arises most naturally; the complete direct sum is, indeed, mostly useful as a source of counterexamples (see Theorem 21 and exercise 33).

(*c*) *Union and intersection.* If S and T are subgroups of a group, we write $S \cap T$ for their intersection, that is, the set of elements lying in both. More generally, if $\{S_i\}$ is a set of subgroups of G, we write $\cap S_i$ for the intersection. Note that we are talking about the set-theoretic intersection and that it is always a subgroup.

As regards the union of subgroups, the situation is different. Consider first two subgroups, S and T. The set-theoretic union, which we might write $S \cup T$, is not generally a subgroup (in fact, $S \cup T$ is a subgroup if and only if one of the two subgroups S and T contains the other). What we wish instead is the smallest subgroup containing S and T, and this is provided by $S + T$, the set of all elements $s + t$, where s and t range over S and T.

Again, let $\{S_i\}$ be any set of subgroups of G. Their union, written ΣS_i, is the smallest subgroup containing them; it may be explicitly described as the set of all *finite* sums of elements extracted from the various subgroups S_i.

(*d*) *Internal direct sums.* In dealing with direct sums we are most often confronted with the problem of showing that a group is isomorphic to the direct sum of certain of its subgroups. Suppose first that the group G has subgroups S and T satisfying $S \cap T = 0$, $S + T = G$. Then it is easy to see that G is isomorphic to the direct sum of S and T, where we are referring to the external direct sum discussed above in (*b*). One may speak of G as being the internal direct sum of S and T, but generally one simply calls G the direct sum of S and T, and writes $G = S \oplus T$.

Consider now any (finite or infinite) set of subgroups $\{S_i\}$. In order to verify that G is the direct sum of these subgroups, the most convenient procedure is generally as follows: Show that $G = \Sigma S_i$, that is, that every element of G can be written as a finite sum of elements from the subgroups S_i; then show that the representation is unique. This uniqueness is equivalent to the statement that each S_i is disjoint from the union of the remaining ones.

In general, if the union ΣS_i of subgroups is their direct sum, we shall call the subgroups S_i *independent.*

A concept of independence for elements will also be useful: We shall say that the elements x_i are *independent* if the cyclic subgroups they generate are independent in the sense just defined, and we write $\Sigma(x_i)$ for the subgroup generated by all the elements.

We should notice the analogy between this concept and linear independence in a vector space. In fact, the elements x_i are independent if and only if the following is true: If a finite sum

$$\Sigma n_i x_i = 0 \qquad (n_i \text{ integers}),$$

then each $n_i x_i = 0$.

(*e*) *Rational numbers.* The most general group so far in our possession is a direct sum of cyclic groups. A classical theorem asserts that this covers all finitely generated groups, and in particular all finite groups. That is to say, any finitely generated group is a direct sum of (a finite number of) cyclic groups.

One might for a moment think that perhaps any abelian group is a direct sum of cyclic groups, the number of summands now being allowed to be infinite, of course. This conjecture is defeated by a very familiar group: the additive group R of rational numbers. That R is not a direct sum of cyclic groups may be seen, for example, from the fact that for any $x \in R$ and any integer n there exists an element $y \in R$ with $ny = x$; this property manifestly cannot hold in a direct sum of cyclic groups. (The property in question is called *divisibility*, and will be studied in §5.)

(*f*) *Rationals mod one.* In the additive group R of rational numbers, there is the subgroup Z of integers. The quotient group R/Z is known as the *rationals* mod one. We note that in R/Z every element has finite order. We argue, just as above, that R/Z is not a direct sum of cyclic groups.

(*g*) *The group* $Z(p^\infty)$. There is an important modification of the two preceding examples. Let p be a fixed prime, and let P denote the additive group of those rational numbers whose denominators are powers of p. The quotient group P/Z will play a dominant role in the ensuing discussion, and we use for it the notation $Z(p^\infty)$.

Let us pause to take a close look at $Z(p^\infty)$. For simplicity we take $p = 2$. We can write the elements of $Z(2^\infty)$ as 0, 1/2, 1/4, 3/4, 1/8, etc., but it is to be understood that addition takes place mod one. Thus

$$1/2 + 1/2 = 0, \quad 1/2 + 3/4 = 1/4, \quad 3/4 + 5/8 = 3/8, \text{ etc.}$$

What are the subgroups of $Z(2^\infty)$? There is a subgroup of order 2 consisting of 0 and 1/2; one of order 4 consisting of 0, 1/4, 1/2, 3/4; and in general a cyclic subgroup (say H_n) of order 2^n generated by $1/2^n$. It is not difficult to see that these are in fact the only subgroups. Thus the array of subgroups can be pictured as follows:

$$0 \subset H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots \subset Z(2^\infty).$$

It is noteworthy that every subgroup of $Z(2^\infty)$ is finite, except for $Z(2^\infty)$ itself. The subgroups form an ascending chain which never terminates. On the contrary, one sees that every *descending* chain of subgroups must be finite. Thus $Z(2^\infty)$ has the so-called "descending-chain condition" but not the "ascending-chain condition."

In conclusion, we give another realization of $Z(p^\infty)$. Consider the set of all p^n-th roots of unity, where p is a fixed prime and $n = 0, 1, 2, \cdots$. These numbers form a group under multiplication, and the group is isomorphic to $Z(p^\infty)$.

This completes our discussion of examples. It will appear that these groups are the fundamental building blocks for some fairly wide classes of infinite abelian groups.

3. TORSION GROUPS

If an abelian group has all its elements of finite order, we shall call it a *torsion* group. (This designation does not convey much algebraically, but it has a suggestive topological background and the merit of brevity.) The other extreme case is that where all the elements (except 0 of course) have infinite order; we then call the group *torsion-free*.

Now let G be an arbitrary abelian group, and T the set of all elements in G having finite order. We leave to the reader the verification of the following two remarks: (*a*) T is a subgroup, (*b*) G/T is torsion-free. We shall call T the torsion subgroup of G.

The study of abelian groups is now seen to split into three parts: (*a*) the classification of torsion groups, (*b*) the classification of torsion-free groups, (*c*) the study of the way the two are put together to form an arbitrary group. Progress has been most notable on the first of these problems, and consequently we shall be chiefly concerned with torsion groups.

Next we define a group (necessarily a torsion group) to be *primary* if, for a certain prime p, every element has order a power of p. The study of torsion groups is reduced to that of primary groups by the following theorem:

Theorem 1. Any torsion group is a direct sum of primary groups.

Proof. Let G be the group, and for every prime p define G_p to be the subset consisting of elements with order a power of p. It is clear that G_p is a subgroup, and that it is primary. We shall now prove that G is isomorphic to the direct sum of the subgroups G_p.

(*a*) We have first to show that G is the union of the subgroups G_p. Take any x in G, say of order n. Then factor n into prime powers: $n = p_1^{r_1} \cdots p_k^{r_k}$, and write $n_i = n/p_i^{r_i}$ $(i = 1, \cdots, k)$. Thus $n_1, \cdots, n_k$ have greatest common divisor 1, and so there exist integers $a_1, \cdots, a_k$ with $a_1 n_1 + \cdots + a_k n_k = 1$. Then

$$x = a_1 n_1 x + \cdots + a_k n_k x. \tag{1}$$

Now $n_i x$ has precisely order $p_i^{r_i}$, and so it is in G_{p_i}. Thus equation (1) is the desired expression of x as a sum of elements in the G_p's.

(*b*) We have further to prove the uniqueness of the expression just found. Suppose

$$\begin{aligned} x &= y_1 + \cdots + y_k \\ &= z_1 + \cdots + z_k \end{aligned}$$

where y_i, z_i lie in the same G_{p_i}. Consider the equation

$$y_1 - z_1 = (z_2 + \cdots + z_k) - (y_2 + \cdots + y_k). \tag{2}$$

We know that $y_1 - z_1$ has order a power of p_1. On the other hand, the right side of (2) is an element whose order is a product of powers of $p_2, \cdots, p_k$. This is possible only if $y_1 - z_1 = 0$. Similarly each $y_i = z_i$. This completes the proof of Theorem 1.

As a general principle, every decomposition theorem should be accompanied by a uniqueness investigation. Such an investigation is particularly easy for the decomposition given by Theorem 1. In fact, there is only one way to express a torsion group as a direct sum of primary subgroups, one for each prime p; for the subgroup attached to p must necessarily consist of all elements whose order is a power of p. In other words, the decomposition is unique not just up to isomorphism; the summands are unique subgroups.

The simplicity of the proof of Theorem 1 is a natural counterpart to this strong uniqueness; for if a decomposition is unique there ought to be a simple natural way to effect it. It is instructive to compare this situation with later ones. For example, under suitable hypotheses of various kinds we shall prove that a primary group is a direct sum of cyclic groups; this decomposition is unique, but only up to isomorphism. The difficulties encountered in the proof are a natural reflection of the large number of arbitrary choices that have to be made in carrying out the decomposition.

We shall conclude this section by giving two illustrations of Theorem 1:

(*a*) Consider the cyclic group Z(n), where $n = p_1^{r_1} \cdots p_k^{r_k}$. Then $Z(n) = Z(p_1^{r_1}) \oplus \cdots \oplus Z(p_k^{r_k})$. (Indeed, this is the fact which really underlies the proof of Theorem 1.)

(*b*) Let G be the additive group of rationals mod one (§ 2). This is a torsion group, and it can be seen that its primary component for the prime p is precisely the group $Z(p^\infty)$ of § 2. Thus G is a direct sum of all the groups $Z(p^\infty)$.

4. ZORN'S LEMMA

Nearly every proof to follow will depend on the use of a transfinite induction. Such an induction is generally best accomplished by the use of Zorn's lemma, which is to be regarded as an axiom like other axioms needed to set up the foundations of mathematics.

We shall make use of a version of Zorn's lemma which refers to the concept of a partially ordered set. A partially ordered set is a set with a binary relation $\geqq$ which satisfies

(*a*) $x \geqq x$ (reflexivity),
(*b*) $x \geqq y$, $y \geqq x$ imply $x = y$ (antisymmetry),
(*c*) $x \geqq y$, $y \geqq z$ imply $x \geqq z$ (transitivity).

Let S be a partially ordered set and T a subset. The element x is said to be the least upper bound of T if $x \geqq y$ for every y in T and if $z \geqq y$ for every y in T implies $z \geqq x$. (The element x itself may or may not be in T.) A least upper bound need not exist, but if it does, it is unique.

An element x of a partially ordered set S is said to be *maximal* if S contains no larger element. It is to be observed that S may contain many maximal elements.

A partially ordered set is a *chain* (also called a simply ordered set or a linearly ordered set) if every two elements are comparable; that is, either $x \geqq y$ or $y \geqq x$.

We now state Zorn's lemma:

Zorn's lemma. Let S *be a partially ordered set in which every chain has a least upper bound. Then* S *has a maximal element.*

This brief account will suffice for the applications we shall make of this lemma. We refer the reader to the literature for details on other forms of Zorn's lemma, and their equivalence to the well-ordering axiom or the axiom of choice.

5. DIVISIBLE GROUPS

In an abelian group any element may be multiplied by an integer. But what about dividing by an integer? The answer is that the result may not exist, and if it exists, it may not be unique. So we shall not attempt to attach a meaning to the symbol $\frac{1}{n}x$, but nevertheless we shall say that x is divisible by n if there exists y with $ny = x$.

Examples. (*a*) The element 0 is divisible by any integer.
(*b*) If x has order m, then it is divisible by any integer prime to m.
(*c*) In the additive group of rational numbers, every element is divisible by every integer.

In this section we are going to study groups which share this last property with the additive group of rational numbers.

Definition. A group G is *divisible* if for every x in G and every integer n there exists an element y in G with $ny = x$.

Alternatively, G is divisible if $G = nG$ for every integer n.

We note that a cyclic group is not divisible. Nor for that matter is a direct sum of cyclic groups. Indeed, it is clear that *a direct sum of groups is divisible if and only if every summand is divisible.* Another easily verified fact is that *a homomorphic image of a divisible group is divisible.* So we note that the group of rationals mod one is divisible, since it is a homomorphic image of the additive group of rationals.

The group $Z(p^\infty)$ is also divisible. This is not apparent from the definition (§ 2) of $Z(p^\infty)$ as P/Z, since P (the group of rationals with denominator a power of p) is not divisible. If we admit, as was claimed at the end of § 3, that $Z(p^\infty)$ is a direct summand of the rationals mod one, then the divisibility of $Z(p^\infty)$ is assured. But let us give a direct argument. Since $Z(p^\infty)$ is a primary group, all of its elements are divisible by any integer prime to p. On the other hand, it is clear that every element of $Z(p^\infty)$ can be divided by arbitrary powers of p. On putting these two statements together, we establish that $Z(p^\infty)$ is divisible.

The theory of divisible groups is based on the theorem below. It is to be understood that by a "divisible subgroup" we mean a subgroup which as a group on its own merits is divisible. In other words, for H to be a divisible subgroup of G, it has to be the case that for every $h \in H$, and every integer n, there exists an element h_1 *again in* H, satisfying $nh_1 = h$.

Theorem 2. A divisible subgroup of an abelian group is a direct summand.

Proof. Let H be a divisible subgroup of G. Our task is to find a subgroup K with $H \cap K = 0$, $H + K = G$. Offhand, it probably seems difficult to imagine how to go about finding such a subgroup. It is rather remarkable that a crude use of Zorn's lemma accomplishes the objective.

We consider the set $\mathfrak{S}$ of all subgroups L which satisfy $H \cap L = 0$. (There is at least one, namely, 0.) We would like to get one as large as possible. So we set out to get a maximal element in $\mathfrak{S}$. We partially order $\mathfrak{S}$ by set-theoretic inclusion. To use Zorn's lemma, we have to verify that every chain in $\mathfrak{S}$ has a least upper bound. Suppose $\{L_i\}$ is a chain in $\mathfrak{S}$. To get the desired least upper bound, we simply take the set-theoretic union of the L_i's, say M. Three things need to be verified:

(*a*) M is a subgroup. We take x and y in M and have to show that $x - y$ is in M. Now x and y got into M only because x was, say, in L_i, y in L_j. Moreover, L_i and L_j are comparable, say $L_i \subseteq L_j$. Then *both* x and y are in L_j, and so is $x - y$. Hence $x - y$ is in M.

(*b*) $H \cap M = 0$. This follows from the fact that every element of M is in one of the L_i's, and each $H \cap L_i = 0$.

(*c*) M is the least upper bound of $\{L_i\}$. This is clear.

We remark that the arguments above are of a routine nature. Indeed the whole would usually be condensed to: "By Zorn's lemma pick a subgroup maximal with respect to disjointness from H." In the future we shall give such a condensed version. But the reader should observe that there is one vital point which must be checked before Zorn's lemma is applicable—that the property of disjointness from H is preserved under taking of least upper bounds of chains.

At any rate, we now have a maximal subgroup K in $\mathfrak{S}$, and we set out to prove $H + K = G$. We suppose the contrary. Then there exists an element x which is not in $H + K$. A fortiori, x is not in K. We now form K', the subgroup generated by K and x. K' is larger than K, and, in fact, K' consists of all elements $k + nx$ where k is in K and n is an integer. By the maximality of K we know that $H \cap K' \neq 0$. Hence there exists a nonzero element h in $H \cap K'$:

$$h = k + nx. \tag{3}$$

From equation (3) we see that nx is in $H + K$. It is interesting to observe that we have not yet used the divisibility of H. In other words, we have proved that if we take any subgroup H and a maximal subgroup

K disjoint from it, then H + K is at any rate large enough so that G/(H + K) is a torsion group.

Now to complete the proof. We may suppose that n is the smallest positive integer such that $nx \in H + K$ (of course $n > 1$). Let p be a prime dividing n, and write $y = (n/p)x$. Thus y is not in H + K, but $py = nx = h - k$. By the divisibility of H we may write $h = ph_1$, with $h_1 \in H$. Let $z = y - h_1$. Then z is not in H + K, but $pz = -k$ is in K. We now repeat the argument above; when we adjoin z to K we must get a subgroup not disjoint from H. Hence we have

$$(4) \qquad h_2 = k_2 + mz$$

where $h_2 \in H$, $h_2 \neq 0$, $k_2 \in K$, and m is an integer. It is impossible that m be a multiple of p, for then the right side of equation (4) is in K, while the left side is a nonzero element of H. Hence m is prime to p, and we may find integers a, b such that $am + bp = 1$. We have $z = amz + bpz \in H + K$, a contradiction. This completes the proof of Theorem 2.

It is worth while to take another glance at the mechanism of the preceding proof. A transfinite induction contains two steps: a passage to the limit, and an argument for pushing one stage further. The first step was accomplished above by the initial application of Zorn's lemma. The second was concealed in an indirect proof, but it is perfectly possible to rewrite this as a direct proof. It is rather typical of the use of Zorn's lemma that it culminates in an indirect argument.

We proceed to a useful theorem applying to an arbitrary group G. In G consider the totality of divisible subgroups and form their union M (this is one of the rare occasions when we could correctly construe M to be the set-theoretic rather than the group-theoretic sum, for there is actually a largest subgroup among them). Now M consists of finite sums $x_1 + \cdots + x_k$ where each x_i lies in some divisible subgroup. Since each x_i is divisible by n (for arbitrary n), so is the sum. Thus M is itself a divisible group. We have proved the first statement of the following theorem:

Theorem 3. Any abelian group G *has a unique largest divisible subgroup* M, *and* $G = M \oplus N$ *where* N *has no divisible subgroups.*

To prove the last statement of Theorem 3 we quote Theorem 2 and deduce that M is a direct summand of G. The other summand N can have no divisible subgroups, for these would be divisible subgroups of G.

The subgroup M is uniquely determined, for it is intrinsically characterized as the maximal divisible subgroup. Suitable examples show that N is not necessarily unique. Of course, N is unique *up to isomorphism*, for it is isomorphic to G/M.

Theorem 3 suggests the following definition:

Definition. An abelian group is *reduced* if it has no (nonzero) divisible subgroups.

To classify all abelian groups it suffices by Theorem 3 to do the divisible and reduced cases. We proceed at once to a complete determination of divisible groups. After this has been done, it will usually be possible to restrict our attention to reduced groups.

Theorem 4. A divisible abelian group is a direct sum of groups each isomorphic to the additive group of rational numbers or to $Z(p^\infty)$ *(for various primes* p).

Proof. Let G be the group, T its torsion subgroup. It is easy to verify that T is again divisible. By Theorem 2, $G = T \oplus F$ where F is isomorphic to G/T, and so is divisible and torsion-free. We now study T and F separately.

The discussion of F is easy to carry out directly, but it will clarify the situation to relate it to standard vector space theory. Let x be any element in F, and n a nonzero integer. Then, since F is divisible and torsion-free, there is *exactly one* element y in F with $ny = x$. Thus we can attach a unique meaning to $\left(\frac{1}{n}\right)x$, and then to rx, where r is a rational number. Now it is routine to check the requisite postulates, and we conclude that F is a vector space over the field of rational numbers. Any vector space (whether finite or infinite-dimensional) has a basis. Translated into group-theoretic terms, this says that F is a direct sum of groups each isomorphic to the additive group of rational numbers.

We turn our attention now to the divisible torsion group T. By Theorem 1, T is a direct sum of primary groups, each of which will again be divisible. So we may as well assume that T itself is a primary group (say for the prime p), and we have to prove that T is a direct sum of groups isomorphic to $Z(p^\infty)$.

A little care must be exercised in applying Zorn's lemma for this purpose. We consider subgroups of T isomorphic to $Z(p^\infty)$. Of course it is not clear at the moment that any such subgroups exist, but this is something to worry about later. Since the objective is to express T as a direct sum of such subgroups, it is appropriate to consider independent sets of these subgroups. So we decide to form $\mathfrak{P}$, the set of all independent sets of subgroups isomorphic to $Z(p^\infty)$. It should be borne in mind that each element of $\mathfrak{P}$ is an independent set of subgroups, that is to say, a set of sets; so $\mathfrak{P}$ is a set of sets of sets! We introduce in $\mathfrak{P}$ the natural ordering given by set-theoretic inclusion. The proof that every chain in $\mathfrak{P}$ has a least upper bound offers no difficulty. Thus we may apply Zorn's lemma to arrive at a maximal independent set of subgroups isomorphic to $Z(p^\infty)$, say $\{S_i\}$. Write $S = \Sigma S_i$. The proof will be finished if we show $S = T$. In any event, S is a divisible group (being a direct sum of divisible groups), and so $T = S \oplus R$ by Theorem 2. Now we come to the crux of the proof. If $R \neq 0$, we shall show that

R contains a subgroup isomorphic to $Z(p^\infty)$; by adjoining this subgroup to $\{S_i\}$ we get a contradiction, for the enlarged set of subgroups is still independent.

We select in R an element x_1 of order p. Using the divisibility of R, we find in succession elements $x_2, x_3, \cdots$ with $px_2 = x_1$, $px_3 = x_2, \cdots$, and in general $px_{i+1} = x_i$. Now map x_1 into $1/p$, x_2 into $1/p^2, \cdots$, x_i into $1/p^i, \cdots$. This gives rise to an isomorphism between the subgroup generated by the x's and the group $Z(p^\infty)$, and completes the proof of Theorem 4.

We conclude this section with two remarks:

(*a*) The particular way in which Zorn's lemma was applied deserves comment. It would perhaps have seemed more natural to consider all subgroups which are direct sums of groups isomorphic to $Z(p^\infty)$, and proceed to use Zorn's lemma to get a maximal one. But there is a catch: Why should the union of an ascending chain of these subgroups be expressible as a direct sum of $Z(p^\infty)$'s? It is true that each subgroup in the chain will have such an expression, but these expressions will presumably be unrelated, and it is impossible to combine them. It was to obviate this difficulty that we chose the fussier formulation above.

(*b*) The uniqueness question that arises in connection with Theorem 4 also warrants attention. We are presented with a set of cardinal numbers: one for the number of rational summands, and then one for every p giving the number of $Z(p^\infty)$ summands. It is a fact that these cardinal numbers are invariants (and consequently, of course, a complete set of invariants). For the rational summands this is simply a restatement of the invariance of the number of elements in a basis of a vector space, a fact which we shall take for granted. For the $Z(p^\infty)$ summands the question of uniqueness of the cardinal number can be rapidly reduced to the case of a vector space: all we have to do is drop down to the subgroup of all elements x satisfying $px = 0$ (this being a vector space over the field of integers mod p).

For an alternative approach to Theorems 2 and 4 the reader is referred to exercises 1-4.

Exercises 1-8

1. Let G be a group, H a subgroup, D a divisible group. Let f be a homomorphism of H into D. Show that f can be extended to a homomorphism of G into D. (First study the task of extending f to the subgroup of G generated by H and one more element; it turns out that the divisibility of D always makes this possible. It remains to prepare the way for application of Zorn's lemma. This may be done as follows: Consider pairs (S_i, f_i), where S_i is a subgroup of G containing H, and f_i is an extension of f. Partially order these pairs by decreeing that $(S_i, f_i) \geq (S_j, f_j)$ means that $S_i \supseteq S_j$ and that f_i is an extension of f_j. Apply Zorn's lemma.)