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Lie Groups and Compact Groups

JOHN F. PRICE

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Preface

The purpose of these notes is twofold: to provide a quick self-contained introduction to the general theory of Lie groups and to give the structure of compact connected groups and Lie groups in terms of certain distinguished 'simple' Lie groups. With regards to the first aim, the notes can be used to provide a general introduction to the fundamentals of Lie groups or as a bridge to more advanced texts. In either case, experience has shown that they are suitable for postgraduate students and, at least the earlier chapters, for senior undergraduates. Concerning the second aim, the existing treatments of the structure results referred to above seem to be all from a fairly advanced point of view (cf. Pontrjagin [1] and Weil [1]). It is hoped that the present, more modern treatment makes these powerful results more generally accessible, in particular to those only wishing to use them as a tool.

The theory of Lie groups lies at the junction of the theories of differentiable manifolds, topological groups and Lie algebras. In keeping with current trends, when dealing with manifolds (and hence with Lie groups) a coordinate-free notation is used, thus removing the necessity for tedious juggling of indices and, hopefully, adding to the clarity and intuitiveness of the theory. In the case of Lie groups, particular emphasis is placed upon results and techniques which educe the interplay between a Lie group and its Lie algebra.

During the past few years a number of important results have been obtained in harmonic analysis on compact groups and compact Lie groups by using the structure of these groups ... the overall orientation of the following notes is to give full details of several of these structure results. The main theorem for Lie groups is that if G is a compact connected Lie group, then G is topologically isomorphic to

$$(G_0 \times G_1 \times \ldots \times G_m)/K,$$

where G_0 is the identity component of the centre of G, the G_j ($j = 1, \ldots, m$) are all the simple, connected, normal Lie subgroups of G, and K is a finite subgroup of the centre of the product. As a corollary, a similar structure theorem is given in which the G_j are also simply connected. This latter result is then generalised to arbitrary compact connected groups.

The decision on whether to include a particular result was based almost entirely on whether or not it was required for the proofs of the above structure theorems. This procedure accounted for the inclusion of most of the fundamental results and concepts in the theory of Lie groups; to round off the notes it only remained to add a few divertimenti such as the contents of Chapter 4 on the geometry of Lie groups or the list in Chapter 6 of necessary and sufficient conditions for a compact group to be Lie.

Chapter 1 contains results in the theory of analytic manifolds which are basic to the study of Lie groups. Chapter 2 begins the study of Lie groups and it is here that most of the fundamental concepts such as Lie algebras, left invariant vector fields, 1-parameter subgroups and the exponential map are introduced. In Chapter 3 the first deep result is presented; this is the Campbell-Baker-Hausdorff formula and it describes a relationship between the group structure of a Lie group and the algebraic structure of its Lie algebra. Chapter 4 introduces the notion of a geodesic on a Lie group and uses the resulting ideas to show that the exponential map is surjective whenever the Lie group is compact and connected. The correspondence between Lie subgroups of a Lie group and subalgebras of its Lie algebra is treated in Chapter 5. Chapter 6 begins with a list of conditions which are necessary and sufficient for a compact group to be Lie and ends with the structure results mentioned above. An appendix contains all the results on locally compact topological groups and their representations used in the body of the notes.

Further remarks, historical and motivational, on the contents of a chapter are given at the end of that chapter, along with related exercises. That a piece of theory is essential to a particular proof is no bar to it being included as an exercise if it is fairly straightforward or if it is fully treated in the literature.

I gave a course on some of the topics treated in these notes during 1973 at the Australian National University to an audience consisting mainly of postgraduate students, and then in 1974 at the University of New South Wales. These notes derive from these courses and in particular from duplicated notes of the earlier chapters. I am grateful to the people attending these courses for improvements of a number of arguments and in particular to Dr. Graham Wood for his reading of Chapter 1 and subsequent discussions. It was he who developed the local coordinate-free formula given in 1. 3. 2 and 1. 3. 3 for the Lie product of two analytic vector fields.

Finally, I feel that this preface would not be complete without some mention of the role of diagrams. Even though a large number of the concepts and results of manifolds and Lie groups have a strong pictorial or diagrammatic aspect, my experience is that diagrams in mathematics books are often of little value without a personal explanation. For this reason and because of widely varying preferences as to style, apart from several 'commutative arrow diagrams', none have been included here. However, without doubt they are valuable in developing an intuition in this area and the reader is strongly encouraged to experiment with them. Also some have found benefit in reformulating key results in terms of coordinates.

In the later chapters a number of substantial results are stated without proof since it is felt that to include them would take us too far afield in a set of lecture notes. However, the omitted proofs are all clearly presented in numerous standard texts to which detailed references are given. This also allows a clearer path to the structure theorems.

Kensington, 1976 J. F. P.

Contents

1·Analytic manifolds

This chapter contains the basic theory of analytic manifolds modelled on finite-dimensional real vector spaces. As promised, a coordinate-free approach will be used with emphasis on global definitions and properties. One of the reasons for including this chapter, instead of referring the reader to one or other of the numerous texts on manifolds, is to allow the reader to gain familiarity with this approach since it will permeate our whole treatment of Lie groups. Once one does away with coordinates it becomes obvious that large chunks of the theory of manifolds can be effortlessly generalised to manifolds modelled on infinite-dimensional spaces. We will have no need here of this degree of generality for reasons explained in the Notes at the end of the chapter, but the interested reader should consult the works of Lang, [1] and [2]. Since the theory of manifolds is one of the three legs on which the study of Lie groups stands, the other two being the theory of locally compact groups and the theory of Lie algebras, it is important that the ideas in this chapter, few though they may be, are well understood.

1.1 Manifolds and differentiability

1.1.1 **Manifolds.** Let M be a nonvoid Hausdorff topological space and E a real finite-dimensional vector space. If $\phi : U \to V$ is a homeomorphism between open subsets U and V of E and M respectively, then we say that ϕ is a chart on M. Also, if $p \in V$, then we say that ϕ is a chart about p. (Thanks to the infiltration of notions from category theory, it is now respectable to suppose that whenever a function is specified, its domain and codomain are automatically specified along with it. Hence there is no need to always explicitly write each function as a triple. We will adopt this convention here and immediately make use of it by supposing that whenever ϕ, ϕ_α and ϕ_β are charts, then their domains are U, U_α and U_β respectively, and their codomains (which in this case are also their ranges) are V, V_α and V_β respectively, unless otherwise specified.)

Suppose that ϕ_α is a chart on M for each α in some index set A. Then this collection, denoted $(\phi_\alpha : \alpha \in A)$, is called an atlas on M provided:

(i) each U_α is contained in the same finite-dimensional space, E say; and

(ii) the union of the V_αs is equal to M.

In this case we say that M is a manifold or M is a manifold modelled on E. (When we wish to be completely explicit we will say that the pair $(M, (\phi_\alpha : \alpha \in A))$ is a manifold. However, when no confusion seems possible, we will write only 'M is a manifold'.) The dimension of M as a manifold is defined as the dimension of E. Regarding the invariance of dimension, see Exercise 1.C(i).

It is obvious that every open subset U of a real finite-dimensional space E is a manifold when equipped with its identity map i. Henceforth, whenever we refer to such a set U as a manifold, its atlas will always be assumed to be $i : U \to U$. Less trivial examples of a manifold will be given in Chapter 2 after the definition of a Lie group.

Convention. It is easily seen that a Hausdorff topological space M can be equipped with a 0-dimensional atlas if and only if the topology of M is discrete. Thus, even though all the ensuing results on manifolds and Lie groups are valid for the 0-dimensional case, they are banal. Hence we will make the convention that the dimensions of all linear spaces, manifolds, and Lie groups are at least 1. In those cases when we want to emphasise that the dimension of a linear real space is n, we will often write it as $\mathbf{R}^n$, where $\mathbf{R}$ denotes the real line. Generally, however, such finite-dimensional real spaces will be denoted by E or F.

1.1.2 **Differentiable maps.** The abstract definition of the derivative of a map between finite-dimensional vector spaces is the main point of departure from the classical approach to differentiable manifolds to one involving no explicit mention of coordinates. Given a function f from an open subset U of a finite-dimensional real space E into another such space F, then we say that f is differentiable at x in U if there exists a linear map $f'(x) : E \to F$ such that

$$(1.1.1) \qquad \lim_{\varepsilon \to 0} \varepsilon^{-1}(f(x + \varepsilon h) - f(x)) = f'(x)h$$

uniformly for h in any bounded subset of U (provided, of course, that $x + \varepsilon h \in U$). This is readily seen to be equivalent to the existence of a linear map $f'(x) : E \to F$ such that

$$(1.1.2) \qquad \lim_{h \to 0, h \neq 0} \frac{\|f(x + h) - f(x) - f'(x)h\|}{\|h\|} = 0 .$$

Here the norm is taken as any one of the equivalent norms which make the finite-dimensional space E into a Banach space. (See Edwards [1, Proposition 1.9.6].) Throughout the sequel, whenever the need for a topology on a finite-dimensional vector space arises, then it will always be taken to be the topology induced from such a norm.

Exercise 1.A collects together some of the elementary properties of this derivative, for example, the uniqueness of the linear map $f'(x)$.

If f is differentiable at each point of U, we say that f is <u>differentiable on</u> U. In this case we have the function

$$f' : U \to \text{hom}(E, F),$$

where $f' : x \mapsto f'(x)$ and $\text{hom}(E, F)$ is the linear space of linear maps from E into F. Continuing in this way it is clear that we may have higher order derivatives $f'' = (f')'$, $f''' = (f'')'$, and so on. In this way we arrive at the notion of a <u>smooth</u> function (at a point or on an open set) being a function which possesses derivatives of all orders (in a neighbourhood of the point or in the open set). Suppose that f is as above and that f'' exists on U, then f'' is a function from U into $\text{hom}(E, \text{hom}(E, F))$. As is customary, we identify this latter space in the canonical manner with $\text{hom}^2(E \times E, F)$, the bilinear maps from $E \times E$ into F. In fact, throughout we adopt the convention that if $f^{(p)}$ exists on U, then its image space is $\text{hom}^p(E \times \ldots \times E$ (p times), $F)$. This simplifies a number of expressions, including Taylor's expansion in 1.1.5 below.

If E, F and G are real finite-dimensional spaces and $f : E \to F$ and $g : F \to G$ are differentiable at x and f(x) respectively, a classical result (included below in Exercise 1.A) states that $g \circ f$ is also differentiable at x and moreover:

$$(1.1.3) \qquad (g \circ f)'(x) = g'(f(x)) \circ f'(x).$$

1.1.3 **Remarks.** The notion of the derivative given above is often called the Fréchet derivative. For Banach spaces the study of this derivative forms Chapter VIII of Dieudonné [1], while Averbukh and Smolyanov [1] study this and related derivatives on topological vector spaces in general. For example, these latter authors show that in a certain sense the Fréchet derivative is the weakest type of differentiation for which the first order chain rule, formula (1.1.3) above, is valid for finite-dimensional spaces [1, p. 74].

1.1.4 **Maps from R.** When considering a differentiable map $f : \mathbf{R} \to E$, then $f'(x)$ satisfies

$$f'(x)(t) = f'(x)(1).t \text{ for each } t \text{ in } \mathbf{R}.$$

Thus $f'(x)$ is completely described by its value at 1 and we often write $f'(x)$ in place of $f'(x)(1)$. (This is precisely what happens in the classical case of functions from **R** into **R** where the derivative $f'(x)$ is taken to be a number as opposed to an operator.)

1.1.5 **Analytic functions.** Suppose that f is a smooth function from an open subset U of a real finite-dimensional space E into another such space F. Let x in U and y in E be such that $x + ty \in U$ for all $t \in [0, 1]$. If $y^{(m)}$ denotes the m-tuple $(y, \ldots, y)$, then

$$(1.1.4) \qquad f(x+y) = f(x) + \frac{1}{1!} f'(x)y + \ldots + \frac{1}{m!} f^{(m)}(x)y^{(m)} + R_{m+1}(y)$$

for each $m \in Z^{+} = \{0, 1, 2, \ldots\}$, where the error term R_{m+1} satisfies $\lim_{y \to 0} R_{m+1}(y). \|y\|^{-m} = 0$. (See Exercise 1.D, where one particular version of the error term is described.) The sum (1.1.4) is called Taylor's formula of degree m.

Just as in the 1-dimensional case, we say that a smooth function $f : U \to F$ is (real) analytic on U if for each x in U there exists an open ball $B \subseteq U$ with centre x such that for all $z = x + y$ in B, the series

$$(1.1.5) \qquad \sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(x)y^{(m)}$$

is absolutely convergent (that is, $\sum_m \frac{1}{m!} \|f^{(m)}(x)y^{(m)}\|$ is convergent, where the norm is that of F) and converges to f(z). The function $f : U \to F$ is said to be analytic at x if it is analytic in some neighbourhood of x.

Examples. (i) If $f : E \to F$ and $g : F \to G$ are analytic at x and f(x) respectively, then $g \circ f$ is analytic at x.

(ii) If $f : U \to F$ is analytic on U, an open subset of E, then $f^{(k)}$ is also analytic on U for each $k \in Z^+$ and its expansion at $x + y \in U$, where $x \in U$, is $\sum_m \frac{1}{m!} f^{(k+m)}(x)y^{(m)}$; in other words,

$$f^{(k)}(x+y)(u_1, \ldots, u_k) = \sum_{m=0}^{\infty} \frac{1}{m!} f^{(k+m)}(x)(\overbrace{y, \ldots, y}^{m \text{ terms}}, u_1, \ldots, u_k).$$

The validation of these two examples is left as an exercise for the interested reader.

(iii) Examples of smooth functions which are not analytic are well known. Even the absolute convergence of (1.1.5) in a ball is not sufficient to ensure the analyticity of the function at the centre of the ball concerned. For example, consider the smooth function $g : \mathbf{R} \to \mathbf{R}$ defined by

$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} .$$

It satisfies $g^{(m)}(0) = 0$ for all $m \in Z^+$ so that the series (1.1.5) is absolutely convergent for all $y \in \mathbf{R}$, but it only converges to g(x) when $x = 0$.

1.1.6 **Smooth atlases and manifolds.** An atlas $(\phi_\alpha : \alpha \in A)$ on the Hausdorff topological space M is said to be smooth if each of the functions $\phi_\beta^{-1} \circ \phi_\alpha$ is smooth on $\phi_\alpha^{-1}(V_\alpha \cap V_\beta)$. Such an atlas is said to be maximal if whenever U and V are open subsets of E and M respectively and $\phi : U \to V$ is a homeomorphism with the property that the functions

$$(1.1.6) \quad \phi_\alpha^{-1} \circ \phi : \phi^{-1}(V \cap V_\alpha) \to \phi_\alpha^{-1}(V \cap V_\alpha)$$

$$(1.1.7) \quad \phi^{-1} \circ \phi_\alpha : \phi_\alpha^{-1}(V \cap V_\alpha) \to \phi^{-1}(V \cap V_\alpha)$$

are smooth for each $\alpha \in A$, then $\phi \in (\phi_\alpha : \alpha \in A)$.

Lemma. Every smooth atlas on M is contained in a unique maximal smooth atlas.

Proof. If $(\phi_\alpha : \alpha \in A)$ is smooth on M, let $(\phi_{\alpha'} : \alpha' \in A')$ denote the collection of all maps ψ which are homeomorphisms between open subsets of E and M and which satisfy (1.1.6) and (1.1.7). This collection is an atlas with the desired properties. //

As a matter of terminology, a smooth atlas is said to generate the unique maximal smooth atlas which contains it.

Definition. A manifold $(M, (\phi_\alpha : \alpha \in A))$ is said to be smooth if the atlas $(\phi_\alpha : \alpha \in A)$ is both smooth and maximal.

In practice it is more usual to work with generating atlases rather than the corresponding maximal atlases since, as for the case of subbases in topology, most of the properties with which we are concerned are valid on a maximal atlas if valid on any of its generating atlases. Thus if we specify a smooth atlas $(\phi_\alpha : \alpha \in A)$ on M and then refer to $(M, (\phi_\alpha : \alpha \in A))$ as a smooth manifold, the precise meaning is that we are to take M equipped with the maximal smooth atlas generated by $(\phi_\alpha : \alpha \in A)$.

For example, if M is a real finite dimensional space equipped with its usual topology and $i : M \to M$ is the identity map, then $(M, \{i\})$ is a smooth manifold. This is simple enough but even here the maximal smooth atlas generated by i contains a superabundance of members. As an exercise describe them.

1.1.7 **Smooth maps.** If M and N are smooth manifolds with smooth atlases $(\phi_\alpha : \alpha \in A)$ and $(\psi_\beta : \beta \in B)$ respectively, we say that a map f from U, an open subset of M, into N is smooth if each of the maps $\psi_\beta^{-1} f \phi_\alpha$, defined on $\phi_\alpha^{-1}(V_\alpha \cap f^{-1}(V_\beta))$, is smooth.

In particular, f is smooth at the point x if and only if

(i) $\psi_\beta^{-1} f \phi_\alpha$ is smooth at $\phi_\alpha^{-1}(x)$ for every pair $(\phi_\alpha, \psi_\beta)$ satisfying

(ii) $x \in \text{codom}\ \phi_\alpha$, $f(x) \in \text{codom}\ \psi_\beta$.

However, because of the smoothness of the atlases involved, we need only consider the smoothness of (i) for any particular pair satisfying (ii). For example, suppose that (i) is satisfied by the pair $(\phi_\alpha, \psi_\beta)$ in (ii), and further suppose that $(\phi_{\alpha'}, \psi_{\beta'})$ is another pair satisfying (ii). Then certainly $\phi_\alpha^{-1}\phi_{\alpha'}$ and $\psi_\beta^{-1}\psi_\beta$ are smooth at $\phi_{\alpha'}^{-1}(x)$ and $\psi_\beta^{-1}(f(x))$ respectively. Thus

$$\psi_{\beta'}^{-1} f \phi_{\alpha'} = (\psi_{\beta'}^{-1}\psi_\beta)(\psi_\beta^{-1} f \phi_\alpha)(\phi_\alpha^{-1}\phi_{\alpha'})$$

is smooth at $\phi_{\alpha'}^{-1}(x)$ showing that (i) is also satisfied by the pair $(\phi_{\alpha'}, \psi_{\beta'})$.

If M and N are smooth manifolds and if $f : M \to N$ is a homeomorphism such that f and f^{-1} are smooth, then f is said to be a diffeomorphism.

1.1.8 **Analytic manifolds.** If in the above definitions of smooth atlases, manifolds and maps between manifolds, we replace 'smooth' by 'analytic', then we arrive at the definitions of analytic atlases, manifolds and maps between manifolds. If a homeomorphism and its inverse are analytic, then the homeomorphism is said to be an analytic homeomorphism or even an analytic diffeomorphism.

1.1.9 **A condition for analyticity.** Let M and N be analytic manifolds with analytic atlases $(\phi_\alpha : \alpha \in A)$ and $(\psi_\beta : \beta \in B)$ respectively. Analogously to 1.1.7, a map $f : M \to N$ is analytic at x in M if and only if there exist charts ϕ_α about x and ψ_β about f(x) such that $\psi_\beta^{-1} f \phi_\alpha$ is analytic at x.

1.2 The tangent bundle

1.2.1 The basic idea. Let U be an open subset of a finite-dimensional real space E. If $\xi : (-\varepsilon, \varepsilon) \to U$, $\varepsilon > 0$, is an analytic curve satisfying $\xi(0) = p$, then, either by calculus or imagination, ξ has a tangent at p. Moreover, two analytic curves passing through p have the same tangent provided they have the same 'direction' and the same 'speed' at p. In mathematical terms, the tangent to ξ at p is defined as $\xi'(0)$ (or, $\xi'(0)(1)$), a vector in E. Thus the tangent space of U at p may be thought of as E and can be given a concrete realisation as the family of analytic curves, equivalent modulo their derivatives at p, passing through p.

Now suppose that M is an analytic manifold modelled on E. Let $\mathcal{C}_p(M)$ denote the set of analytic maps ξ from open neighbourhoods of 0 in **R** into M which satisfy $\xi(0) = p$. The derivative of ξ is not defined, but we circumvent this difficulty by considering the derivative of $\phi_\alpha^{-1}\xi$ at 0, where ϕ_α is a chart about p. Define an equivalence relation on $\mathcal{C}_p(M)$ by

$$(1.2.1) \quad \xi \sim \eta \text{ if } (\phi_\alpha^{-1}\xi)'(0) = (\phi_\alpha^{-1}\eta)'(0).$$

Let $[[\xi]]_p$ denote the class of curves equivalent to ξ (and note that if two curves are equivalent as in (1.2.1), then they are equivalent for each chart about p); the <u>tangent space</u> to M at p is defined as the set of all such equivalence classes.

Proceeding in the opposite direction, given any v in E and any analytic chart ϕ_α about p, can we always find a curve ξ in $\mathcal{C}_p(M)$ such that $(\phi_\alpha^{-1}\xi)'(0) = v$? The answer is 'yes' and forms part of Exercise 1.B.

Thus this fairly intuitive approach to the idea of a tangent space at a point of a manifold modelled on E shows that it is always isomorphic to E. Once this point is seen, it becomes notationally easier to proceed straight to E without any consideration of analytic curves. This will be done in the next subsection. A further approach to tangent spaces via