Multivariable and Optimal Systems

D.H. Owens

ACADEMIC PRESS
A Subsidiary of Harcourt Brace Jovanovich, Publishers
London New York Toronto Sydney San Francisco

Multivariable and Optimal Systems

D. H. OWENS

Department of Control Engineering, University of Sheffield, Sheffield, U.K.

1981

ACADEMIC PRESS

London New York Toronto Sydney San Francisco

A Subsidiary of Harcourt Brace Jovanovich, Publishers

ACADEMIC PRESS INC. (LONDON) LTD.
24/28 Oval Road
London NW1

United States Edition published by
ACADEMIC PRESS INC.
111 Fifth Avenue
New York, New York 10003

British Library Cataloguing in Publication Data

Owens, D.H.
Multivariable and optimal systems.
1. Control theory
I. Title
629.8′312 QA402.3
ISBN Hardback: 0 12 531720 4
ISBN Paperback: 0 12 531722 0

LCCCN: 81–67886

Printed in Great Britain at the Alden Press
Oxford London and Northampton

Preface

Multivariable and optimal systems are now an established part of systems science and control engineering degree courses. This text aims to provide a course and self study textbook for undergraduate and Master's degree control engineering students in universities and polytechnics covering the conceptual basis of multivariable systems control theory and optimal control and illustrating its application to simple multivariable process plant. It is motivated by the observation that available texts either lie at the research level demanding a degree of mathematical sophistication that few undergraduates can cope with, or are fairly general control texts containing some information on multivariable and optimal systems but, due to lack of space, tending to make the treatment rather superficial. The proposed text attempts to bring depth whilst using, in its simplest form, mathematics normally included in undergraduate engineering courses. In this way the reader can obtain a (relatively) painless and firm basis for future specialist studies.

In my experience, the major obstacles met by the typical student are the steps from classical control design methods to "thinking multivariable-style" and from there to expressing multivariable concepts in matrix form. The approach taken, therefore, is to introduce concepts in the context of dynamics studies of simple process plant to illustrate the natural source of the techniques. I have also found that engineering students, in general, respond best if the engineering design applications aspects of the material are emphasized rather than the general systems theoretical topics. Thus the text takes the view that design principles and design practice are the most important part of the armoury of the future control engineer in the sense that design decisions are made by human judgement based on experience of synthesis procedures seen to work in elementary cases.

An important comment is that, despite the elementary style of the text, the standard of mathematical rigour is, on the whole, high and based on the maxim, "He who does not know the limitations of the theory is lost" (anon.). In this sense the reader is presented with a rigorous development of most of the essential results forming the cornerstones of applications studies and techniques described in more advanced texts. The validity and potential of the concepts are also illustrated by simple but meaningful physical examples. The subtleties of more advanced computer-aided design and synthesis procedures are, however, left for further study in more advanced texts.

The text is divided naturally into three parts. Chapters 1 and 2 provide a sound introduction to the basic ideas of modelling systems behaviour by continuous or discrete state variable models and the effect of a general form of state feedback. Chapters 3 and 4 lay a firm foundation for control studies using Laplace transforms, poles, zeros and design criteria of systems with more than one output. A number of essentially multivariable design concepts are illustrated by a detailed consideration of multivariable generalizations of a first order lag. Finally, Chapters 5 to 7 return to the time-domain and formulate and solve a number of continuous and discrete optimal control problems. Emphasis is placed on the linear quadratic optimal control problem and minimum energy and minimum fuel problems to illustrate principles. A statement of the Euler-Lagrange equations and the Minimum Principle of Pontryagin are also included for completeness.

This text is the product of several courses given by the author at the University of Sheffield since 1973 and the comments and questions of the undergraduates and postgraduates who attended them. Many thanks go to Mrs P. Turner and Mrs J. Stubbs for typing the draft manuscript and to my wife and family for enduring its preparation.

D.H. Owens
August 1981

Contents

1. Systems and Dynamics

This chapter introduces the fundamental language of systems models, systems dynamics, systems structure and simulation used in the multivariable and optimal control studies described in the remaining chapters. The material is frequently initiated by illustrating how it naturally arises in practice and its distinctive differences from, and similarities to, classical ordinary differential equation methods are highlighted to emphasize fundamentally new concepts and to reassure the timid reader that he is not too far from familiar territory.

1.1 Introduction and Review of Basic Concepts

The schematic diagram shown in Fig. 1 is the familiar block diagram representation of a classical dynamic system with manipulable *input* u driving the system from given initial conditions to produce the consequent measured *output* y. Either or both could be continuously varying with time or be represented by discrete/sampled values at defined points of time. For simplicity at this stage we will assume that all signals are continuous. The only significant exclusions from this general picture are the possibilities of other measurable or stochastic (i.e. noise) inputs to the process from the environment.

The mathematical building blocks used to express, analyse and predict the relationships between input, output and initial conditions are surprisingly few in number. The most fundamental branch of knowledge is the theory of nth

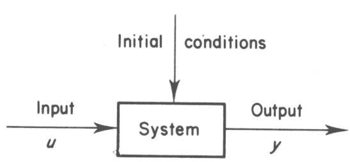

Fig. 1. Deterministic representation of system dynamics.

order ordinary differential equations. More precisely the most commonly used method of expressing the relationship between the input and output uses an nth order ordinary differential equation of the general (nonlinear) form

$$\frac{d^n y(t)}{dt^n} = \psi\left(y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{n-1}y(t)}{dt^{n-1}}, u(t), \ldots, \frac{d^n u(t)}{dt^n}, t\right) \quad (1.1)$$

with initial conditions of the form

$$y(0) = d_0, \left.\frac{dy}{dt}\right|_{t=0} = d_1, \ldots, \left.\frac{d^{n-1}y(t)}{dt^{n-1}}\right|_{t=0} = d_{n-1} \quad (1.2)$$

These models are normally obtained either by "analytical modelling" of the system dynamics directly from the fundamental and empirical laws governing the process or by "indirect" model-fitting techniques based on available plant transient or frequency response data.

In the case of equation (1.1) being linear of the form (after a little reorganization)

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_{n-1}\frac{dy}{dt} + a_n y = b_0 \frac{d^n u}{dt^n} + b_1 \frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_{n-1}\frac{du}{dt} + b_n u, \quad (a_0 \neq 0) \quad (1.3)$$

the mathematical machinery of analysis is extremely highly developed. In particular the use of Laplace transform techniques has long been established as a powerful tool. Perhaps the most significant idea is that of the system transfer function. Introducing the differential operator $D = d/dt$ and the polynomials

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n \quad (1.4)$$

$$Q(\lambda) = b_0\lambda^n + b_1\lambda^{n-1} + \ldots + b_{n-1}\lambda + b_n \quad (1.5)$$

then the system of equation (1.3) has the compact form

$$P(D)y(t) = Q(D)u(t) \quad (1.6)$$

The system *transfer function* is the rational function of the complex variable s defined by

$$g(s) = \frac{Q(s)}{P(s)} \quad (1.7)$$

and the system is frequently represented by the block diagram of Fig. 2. A fundamental property of the system transfer function is obtained for the case of zero initial conditions, $d_{k-1} = 0$, $1 \leqslant k \leqslant n$, and the case of inputs possessing

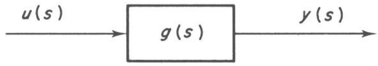

Fig. 2. Transfer function representation of system dynamics.

the property that $D^{k-1}u(t)|_{t=0} = 0, 1 \leqslant k \leqslant n$, namely

$$y(s) = g(s)u(s) \tag{1.8}$$

where, for notational simplicity, the Laplace transform of $u(t)$ and $y(t)$ are denoted $u(s)$ and $y(s)$ respectively. If $\mathscr{L}$ and $\mathscr{L}^{-1}$ denote the operations of taking the Laplace transform and taking inverse Laplace transforms respectively, then (1.8) takes the form

$$y(t) = \mathscr{L}^{-1}[g(s)u(s)] = \int_0^t h(t-t')u(t')\mathrm{d}t' \tag{1.9}$$

where $h(t) = \mathscr{L}^{-1}[g(s)]$ is the system *impulse response* and the final integral is termed the *convolution* of h and u.

The power of the transfer function representation of system dynamics originates in the simplicity of representation of the dynamics of composite systems. Consider the parallel, series and feedback configurations illustrated in Fig. 3., the reader will easily verify that the transfer functions in each case are

(a) $$g(s) = g_1(s) + g_2(s) + \ldots + g_m(s)$$

(b) $$g(s) = g_1(s)g_2(s)\ldots g_m(s)$$

(c) $$g(s) = \frac{g_1(s)}{1 + g_1(s)g_2(s)} \tag{1.10}$$

The major impact of transfer function methods are felt, however, in the design of the forward path and minor loop elements in the general scalar *feedback system* illustrated in Fig. 4., by the use of the *frequency response* $g(j\omega)$ $(j^2 = -1)$ and the *Nyquist stability criterion.* Alternatively, the factorization

$$g(s) = \frac{g_0(s-z_1)(s-z_2)\ldots(s-z_{n_z})}{(s-p_1)(s-p_2)\ldots(s-p_n)} = g_0 \prod_{l=1}^{n_z}(s-z_l) \Big/ \prod_{l=1}^{n}(s-p_l) \tag{1.11}$$

and consideration of the poles $\{p_1, p_2, \ldots, p_n\}$, zeros $\{z_1, z_2, \ldots, z_{n_z}\}$, order n, rank $n - n_z$ and gain g_0 of the transfer function (a) provide valuable information on open-loop stability and transient performance and (b) can be used as the basis for the choice of control elements using the well known *root-locus method.*

The following chapters can be regarded as a self-contained introduction to the

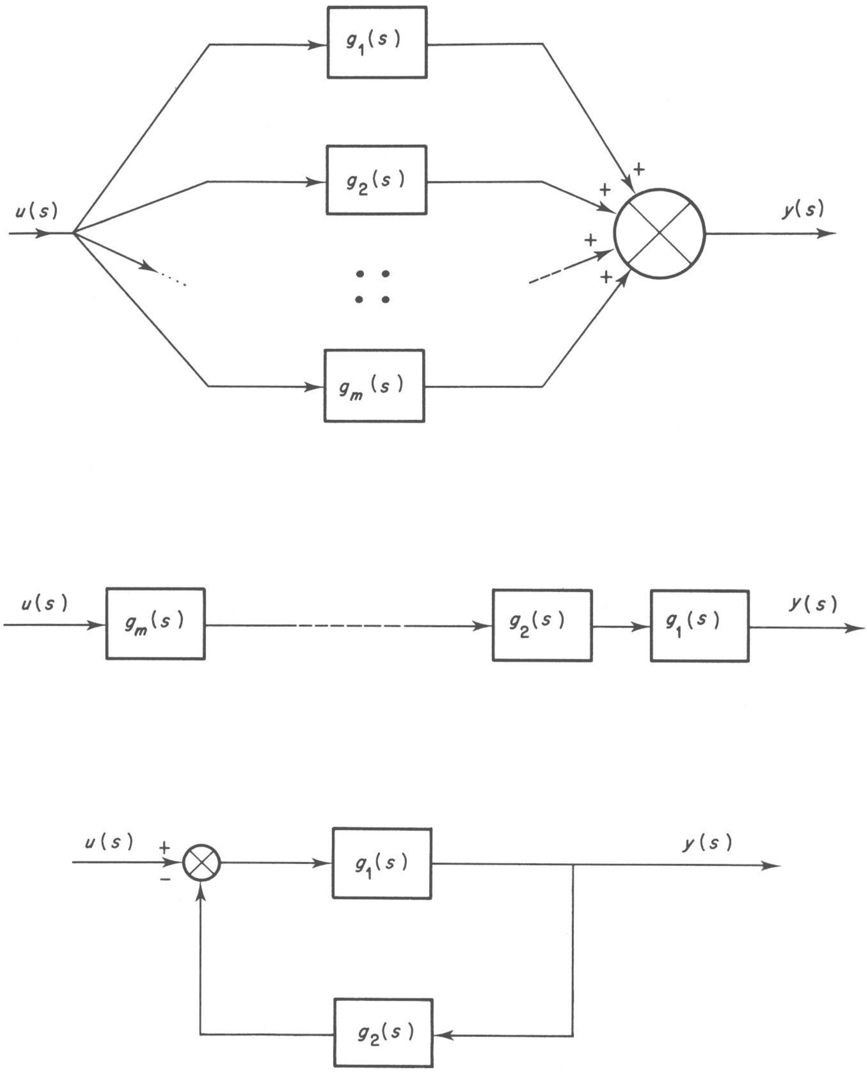

Fig. 3. Parallel, series and feedback configurations.

methodology required for the extension of the above ideas to cover the modelling, analysis and design of control systems for engineering systems with more than one input and more than one output. Such systems are commonly called multi-input/multi-output or MULTIVARIABLE systems and are represented schematically as shown in Fig. 5. They reduce to the single-input/single-output case wherever $m = l = 1$. An important general property of such systems (and a

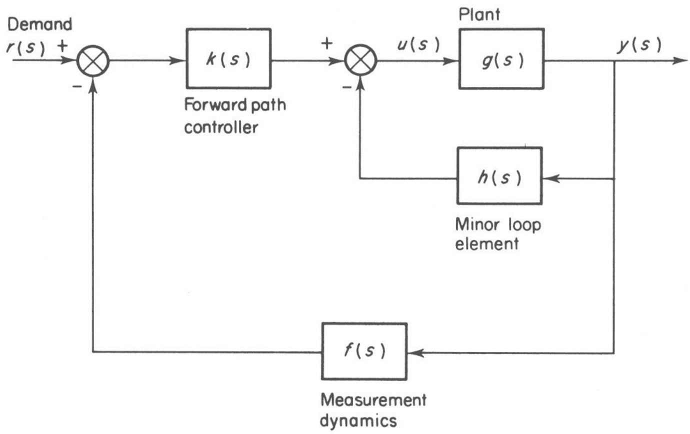

Fig. 4. Feedback system with minor loop and transducer dynamics.

major source of design difficulties) is INTERACTION between inputs and outputs in the sense that any input u_k (say) will have a dynamic effect on all outputs y_i, $1 \leqslant i \leqslant m$. In the case of $m = l > 1$ (i.e. we have equal numbers of inputs and outputs) and each input u_k only has a dynamic effect on y_k, $1 \leqslant k \leqslant m$, the system is said to be NON-INTERACTING. Examples of interacting and non-interacting two-input/two-output systems are illustrated in Figs 6 and 7 respectively. The source of the interaction is easily identified by noting that Fig. 6 reduces to Fig. 7 in the case of $\alpha = \beta = 0$. Note that non-interacting systems can be regarded as a collection of distinct single-input/single-output systems, each of which can be controlled independently of the others.

The success of the classical ideas of feedback and transfer functions in the analysis and design of single-input/single-output systems is a major motivation for the attempt to generalize them to cover the case of multi-input/multi-output

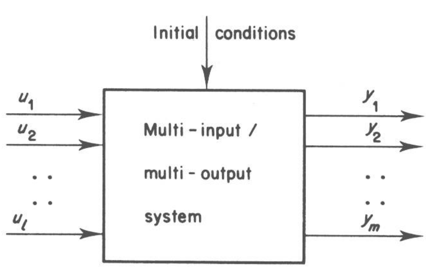

Fig. 5. Deterministic representation of a multi-input/multi-output system.

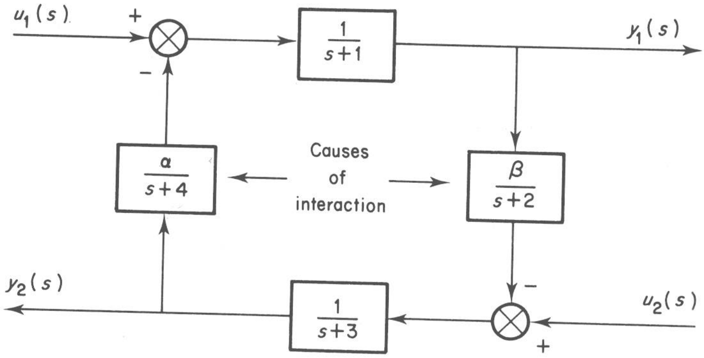

Fig. 6. An interacting multi-input/multi-output system.

systems. Such an attempt must formulate and provide precise answers to questions such as

(i) What form of mathematical model is most suitable for multi-input/multi-output studies?

(ii) What is the generalization of the notion of transfer function?

(iii) What do we mean by the ideas of poles, zeros and frequency response?

(iv) What is feedback in this general case?

(v) What is the generalization of the notion of stability?

(vi) Are there useful generalizations of the ideas of Nyquist diagrams and root-loci?

Of course the answers to these questions will apply to single-input/single-output systems also.

In many cases the required generalizations take their most natural form when expressed in the language of matrix theory. This formal simplicity is simultaneously of great value in removing mathematical clutter and revealing the basic structure of the problem and in the conversion of design relationships into a form suitable for evaluation by a digital computer. The reader should beware,

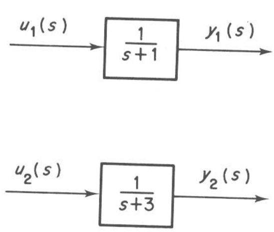

Fig. 7. A non-interacting multi-input/multi-output system.

however, of its tendency to mask the difficulties inherent in multi-input/multi-output control problems. In other cases, such as the development of optimal controllers and a consideration of the integrity of the design, the development has no counterpart in classical theory.

A final word on the use of matrix theory: there is no doubt that it is a powerful tool for analysis and a natural setting for converting design relations into a form suitable for digital computer calculations. It can even be agreed that matrix methods are a major contributor to the rapid rate of advance of control science over the last two decades. The methods do, however, take a little getting used to and, for this reason, the text restricts its attention to design methods that can be formulated using only simple matrix operations, such as addition, multiplication by scalars, inversion and calculation of eigenvalues and eigenvectors. Subroutines performing such operations are commonly available as library software on digital computers.

1.2 State Variable Models of Process Dynamics

Although the scalar nth order differential equations of (1.3) and (1.6) have been used frequently in classical design studies, it is true that modelling of process dynamics directly from known physical laws rarely produces such equations. In contrast, the system model frequently takes the form of a set of ordinary differential equations, each obtained from modelling individual system compo-

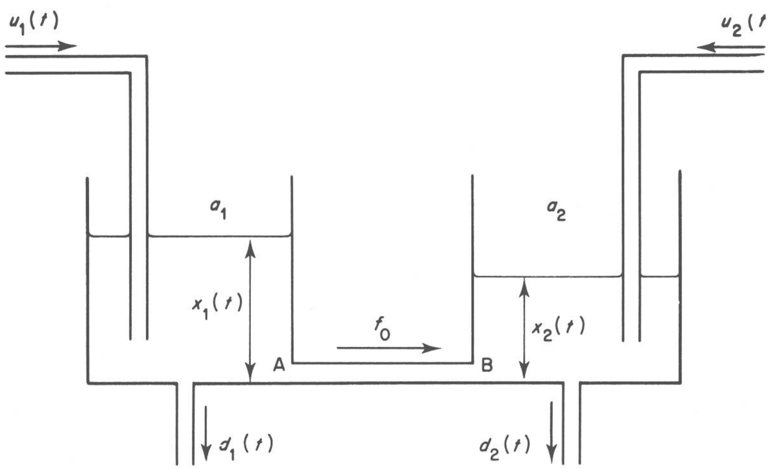

Fig. 8. Two-input liquid level system.

nents and their interconnections. Only in the case of single-input/single-output ($m = l = 1$) systems where all such equations are linear with constant coefficients can these equations be reduced to a single ordinary differential equation. In many cases, this can require considerable computational effort. For these, and other reasons, the mathematical models used in the study of the dynamics and control of multi-input/multi-output systems take a general form that bears little formal similarity to (1.3) and (1.6). The following examples will be used to motivate the general form to be used.

EXAMPLE 1.2.1 (nonlinear model of a two-vessel liquid level system). Consider the elementary dynamic system illustrated in Fig. 8 consisting of two interconnected vessels of uniform cross-section and areas a_1, a_2 respectively. Let $x_i(t)$, $i = 1,2$, denote the height of liquid in vessel i, $u_i(t)$, $i = 1,2$, denote the input flow rate into vessel i (m^3/s), and let $d_i(t)$, $i = 1,2$, denote known disturbance outlet flow rates (m^3/s) from the bottom of the vessels. The intervessel flow (m^3/s) is assumed to be a function only of the pressure drop between A and B and is hence taken as a function $f_0(x_1(t) - x_2(t))$ of the height difference $x_1(t) - x_2(t)$.

Elementary considerations of the mass balance in each vessel yields the following (nonlinear) differential equations describing the dynamics of the process:

$$a_1 \frac{dx_1(t)}{dt} = u_1(t) - f_0(x_1(t) - x_2(t)) - d_1(t)$$

$$a_2 \frac{dx_2(t)}{dt} = u_2(t) + f_0(x_1(t) - x_2(t)) - d_2(t) \qquad (1.12)$$

The system initial conditions can be taken as the known values of $x_1(t)$ and $x_2(t)$ at a given time $t = t_0$. Note that each equation cannot be solved independently of the other, i.e. the vessels dynamics are coupled by the intervessel flow.

The model will be complete when the system ouputs are defined. There are a number of possibilities here. For example, if separate measurements of the levels $x_1(t)$ and $x_2(t)$ are available, the outputs could be defined as

$$y_1(t) = x_1(t), \qquad y_2(t) = x_2(t) \qquad (1.13)$$

Alternatively, the outputs could be taken to be the total volume of liquids in both vessels and the difference in head,

$$y_1(t) = a_1 x_1(t) + a_2 x_2(t), \qquad y_2(t) = x_1(t) - x_2(t) \qquad (1.14)$$

EXERCISE 1.2.1. In the case of $a_1 = a_2 = a$ with outputs defined by (1.14) show that the model (1.12) reduces to the equations

$$\frac{dy_1(t)}{dt} = (u_1(t) + u_2(t)) - (d_1(t) + d_2(t))$$

$$\frac{dy_2(t)}{dt} = a^{-1}(u_1(t) - u_2(t)) - 2a^{-1}f(y_2(t)) - a^{-1}(d_1(t) - d_2(t)) \tag{1.15}$$

Note that each equation can be solved independently of the other if the initial conditions and inputs are known. (This is not the case for the outputs defined by (1.13).) Note also that, if we used the input variables $\hat{u}_1 = u_1 + u_2$ and $\hat{u}_2 = u_1 - u_2$, equation (1.15) indicates that the resulting system is non-interacting.

EXAMPLE 1.2.2 (model of a double spring-mass system). Consider the mechanical system of Fig. 9 consisting of two masses m_1, m_2 connected by lossless springs with linear displacement/force characteristics. The system has one input $u_1(t)$ equal to the vertical displacement of the supporting platform from an equilibrium position. Applying the normal laws of motion, the following linear equations are obtained

$$m_1 \frac{d^2x_1(t)}{dt^2} = k_1(u_1(t) - x_1(t)) - k_2(x_1(t) - x_2(t))$$

$$m_2 \frac{d^2x_2(t)}{dt^2} = k_2(x_1(t) - x_2(t)) \tag{1.16}$$

i.e. two coupled second order ordinary differential equations. An equivalent set of four first order ordinary differential equations are obtained by defining auxiliary variables $x_3(t) = dx_1(t)/dt$ and $x_4(t) = dx_2(t)/dt$,

$$\frac{dx_1(t)}{dt} = x_3(t)$$

$$\frac{dx_2(t)}{dt} = x_4(t)$$

$$\frac{dx_3(t)}{dt} = \frac{k_1}{m_1}(u_1(t) - x_1(t)) - \frac{k_2}{m_1}(x_1(t) - x_2(t))$$

$$\frac{dx_4(t)}{dt} = \frac{k_2}{m_2}(x_1(t) - x_2(t)) \tag{1.17}$$