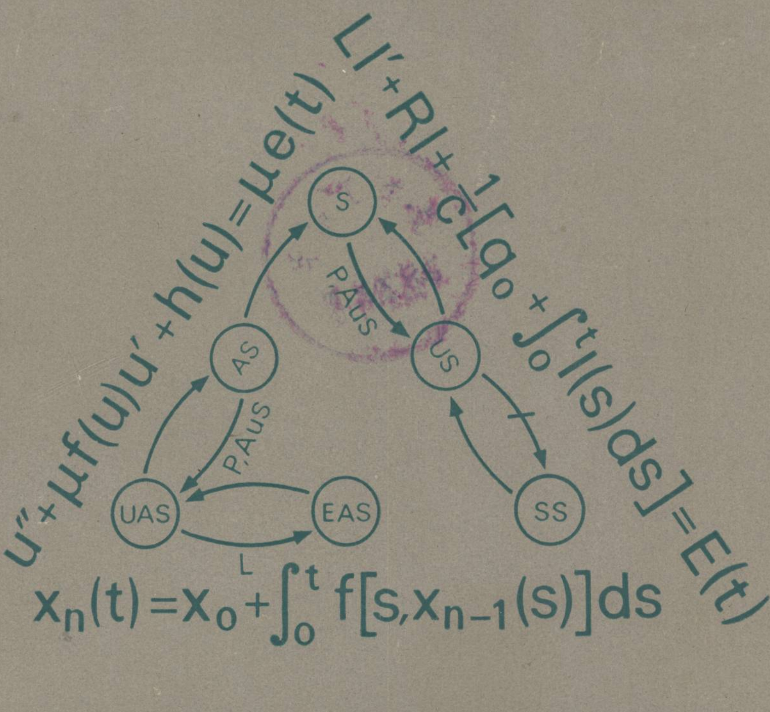
M. RAMA MOHANA RAO

Ordinary Differential Equations

THEORY AND APPLICATIONS



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ORDINARY DIFFERENTIAL EQUATIONS

THEORY AND APPLICATIONS

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ORDINARY DIFFERENTIAL EQUATIONS

to

SATYAVATHI

who departed from our midst when the book was half-written
and whose guiding spirit alone enthused
its completion

Foreward

The importance of ordinary differential equations in other areas of science lies in their power to motivate, unify, and give purport to those areas. This book has been designed as a bridge between the theory and applications of differential equations. A fairly elementary treatment of several topics of importance and of current interest and the inclusion of a large number of examples and exercises should give the reader a better insight into and understanding of the subject. I am sure that this work will be of keen interest to mathematicians, engineers, and applied scientists.

V. LAKSHMIKANTHAM
Professor and Chairman
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Preface

Ordinary differential equations find a wide range of application in biological, physical, social, and engineering systems which are dynamic in character. They can be used to effectively analyze the evolutionary trend of such systems; they also aid in the formulation of these systems and the qualitative examination of their stability under and adaptability to external stimuli. Any attempt in these directions by either the applied scientist or the mathematician has been largely one-sided. Generally, the applied scientist concerns himself with specific instances of application without going far into the fundamentals of the theory, whereas the mathematician concentrates on the abstract theory, paying little or no attention to the practical aspects. This text is designed to provide a bridge between the applied and the theoretical aspects.

The volume is divided into five chapters. Chapter 1 introduces the preliminaries needed for an understanding of what follows. In addition to the existence and uniqueness theory, several useful and vital differential and integral inequalities are included. The importance of the applications of elementary functional analysis and topology is brought out by using the fixed-point techniques to prove the existence and uniqueness theorems.

Chapter 2 deals with the fundamental properties of homogeneous and nonhomogeneous linear systems as also the asymptotic behaviour of linear systems with constant and variable coefficients.

One of the important aspects of the qualitative theory of differential equations is the stability behaviour of solutions. In Chapter 3, the concepts of Liapunov's stability are introduced and theorems with worked examples provided to distinguish various types of stability. The stability of perturbed linear systems and an elementary treatment of two-dimensional systems also feature in this chapter.

Chapter 4 is devoted exclusively to second order equations since such a study yields, as in many areas of mathematics, much more information than that obtained from general theorems. The stability, boundedness, and oscillatory properties of solutions of second order equations relevant in engineering and physics are also discussed in detail.

Chapter 5 discusses the application of Liapunov's direct method to the stability theory and the sufficient conditions for the stability and instability of autonomous and nonautonomous systems. Also, converse theorems on stability are developed and utilized to derive the properties of solutions of perturbed systems.

The book is the outcome of a course offered for several years at the Indian Institute of Technology, Kanpur, to students of mathematics, engineering, and physics. The content is so designed that it can serve primarily as a basic text for a one-year course on ordinary differential equations at the senior undergraduate or master's level; by a judicious omission of certain topics, it can be used also for a one-semester course. The subject is developed keeping in mind the minimum exposure of any student in the biological, physical, social, and engineering sciences to the techniques of solution of ordinary differential equations, i.e., the elements of calculus, linear algebra, and matrix analysis. All the functions and matrices considered are real, unless otherwise indicated. The large number of examples throughout the study and the exercises at the end of every chapter are meant for the student's better understanding of and self-evaluation on the subject.

I recall with pleasure my association with Professors R. Bellman, V. Lakshmikantham, and B. Viswanatham, who were the first to introduce me to research in the theory of differential equations, and I am grateful to them for their inspired guidance. Warm thanks are due also to Professor Chris P. Tsokos for his painstaking reading of the manuscript and his valuable suggestions for its improvement. The fruitful discussions with my colleagues at the Indian Institute of Technology, Kanpur—Professors P. C. Das, P. K. Kamthan, J. N. Kapur, J. B. Shukla, and R. S. L. Srivastava (of the mathematics department) and Professors K. V. G. K. Gokhale, M. A. Pai, R. Subramanian, and V. Sundararajan (of the engineering departments)—helped to put in perspective the different aspects of the areas covered; I gratefully acknowledge their helpful comments. I take this opportunity to thank my students Dr. V. Sree Hari Rao, Mr. P. Srinivas, and Mr. Mohd. Faheem for their patient scrutiny of the manuscript. I record too my appreciation of the financial assistance granted by the Educational Development Centre at the Indian Institute of Technology, Kanpur, for the preparation of the manuscript.

I will greatly value any improvements in the text that readers would like to suggest.

November 1979 M. Rama Mohana Rao
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Kanpur

Contents

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Existence, Uniqueness, and Continuation of Solutions

1.1 INTRODUCTION

Most dynamical systems—physical, social, biological, engineering—are often conveniently expressed in the form of differential equations. Such equations can provide an insight into the behaviour of a system if they represent the various important factors governing the system. For instance, when a system is known to perform efficiently over a certain range of input, the existence of the solutions of the differential equation governing the system over the interval concerned is an important consideration in the understanding of its behaviour. A scientist or an engineer can use differential equations in his work more confidently if he is conversant with the theory of existence, uniqueness, and continuation of solutions. Similarly, a mathematician who is familiar with these properties of solutions is better equipped to develop further mathematical methods for examining the behaviour of solutions of differential equations.

This chapter introduces the existence, uniqueness, and continuation of solutions. Besides the classical methods, fixed-point techniques are employed in proving some of the existence and uniqueness theorems.

1.2 NOTATION AND DEFINITIONS

In our discussion, the independent variable is always treated as real and is denoted by t. Further, the dependent variables, u for scalar equations and x for vector-valued equations, as also all the functions are assumed to be real. However, the theory developed in this chapter can, with minor modifications, be extended to the complex case.

Let R be the set of all real numbers, and I be an open interval on the real line R, that is, $I = \{t : t \in R, r_1 < t < r_2\}$, where r_1 and r_2 are any two fixed points in R. Also, let R^n denote the real n-dimensional euclidean space with elements $x = (x_1, x_2, \ldots, x_n)$, and let R^{n+1} be the space of elements of $(n + 1)$-triple $(t, x_1, x_2, \ldots, x_n)$ or (t, x). We shall often use R instead of R^1. Let B be a domain, i.e., an open-connected set in R^{n+1}, and $C[B, R^n]$ be a class of functions defined and continuous on B, taking values in R^n. When f is a member of this class, we shall write $f \in C[B, R^n]$.

An ordinary differential equation of the n-th order and of the form

$$F(t, u, u', u'', \ldots, u^{(n)}) = 0, \tag{1.2.1}$$

where $u^{(n)}$ is the n-th derivative of the unknown function u with respect to t and F is defined in some subset of R^{n+2}, expresses a relation between the $(n + 2)$-variables $t, u, u', u'', \ldots, u^{(n)}$. Because of its implicit nature, (1.2.1) may represent a collection of differential equations. For example, the implicit equation $u'^3 - 3t^2u'^2 + 3uu' = 0$ leads to three equations, namely,

$$u' = 0, \qquad u' = (3t^2 + (9t^2 - 12u)^{1/2})/2,$$

$$u' = (3t^2 - (9t^2 - 12u)^{1/2})/2.$$

In order to avoid the ambiguity the implicit equation (1.2.1) may exhibit, we shall assume that this equation is solvable for $u^{(n)}$; then, it can be written in the form

$$u^{(n)} = g(t, u, u', \ldots, u^{(n-1)}), \tag{1.2.2}$$

where g is a given function defined on B. If g is linear in $u, u', \ldots, u^{(n-1)}$, then the differential equation (1.2.2) is called *linear*; otherwise it is referred to as *nonlinear*.

Definition 1.2.1 (solution) A function $u = \phi(t)$ is called a solution of (1.2.2) on $r_1 < t < r_2$ if ϕ is defined and n-times differentiable on $r_1 < t < r_2$ and satisfies

$$\phi^{(n)}(t) = g(t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)), \qquad t \in (r_1, r_2).$$

The functions $u_1(t) = t^2$ and $u_2(t) = 1/t$ are the solutions of the differential equation $u'' = 2u/t^2$, $t > 0$. Similarly, the functions $u_1(t) = 1$, $u_2(t) = \cos t$, and $u_2(t) = \sin t$ are the solutions of $u''' + u' = 0$ for all t.

System of Differential Equations

We shall consider a system of first order differential equations of the form

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, \ldots, x_n) \\ x_2' &= f_2(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, x_2, \ldots, x_n) \end{aligned} \tag{1.2.3}$$

where $f_1, f_2, \ldots, f_n$ are n given functions in some domain B of $(n + 1)$-dimensional euclidean space R^{n+1} and $x_1, x_2, \ldots, x_n$ are n unknown functions. A set of n-functions $\phi_1, \phi_2, \ldots, \phi_n$ defined on I is said to be a solution of (1.2.3) on I if, for $t \in I$,

(i) $\phi_1'(t), \ldots, \phi_n'(t)$ exist;

(ii) the point $(t, \phi_1(t), \ldots, \phi_n(t))$ remains in B; and

(iii) $\phi_i'(t) = f_i(t, \phi_1(t), \ldots, \phi_n(t))$, $i = 1, 2, \ldots, n$.

Geometrically, this amounts to saying that a solution of (1.2.3) is a curve in the $(n+1)$-dimensional region B with each point p on the curve and has the coordinates $(t, \phi_1(t), \ldots, \phi_n(t))$, where $\phi_i'(t)$ is the i-th component of the tangent vector to the curve in the direction x_i. When $n = 1$, this interpretation is clear, and thus the curve in B defined by any solution of (1.2.3) is again a solution curve.

An n-th order differential equation of the form (1.2.2) may also be treated as a system of the type (1.2.3). To see this, let

$$u = u_1, u' = u_2, \ldots, u^{(n-1)} = u_n.$$

Then, (1.2.2) is equivalent to

$$u_i' = u_{i+1}, \qquad i = 1, 2, \ldots, n-1,$$
$$u_n' = g(t, u_1, u_2, \ldots, u_n).$$

This set of equations is indeed of the form (1.2.3). In particular, consider the second order differential equation

$$u'' + u'^2 = g(t, u), \tag{1.2.4}$$

where g is a given function. Setting $u = u_1$ and $u' = u_2$, we have the system

$$u_1' = u_2, \qquad u_2' = -u_2^2 + g(t, u_1). \tag{1.2.5}$$

This is a special case of (1.2.3) with $n = 2$, $f_1(t, u_1, u_2) = u_2$, and $f_2(t, u_1, u_2) = -u_2^2 + g(t, u_1)$. It can be easily verified that (1.2.4) and (1.2.5) are equivalent. For this, let ϕ be a solution of (1.2.4) on I. Then, $u_1 = \phi(t)$, $u_2 = \phi'(t)$ is a solution of (1.2.5) on I since

$$u_1' = \phi' = u_2,$$
$$u_2' = \phi'' = -\phi'^2 + g(t, \phi) = -u_2^2 + g(t, u_1).$$

Conversely, let (ϕ_1, ϕ_2) be a solution of (1.2.5) on I. Then, $u = \phi_1(t)$, that is, the first component, is a solution of (1.2.4) on I since

$$u'' = \phi_1'' = (\phi_1')' = \phi_2' = -\phi_2^2 + g(t, \phi_1) = -u'^2 + g(t, u).$$

Vector-matrix notation

A system of equations of the form (1.2.3) can always be written as a single vector-valued equation by introducing the n-dimensional column vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \text{col}\,(x_1, x_2, \ldots, x_n).$$

Let $x(t)$ be the vector-valued function defined by

$$x(t) = \text{col}\,(x_1(t), \ldots, x_n(t)).$$

Similarly, let f be the vector-valued function given by

$$f(t,x)=\begin{pmatrix} f_1(t,x_1,x_2,\ldots,x_n)\\ f_2(t,x_1,x_2,\ldots,x_n)\\ \vdots\\ f_n(t,x_1,x_2,\ldots,x_n)\end{pmatrix}=\operatorname{col}(f_1(t,x),f_2(t,x),\ldots,f_n(t,x)).$$

Then, (1.2.3) can be expressed as

$$x'=f(t,x). \tag{1.2.6}$$

By a solution of (1.2.6) on I we mean a vector-valued function ϕ with components $\phi_1,\phi_2,\ldots,\phi_n$ which satisfies

$$(t,\phi(t))=(t,\phi_1(t),\ldots,\phi_n(t))\in B,\qquad t\in I,$$
$$\phi'(t)=f(t,\phi(t)),\qquad t\in I.$$

Equation (1.2.6) is usually referred to as a *nonautonomous differential system*. A differential system of the form

$$x'=f(x), \tag{1.2.7}$$

in which the right-hand side does not involve the independent variable t, is said to be *autonomous*. An important feature of (1.2.7) is that if $\phi(t)$ is a solution of (1.2.7) on $r_1<t<r_2$, then $\phi(t-t_0)$ is a solution on $t_0+r_1<t<t_0+r_2$. Further, it is sometimes convenient to represent the solutions of (1.2.7) in the (t,x)-space as curves in the x-space with t as a curve parameter. Such curves are called *trajectories* and the space that contains these is known as the *phase space* of (1.2.7).

Linear case

Consider a system of first order linear differential equations of the form

$$\begin{aligned} x_1'&=a_{11}(t)x_1+\ldots+a_{1n}(t)x_n+b_1(t)\\ x_2'&=a_{21}(t)x_1+\ldots+a_{2n}(t)x_n+b_2(t)\\ &\vdots\\ x_n'&=a_{n1}(t)x_1+\ldots+a_{nn}(t)x_n+b_n(t)\end{aligned}$$

or

$$x_i'=\sum_{j=1}^{n}a_{ij}(t)x_j+b_i(t),\qquad i=1,2,\ldots,n, \tag{1.2.8}$$

where $a_{ij}(t)$, $i,j=1,2,\ldots,n$, $b_i(t)$, $i=1,2,\ldots,n$, are real-valued functions defined on I, and $x(t)=(x_1(t),\ldots,x_n(t))$ is an unknown n-dimensional vector-valued function. Let $A(t)=(a_{ij}(t))$ be an $n\times n$ matrix and $B(t)$ be an n-vector $(b_1(t),b_2(t),\ldots,b_n(t))$. Then, (1.2.8) can be written as

$$x'=A(t)x+B(t). \tag{1.2.9}$$

This is a special case of (1.2.6) with $f(t,x)=A(t)x+B(t)$, $A(t)x$ being the usual matrix-vector product. Equation (1.2.9) is referred to as a *nonhomogeneous linear differential system*, but when $B(t) \equiv 0$, it is called a *homogeneous linear system.*

An important special case of (1.2.9) is the n-th order linear differential equation

$$u^{(n)} + a_1(t)u^{(n-1)} + \ldots + a_n(t)u = b(t). \tag{1.2.10}$$

This is of the type (1.2.8). To see this, let

$$u = u_1,\ u' = u_2,\ \ldots,\ u^{(n-1)} = u_n.$$

Then, (1.2.10) is equivalent to

$$u_i' = u_{i+1}, \qquad i = 1, 2, \ldots, n-1,$$
$$u_n' = -a_n(t)u_1 - a_{n-1}(t)u_2 - \ldots - a_1(t)u_n + b(t).$$

When $n = 3$, (1.2.10) takes the form (1.2.9) with

$$x = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3(t) & -a_2(t) & -a_1(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ 0 \\ b(t) \end{pmatrix}.$$

Functional Analysis—Brief Review

For any vector $x \in R^n$, we define the scalar quantity by

$$\|x\| = \sum_{i=1}^{n} |x_i| \tag{1.2.11}$$

and call it the norm of x.

Definition 1.2.2 The norm $\|x\|$ of a vector $x \in R^n$ is a real-valued function satisfying the properties

(i) $\|x\| \geqslant 0$ with $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\| = |\lambda|\, \|x\|$ for any scalar λ; and

(iii) $\|x + y\| \leqslant \|x\| + \|y\|$, $x,y \in R^n$.

Besides the norm defined by relation (1.2.11), we shall use, in Chapter 5, the usual euclidean norm

$$\|x\|_e^2 = \sum_{i=1}^{n} x_i^2$$

which also satisfies properties (i), (ii), and (iii).

Definition 1.2.3 A linear space or vector space X is a nonempty collection of objects, called vectors, of which one is a zero vector (denoted by 0), which can be added pairwise and multiplied by any scalar, consistent with axioms (i) to (ix). That is, for all vectors $x,y,z \in X$ and for all scalars $\alpha,\beta \in R$,

(i) $x + y = y + x$;

(ii) $(x + y) + z = x + (y + z)$;

(iii) $\alpha(x + y) = \alpha x + \alpha y$;

(iv) $\alpha(\beta x) = (\alpha\beta)x$;

(v) $(\alpha + \beta)x = \alpha x + \beta x$;

(vi) $1x = x$;

(vii) $0 + x = x$;

(viii) $0x = 0$; and

(ix) $x - x = 0$.

A *normed linear space* is a linear space X if, for every x in X, there exists a real-valued function, denoted by $\|x\|$ and called the norm of x, satisfying properties (i), (ii), and (iii) of Definition 1.2.2. For convenience, the notation $\|\cdot\|_X$ is sometimes used for the norm on X. A sequence $\{x_n\}$ of vectors in a normed linear space X is said to converge to the point x in X if, for every positive number ϵ, there exists a positive integer $N = N(\epsilon)$ such that $\|x - x_n\|_X < \epsilon$ for all (and not merely some) integers n exceeding N. This is often written as $\lim_{n\to\infty} x_n = x$.

Definition 1.2.4 (Cauchy's sequence) A sequence $\{x_n\}$ in X is called a Cauchy (or fundamental) sequence if, for every positive number ϵ, there exists a positive integer $N = N(\epsilon)$ such that $\|x_n - x_m\|_X < \epsilon$ whenever both m and n exceed N.

A normed linear space X is said to be *complete* if every Cauchy sequence in X converges to an element in X.

A sequence may be a Cauchy sequence in X but need not converge to an element in X. For example, consider the subspace $X = (0, 1]$ of the real line. The sequence defined by $x_n = 1/n$ is easily seen to be a Cauchy sequence in this subspace but it is not convergent since 0 (to which it converges) is not a point in this subspace. Thus, the subspace $(0, 1]$ is not complete.

Definition 1.2.5 A complete normed linear space is called a Banach space.

Example 1.2.1 (i) The space R of real numbers is the simplest of all normed linear spaces. The norm of an element x in R is defined by $\|x\| = |x|$ and this space is a Banach space.

(ii) The space R^n of all n-tuples $x = (x_1, x_2, \ldots, x_n)$ of real numbers is a linear space with addition and scalar multiplication defined componentwise. R^n is a Banach space if the norm $\|x\|$ is given by

$$\sup_i |x_i| \quad \text{or} \quad \sum_{i=1}^{n} |x_i| \quad \text{or} \quad \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

since R^n is complete.

(iii) The space

$$L^p(a, b) = \left\{ f : f \text{ is measurable and } \int_a^b |f(t)|^p \, dt < \infty \right\}$$

is a Banach space with the norm of f given by

$$\|f\| = \left[\int_a^b |f(t)|^p \, dt\right]^{1/p}.$$

(iv) Let p be a real number such that $1 \leqslant p < \infty$. The space l_p of all sequences $\{x_n\}$ of scalars such that

$$\sum_{i=1}^{\infty} |x_i|^p < \infty$$

with the norm defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

is a Banach space since it is complete.

Definition 1.2.6 A set $\{y \in X : \|y - x\|_X < \epsilon\}$ for any element x of a normed linear space X is said to be an ϵ-neighbourhood of x.

A set S in X is *open* if every element x of S has an ϵ-neighbourhood, every element of which belongs to S.

Given a subset S of X and an element x of X (not necessarily contained in S), we say that x is a *limit point* of S if, for every positive number ϵ, there exists at least one element y belonging to S and distinct from x such that $\|x - y\|_X < \epsilon$.

A set S is said to be *closed* if it contains all its limit points.

A set S, together with all its limit points, is called the *closure* of S and is denoted by $\bar{S}$.

A set S is *bounded* if all its elements lie within a circle of sufficiently large radius. For example, the points in a rectangle form a bounded set whereas those on a straight line do not.

If $S \subset X$, $A \subset R$, and $\{V_a\}_{a \in R}$ is a collection of open sets in X such that $S \subset \bigcup_{a \in R} V_a$, then V_a is called an *open covering* of S.

Definition 1.2.7 A set S is said to be compact if every open covering of S has a finite subcover, that is, every open covering of S contains a finite number of open sets which cover S.

In particular, if S is a subset of a Banach space, then S is *compact* if every sequence $\{x_n\}$ in S contains a subsequence which converges to an element in S. For example, a set S in R^n is compact if and only if it is closed and bounded.

Definition 1.2.8 A nonempty subset S of a linear space X (not necessarily normed) is said to be convex if, for every pair of vectors $\{x, y\}$ belonging to S and for every real number α in the interval $[0, 1]$, the vector $\alpha x + (1 - \alpha)y$ also belongs to S.

Any ball, open or closed, in a normed linear space is always convex.