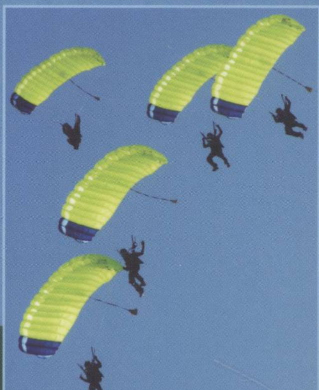
INSIGHTS INTO GAME THEORY

An Alternative Mathematical Experience



EIN-YA GURA and MICHAEL B. MASCHLER

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AND
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Insights into Game Theory

Few branches of mathematics have been more influential in the social sciences than game theory. In recent years, it has become an essential tool for all social scientists studying the strategic behavior of competing individuals, firms, and countries. However, the mathematical complexity of game theory is often very intimidating for students who have only a basic understanding of mathematics. *Insights into Game Theory* addresses this problem by providing students with an understanding of the key concepts and ideas of game theory without using formal mathematical notation. The authors use four very different topics (college admissions, social justice and majority voting, coalitions and cooperative games, and a bankruptcy problem from the Talmud) to investigate four areas of game theory. The result is a fascinating introduction to the world of game theory and its increasingly important role in the social sciences.

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MICHAEL B. MASCHLER is a Professor of Mathematics and a member of the Center for the Study of Rationality at the Hebrew University of Jerusalem. Professor Maschler is a world-renowned game theorist. He has had a long and fruitful collaboration with Robert Aumann (co-winner of the 2005 Nobel Prize for Economics), including their solution to a 2000-year-old puzzle on inheritance laws in the Talmud, discussed in this book.

This book is dedicated to the memory of Michael Maschler,
who passed away on July 20, 2008.

Preface

This book is a *tour de force* of what may be called "verbal mathematics." It demonstrates conclusively that mathematics is not a matter of symbols and equations; rather, it may be characterized as "precise reasoning that has considerable depth, complexity, or sophistication." The book is accessible to everyone who can think.

Also, it is a wonderful introduction to game theory; rather than "explaining" what the theory is about, it simply *does* it. If somebody came from Mars and wanted to know what we mean by "music," you could try to "explain" it; but it would be better to play a Bach fugue, a Verdi aria, some Louis Armstrong jazz, and "Lucy in the Sky with Diamonds." The second alternative is what Gura and Maschler do. Enjoy!

Robert J. Aumann

Introduction

Game theory is a relatively young branch of mathematics that goes back to the publication of *Theory of Games and Economic Behavior* by John von Neumann and Oskar Morgenstern in 1944.[1]

Game theory undertakes to build mathematical models and draw conclusions from these models in connection with interactive decision-making: situations in which a group of people not necessarily sharing the same interests are required to make a decision.

The choice of the topics reflects our purpose: we wanted to present material that does not require mathematical prerequisites and yet involves deep game-theoretic ideas and some mathematical sophistication. Thus, we ruled out topics from non-cooperative game theory, which requires some knowledge of probability, matrices, and point-set topology.

Broadly speaking, the topics chosen are all related to the various meanings that can be given to the concept of "fair division." The four chapters illustrate this.

The first, "Mathematical Matching," concerns, among other things, the problem of assigning applicants to institutions of higher learning. Each applicant ranks the universities according to his scale of preferences. The institutions of higher learning, in turn, rank the applicants for admission according to their own scale of preferences. The question is how to effect the "matching" between the applicants and the universities. The reader will discover that this problem leads to unexpected solutions.

The second chapter, "Social Justice," concerns social decision rules. In a democratic society it is customary to make decisions by

[1] Several "game-theoretic" topics had been discussed before this publication, but not in any systematic way.

a vote. The decision supported by the majority of voters is adopted. But the reader will discover that "majority rule" does not always yield clear-cut solutions. The attempt to find other voting rules raises unexpected difficulties.

The third chapter, "The Shapley Value in Cooperative Games," addresses, among other things, the following problem: a group of people come before an arbitrator and inform him of the expected profits of every subgroup, as well as of the whole group, if the groups operate independently. It seems that these data are sufficient for the arbitrator to decide how to divide the profits if all the litigants operate jointly.

The fourth chapter, "Analysis of a Bankruptcy Problem from the Talmud," addresses the following problem: several creditors have claims to an estate, but the total amount of the claims exceeds the value of the estate. How should the estate be divided among the creditors? In the chapter several solutions are accepted, two of which are discussed in the Talmud.

As explained above, this book is not a textbook in game theory. Rather, it is a collection of a few topics from the theory intended to open a window onto a new and fascinating world of mathematical applications to the social sciences. Our hope is that it will motivate the reader to take a solid course in game theory.

One of the aims of the book is to acquaint the reader and the student with "a different mathematics" – a mathematics that is not buried under complicated formulas, yet contains deep mathematical thinking. Another aim is to show that mathematics can efficiently handle social issues. A third aim is to deepen the mathematical thinking of the person who studies this book.

We believe that by studying the topics of this book, the mathematical thinking of the student will be enriched.

This book selects a small number of topics and studies them in depth. It shows the student of the social sciences how a mathematical model can be constructed for real-life issues.

The chapters are independent. A teacher and a student can choose one chapter or several and cover them in any order.

In high schools, the book can be used by students on any program track or as extracurricular material. The teacher can proceed to the deeper parts of each chapter if she has a mathematically inclined class or skip some of the proofs if the class cannot handle them. The book can also be used by students who want to read independently or under the guidance of a teacher beyond what is required in school.

At universities and colleges, the book can be used in courses whose aim is to introduce general game-theoretic topics and deepen mathematical thinking.

This book owes its origin to the PhD thesis of co-author Ein-Ya Gura. We thank the Science Teaching Center at the Hebrew University of Jerusalem for permission to publish this translation from Hebrew, the Center for the Study of Rationality for funding the translation, and the translator, Michael Borns, not only for the accuracy of his translation, but for the competence of his editing. We thank James Morrow for taking the time to read and comment on the manuscript, Zur Shapira for recommending it to Cambridge University Press, and Chris Harrison and the staff of Cambridge University Press for their encouragement and help in bringing the book to its final form. Last, but not least, we thank Robert Aumann for providing the impetus for both the Hebrew and the English publication of this book.

Ein-Ya Gura and Michael Maschler,
The Hebrew University of Jerusalem,
April 2008

Contents

1 Mathematical Matching

1.1 INTRODUCTION

In 1962 a paper by David Gale and Lloyd S. Shapley[1] appeared at the RAND Corporation, whose title, "College Admissions and the Stability of Marriage," raised eyebrows. Actually, the paper dealt with a matter of some urgency.

According to Gale,[2] the paper owes its origin to an article in the *New Yorker*, dated September 10, 1960, in which the writer describes the difficulties of undergraduate admissions at Yale University. Then as now, students would apply to several universities and admissions officers had no way of telling which applicants were serious about enrolling. The students, who had every reason to manipulate, would create the impression that each university was their top choice, while the universities would enroll too many students, assuming that many of them would not attend. The whole process became a guessing game. Above all, there was a feeling that actual enrollments were far from optimal.

Having read the article, Gale and Shapley collaborated. First, they defined the concept of stable matching, and then proved that stable matching between students and universities always exists. This and further developments will be discussed in this chapter.

For simplicity, Gale and Shapley started with the unrealistic case in which there are exactly n universities and n applicants and each university has exactly one vacancy. A more realistic description of this case is a matching between men and women – hence the title of their paper.

[1] Gale, D. and Shapley, L. S. 1962. "College admissions and the stability of marriage," American Mathematical Monthly 69: 9–15.

[2] Gale, D. 2001. "The two-sided matching problem: origin, development and current issues," International Game Theory Review 3: 237–52.

1.2 THE MATCHING PROBLEM

Consider a community of men and women where the number of men equals the number of women.

Objective: Propose a good matching system for the community.[3] To be able to propose such a system, we shall need relevant data about the community. Accordingly, we shall ask every community member to rank members of the opposite sex in accordance with his or her preferences for a marriage partner. We shall assume that no man or woman in the community is indifferent to a choice between two or more members of the opposite sex.[4] For example, if Al's list of preferences consists of Ann, Beth, Cher, and Dot, in that order, then Al ranks Ann first, Beth second, Cher third, and Dot fourth.[5] Again, we shall assume that Al is not indifferent to a choice between two or more of the four women on his list.

Example:
The men are Al, Bob, Cal, Dan.
The women are Ann, Beth, Cher, Dot.
Their list of preferences is:

Women's Preferences:

	Ann	Beth	Cher	Dot
Al	1	1	3	2
Bob	2	2	1	3
Cal	3	3	2	1
Dan	4	4	4	4

Men's Preferences:

	Ann	Beth	Cher	Dot
Al	3	4	1	2
Bob	2	3	4	1
Cal	1	2	3	4
Dan	3	4	2	1

Explanation: The numbers in the table indicate what rank a man or woman occupies in the order of preferences. For example, according to the men's ranking of the women, Al ranks Cher first, Dot second,

[3] The meaning of "good" will become clear presently.

[4] This assumption is introduced to simplify our task. In Section 1.10 we shall see how to dispense with it.

[5] If Al prefers Ann to Beth and Beth to Cher, it follows that he prefers Ann to Cher. Accordingly, we may list all his preferences in a row.

Ann third, and Beth last. And according to the women's ranking of the men, Cher ranks Bob first, Cal second, Al third, and Dan last. Thus Al ranks Cher first, while Cher ranks Al just third. If we pair them off, the match will not work out, if the first or second candidate on Cher's preference list agrees to be paired off with her.

Given everyone's preferences, can you propose a matching system for the community?

A Possible Proposal:

$$\begin{pmatrix} \text{Al} & \text{Bob} & \text{Cal} & \text{Dan} \\ | & | & | & | \\ \text{Dot} & \text{Ann} & \text{Beth} & \text{Cher} \end{pmatrix}$$
$$2 \times 2 \quad 2 \times 2 \quad 2 \times 3 \quad 2 \times 4$$

The numbers below each couple indicate what rank one member of a couple assigns to the other member. The number on the left indicates what rank the man assigns to the woman; the number on the right, what rank the woman assigns to the man. (Verify it!)

Argument for the Proposal:

(1) No members of any couple rank each other first.
(2) No members of any couple rank each other 1×2 or 2×1.
(3) The members of two couples rank each other second.
(4) Cal can be paired off with Cher or Beth, but he prefers Beth.
(5) That leaves Dan and Cher, who can be paired off.

This is indeed a possible proposal, but it is not a good one.

Cher is displeased, because she is paired off with her last choice. She can propose to Bob, but she will be turned down because she is his last choice. She will fare no better with Cal, because she is his third choice while he is paired off with his second choice. On the other hand, if Cher proposes to Al, he will be very pleased, because she is his first choice.

The proposal is rejected, because Cher and Al prefer each other to their actual mates, and one can reasonably assume that they will reject the matchmaker's proposal.

Another Possible Proposal: Let us try to pair off all the men with their first choice.

Al's first choice is Cher.
Bob's first choice is Dot.
Cal's first choice is Ann.
Dan's first choice is Dot.

We see that there is a problem: both Bob and Dan prefer Dot. We can try to pair off Dan with his second choice, Cher, but she is already paired off with Al. Will Dan's third choice work out? Dan's third choice is Ann, but she is already paired off with Cal. That leaves Dan with his last choice, Beth.

$$\begin{pmatrix} \text{Al} & \text{Bob} & \text{Cal} & \text{Dan} \\ | & | & | & | \\ \text{Cher} & \text{Dot} & \text{Ann} & \text{Beth} \end{pmatrix}$$

$$1 \times 3 \quad 1 \times 3 \quad 1 \times 3 \quad 4 \times 4$$

Three of the four men are paired off with their first choice. Do you think this proposal will be accepted or rejected?

Still Another Possible Proposal: Now we shall try to pair off all the women with their first choice. Is it possible?

Ann's first choice is Al.
Beth's first choice is Al.
Cher's first choice is Bob.
Dot's first choice is Cal.

We see that if we pair off Ann with her first choice, Al, then Beth cannot be paired off with him too. We can pair off Beth with her second choice, Bob, but he is already paired off with Cher. And Beth's third choice, Cal, is already paired off with Dot. Beth is therefore left with her last choice, Dan.