A Treatise on the Theory of Bessel Functions

By G. N. WATSON

Second Edition

A TREATISE ON THE THEORY OF BESSEL FUNCTIONS

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PREFACE

THIS book has been designed with two objects in view. The first is the development of applications of the fundamental processes of the theory of functions of complex variables. For this purpose Bessel functions are admirably adapted; while they offer at the same time a rather wider scope for the application of parts of the theory of functions of a real variable than is provided by trigonometrical functions in the theory of Fourier series.

The second object is the compilation of a collection of results which would be of value to the increasing number of Mathematicians and Physicists who encounter Bessel functions in the course of their researches. The existence of such a collection seems to be demanded by the greater abstruseness of properties of Bessel functions (especially of functions of large order) which have been required in recent years in various problems of Mathematical Physics.

While my endeavour has been to give an account of the theory of Bessel functions which a Pure Mathematician would regard as fairly complete, I have consequently also endeavoured to include all formulae, whether general or special, which, although without theoretical interest, are likely to be required in practical applications; and such results are given, so far as possible, in a form appropriate for these purposes. The breadth of these aims, combined with the necessity for keeping the size of the book within bounds, has made it necessary to be as concise as is compatible with intelligibility.

Since the book is, for the most part, a development of the theory of functions as expounded in the *Course of Modern Analysis* by Professor Whittaker and myself, it has been convenient to regard that treatise as a standard work of reference for general theorems, rather than to refer the reader to original sources.

It is desirable to draw attention here to the function which I have regarded as the canonical function of the second kind, namely the function which was defined by Weber and used subsequently by Schläfli, by Graf and Gubler and by Nielsen. For historical and sentimental reasons it would have been pleasing to have felt justified in using Hankel's function of the second kind; but three considerations prevented this. The first is the necessity for standardizing the function of the second kind; and, in my opinion, the authority of the group of mathematicians who use Weber's function has greater weight than the authority of the mathematicians who use any other one function of the second kind. The second is the parallelism which the use of Weber's function exhibits between the two kinds of Bessel functions and the two kinds (cosine and sine) of trigonometrical functions. The third is the existence of the device by which interpolation is made possible in Tables I and III at the end of Chapter XX, which seems to make the use of Weber's function inevitable in numerical work.

It has been my policy to give, in connexion with each section, references to any memoirs or treatises in which the results of the section have been previously enunciated; but it is not to be inferred that proofs given in this book are necessarily those given in any of the sources cited. The bibliography at the end of the book has been made as complete as possible, though doubtless omissions will be found in it. While I do not profess to have inserted every memoir in which Bessel functions are mentioned, I have not consciously omitted any memoir containing an original contribution, however slight, to the theory of the functions; with regard to the related topic of Riccati's equation, I have been eclectic to the extent of inserting only those memoirs which seemed to be relevant to the general scheme.

In the case of an analytical treatise such as this, it is probably useless to hope that no mistakes, clerical or other, have remained undetected; but the number of such mistakes has been considerably diminished by the criticisms and the vigilance of my colleagues Mr C. T. Preece and Mr T. A. Lumsden, whose labours to remove errors and obscurities have been of the greatest value. To these gentlemen and to the staff of the University Press, who have given every assistance, with unfailing patience, in a work of great typographical complexity, I offer my grateful thanks.

G. N. W.

August 21, 1922.

PREFACE TO THE SECOND EDITION

To incorporate in this work the discoveries of the last twenty years would necessitate the rewriting of at least Chapters XII—XIX; my interest in Bessel functions, however, has waned since 1922, and I am consequently not prepared to undertake such a task to the detriment of my other activities. In the preparation of this new edition I have therefore limited myself to the correction of minor errors and misprints and to the emendation of a few assertions (such as those about the unproven character of Bourget's hypothesis) which, though they may have been true in 1922, would have been definitely false had they been made in 1941.

My thanks are due to many friends for their kindness in informing me of errors which they had noticed; in particular, I cannot miss this opportunity of expressing my gratitude to Professor J. R. Wilton for the vigilance which he must have exercised in the compilation of his list of corrigenda.

March 31, 1941. G. N. W.

To stand upon every point, and go over things at large, and to be curious in particulars, belongeth to the first author of the story: but to use brevity, and avoid much labouring of the work, is to be granted to him that will make an abridgement.

2 Maccabees ii. 30, 31.

CONTENTS

CHAPTER I

BESSEL FUNCTIONS BEFORE 1826

1·1. *Riccati's differential equation.*

The theory of Bessel functions is intimately connected with the theory of a certain type of differential equation of the first order, known as Riccati's equation. In fact a Bessel function is usually defined as a particular solution of a linear differential equation of the second order (known as Bessel's equation) which is derived from Riccati's equation by an elementary transformation.

The earliest appearance in Analysis of an equation of Riccati's type occurs in a paper* on curves which was published by John Bernoulli in 1694. In this paper Bernoulli gives, as an example, an equation of this type and states that he has not solved it†.

In various letters‡ to Leibniz, written between 1697 and 1704, James Bernoulli refers to the equation, which he gives in the form

$$dy = yy\,dx + xx\,dx,$$

and states, more than once, his inability to solve it. Thus he writes (Jan. 27, 1697): "Vellem porro ex Te scire num et hanc tentaveris $dy = yy\,dx + xx\,dx$. Ego in mille formas transmutavi, sed operam meam improbum Problema perpetuo lusit." Five years later he succeeded in reducing the equation to a linear equation of the second order and wrote§ to Leibniz (Nov. 15, 1702): "Qua occasione recordor aequationes alias memoratae $dy = yy\,dx + x^2 dx$ in qua nunquam separare potui indeterminatas a se invicem, sicut aequatio maneret simpliciter differentialis: sed separavi illas reducendo aequationem ad hanc differentio-differentialem|| $ddy : y = -x^2 dx^2$."

When this discovery had been made, it was a simple step to solve the last equation in series, and so to obtain the solution of the equation of the first order as the quotient of two power-series.

* *Acta Eruditorum publicata Lipsiae*, 1694, pp. 435—437.

† "Esto proposita aequatio differentialis haec $x^2dx + y^2dx = a^2dy$ quae an per separationem indeterminatarum construi possit nondum tentavi" (p. 436).

‡ See *Leibnizens gesamellte Werke*, Dritte Folge (Mathematik), III. (Halle, 1855), pp. 50—87.

§ *Ibid.* p. 65. Bernoulli's procedure was, effectively, to take a new variable u defined by the formula

$$-\frac{1}{u}\frac{du}{dx} = y$$

in the equation $dy/dx = x^2 + y^2$, and then to replace u by y.

|| The connexion between this equation and a special form of Bessel's equation will be seen in § 4·3.

And, in fact, this form of the solution was communicated to Leibniz by James Bernoulli within a year (Oct. 3, 1703) in the following terms*:

"Reduco autem aequationem $dy = yy\,dx + xx\,dx$ ad fractionem cujus uterque terminus per seriem exprimitur, ita

$$y = \frac{\dfrac{x^3}{3} - \dfrac{x^7}{3.4.7} + \dfrac{x^{11}}{3.4.7.8.11} - \dfrac{x^{15}}{3.4.7.8.11.12.15} + \dfrac{x^{19}}{3.4.7.8.11.12.15.16.19} - \text{etc.}}{1 - \dfrac{x^4}{3.4} + \dfrac{x^8}{3.4.7.8} - \dfrac{x^{12}}{3.4.7.8.11.12} + \dfrac{x^{16}}{3.4.7.8.11.12.15.16} - \text{etc.}}$$

quae series quidem actuali divisione in unam conflari possunt, sed in qua ratio progressionis non tam facile patescat, scil.

$$y = \frac{x^3}{3} + \frac{x^7}{3.3.7} + \frac{2x^{11}}{3.3.3.7.11} + \frac{13x^{15}}{3.3.3.3.5.7.7.11} + \text{etc."}$$

Of course, at that time, mathematicians concentrated their energy, so far as differential equations were concerned, on obtaining solutions *in finite terms*, and consequently James Bernoulli seems to have received hardly the full credit to which his discovery entitled him. Thus, twenty-two years later, the paper†, in which Count Riccati first referred to an equation of the type which now bears his name, was followed by a note‡ by Daniel Bernoulli in which it was stated that the solution of the equation§

$$ax^n\,dx + uu\,dx = b\,du$$

was a hitherto unsolved problem. The note ended with an announcement in an anagram of the solution: "Solutio problematis ab Ill. Riccato proposito characteribus occultis involuta $24a$, $6b$, $6c$, $8d$, $33e$, $5f$, $2g$, $4h$, $33i$, $6l$, $21m$, $26n$, $16o$, $8p$, $5q$, $17r$, $16s$, $25t$, $32u$, $5x$, $3y$, $+$, $-$, ——, $\pm$, $=$, 4, 2, 1."

The anagram appears never to have been solved; but Bernoulli published his solution‖ of the problem about a year after the publication of the anagram. The solution consists of the determination of a set of values of n, namely $-4m/(2m \pm 1)$, where m is any integer, for any one of which the equation is soluble in finite terms; the details of this solution will be given in §§ 4·1, 4·11.

The prominence given to the work of Riccati by Daniel Bernoulli, combined with the fact that Riccati's equation was of a slightly more general type than

* See *Leibnizens gesamellte Werke*, Dritte Folge (Mathematik), III. (Halle, 1855), p. 75.

† *Acta Eruditorum, Suppl.* VIII. (1724), pp. 66—73. The form in which Riccati took the equation was

$$x^m dq = du + uu\,dx : q,$$

where $q = x^n$.

‡ *Ibid.* pp. 73—75. Daniel Bernoulli mentioned that solutions had been obtained by three other members of his family—John, Nicholas and the younger Nicholas.

§ The reader should observe that the substitution

$$u = -\frac{b}{z}\frac{dz}{dx}$$

gives rise to an equation which is easily soluble in series.

‖ *Exercitationes quaedam mathematicae* (Venice, 1724), pp. 77—80; *Acta Eruditorum*, 1725, pp. 465—473.

John Bernoulli's equation* has resulted in the name of Riccati being associated not only with the equation which he discussed without solving, but also with a still more general type of equation.

It is now customary to give the name† *Riccati's generalised equation* to any equation of the form

$$\frac{dy}{dx} = P + Qy + Ry^2,$$

where P, Q, R are given functions of x.

It is supposed that neither P nor R is identically zero. If $R=0$, the equation is linear; if $P=0$, the equation is reducible to the linear form by taking $1/y$ as a new variable.

The last equation was studied by Euler‡; it is reducible to the general linear equation of the second order, and this equation is sometimes reducible to Bessel's equation by an elementary transformation (cf. §§ 3·1, 4·3, 4·31).

Mention should be made here of two memoirs by Euler. In the first§ it is proved that, when a particular integral y_1 of Riccati's generalised equation is known, the equation is reducible to a linear equation of the first order by replacing y by $y_1 + 1/u$, and so the general solution can be effected by two quadratures. It is also shewn (*ibid.* p. 59) that, if two particular solutions are known, the equation can be integrated completely by a single quadrature; and this result is also to be found in the second‖ of the two papers. A brief discussion of these theorems will be given in Chapter IV.

1·2. *Daniel Bernoulli's mechanical problem.*

In 1738 Daniel Bernoulli published a memoir¶ containing enunciations of a number of theorems on the oscillations of heavy chains. The eighth** of these is as follows: "*De figura catenae uniformiter oscillantis.* Sit catena AC uniformiter gravis et perfecte flexilis suspensa de puncto A, eaque oscillationes facere uniformes intelligatur: pervenerit catena in situm AMF; fueritque longitudo catenae $= l$: longitudo cujuscunque partis $FM = x$, sumatur n ejus valoris†† ut fit

$$1 - \frac{l}{n} + \frac{ll}{4nn} - \frac{l^3}{4\,.\,9n^3} + \frac{l^4}{4\,.\,9\,.\,16n^4} - \frac{l^5}{4\,.\,9\,.\,16\,.\,25n^5} + \text{etc.} = 0.$$

* See James Bernoulli, *Opera Omnia*, II. (Geneva, 1744), pp. 1054—1057; it is stated that the point of Riccati's problem is the determination of a solution in finite terms, and a solution which resembles the solution by Daniel Bernoulli is given.

† The term 'Riccati's equation' was used by D'Alembert, *Hist. de l'Acad. R. des Sci. de Berlin*, XIX. (1763), [published 1770], p. 242.

‡ *Institutiones Calculi Integralis*, II. (Petersburg, 1769), § 831, pp. 88—89. In connexion with the reduction, see James Bernoulli's letter to Leibniz already quoted.

§ *Novi Comm. Acad. Petrop.* VIII. (1760—1761), [published 1763], p. 32.

‖ *Ibid.* IX. (1762—1763), [published 1764], pp. 163—164.

¶ "Theoremata de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae," *Comm. Acad. Sci. Imp. Petrop.* VI. (1732—3), [published 1738], pp. 108—122.

** *Loc. cit.* p. 116.

†† The length of the simple equivalent pendulum is n.

Ponatur porro distantia extremi puncti F ab linea verticali $= 1$, dico fore distantiam puncti ubicunque assumpti M ab eadem linea verticali aequalem

$$1 - \frac{x}{n} + \frac{xx}{4nn} - \frac{x^3}{4 \cdot 9n^3} + \frac{x^4}{4 \cdot 9 \cdot 16n^4} - \frac{x^5}{4 \cdot 9 \cdot 16 \cdot 25n^5} + \text{etc."}$$

He goes on to say: "Invenitur brevissimo calculo $n =$ proxime $0{\cdot}691\, l$.... Habet autem littera n infinitos valores alios."

The last series is now described as a Bessel function* of order zero and argument $2\sqrt{(x/n)}$; and the last quotation states that this function has an infinite number of zeros.

Bernoulli published† proofs of his theorems soon afterwards; in theorem VIII, he obtained the equation of motion by considering the forces acting on the portion FM of length x. The equation of motion was also obtained by Euler‡ many years later from a consideration of the forces acting on an element of the chain.

The following is the substance of Euler's investigation:

Let ρ be the line density of the chain (supposed uniform) and let T be the tension at height x above the lowest point of the chain in its undisturbed position. The motion being transversal, we obtain the equation $\delta T = g\rho\,\delta x$ by resolving vertically for an element of chain of length δx. The integral of the equation is $T = g\rho x$.

The horizontal component of the tension is, effectively, $T\,(dy/dx)$ where y is the (horizontal) displacement of the element; and so the equation of motion is

$$\rho\,\delta x \frac{d^2y}{dt^2} = \delta\left(T\frac{dy}{dx}\right).$$

If we substitute for T and proceed to the limit, we find that

$$\frac{d^2y}{dt^2} = g\frac{d}{dx}\left(x\frac{dy}{dx}\right).$$

If f is the length of the simple equivalent pendulum for any one normal vibration, we write

$$y = A\,\Pi\left(\frac{x}{f}\right)\sin\left(\zeta + t\sqrt{\frac{g}{f}}\right),$$

where A and ζ are constants; and then $\Pi\,(x/f)$ is a solution of the equation

$$\frac{d}{dx}\left(x\frac{dv}{dx}\right) + \frac{v}{f} = 0.$$

If $x/f = u$, we obtain the solution in the form of Bernoulli's series, namely

$$v = 1 - \frac{u}{1} + \frac{u^2}{1 \cdot 4} - \frac{u^3}{1 \cdot 4 \cdot 9} + \frac{u^4}{1 \cdot 4 \cdot 9 \cdot 16} - \dots.$$

* On the Continent, the functions are usually called *cylinder functions*, or, occasionally, *functions of Fourier-Bessel*, after Heine, *Journal für Math.* LXIX. (1868), p. 128; see also *Math. Ann.* III. (1871), pp. 609—610.

† *Comm. Acad. Petrop.* VII. (1734—5), [published 1740], pp. 162—179.

‡ *Acta Acad. Petrop.* V. pars 1 (Mathematica), (1781), [published 1784], pp. 157—177. Euler took the weight of length e of the chain to be E, and he defined g to be the measure of the distance (not twice the distance) fallen by a particle from rest under gravity in a second. Euler's notation has been followed in the text apart from the significance of g and the introduction of ρ and δ (for d).

The general solution of the equation is then shewn to be $Dv+Cv\int^u \frac{du}{uv^2}$, where C and D are constants. Since y is finite when $x=0$, C must be zero.

If a is the whole length of the chain, $y=0$ when $x=a$, and so the equation to determine f is

$$1-\frac{a}{1\,.\,f}+\frac{a^2}{1\,.\,4f^2}-\frac{a^3}{1\,.\,4\,.\,9f^3}+\ldots=0.$$

By an extremely ingenious analysis, which will be given fully in Chapter XV, Euler proceeded to shew that the three smallest roots of the equation in a/f are 1·445795, 7·6658 and 18·63. [More accurate values are 1·4457965, 7·6178156 and 18·7217517.]

In the memoir* immediately following this investigation Euler obtained the general solution (in the form of series) of the equation $\frac{d}{du}\left(u\frac{dv}{du}\right)+v=0$, but his statement of the law of formation of successive coefficients is rather incomplete. The law of formation had, however, been stated in his *Institutiones Calculi Integralis*†, II. (Petersburg, 1769), § 977, pp. 233–235.

1·3. *Euler's mechanical problem.*

The vibrations of a stretched membrane were investigated by Euler‡ in 1764. He arrived at the equation

$$\frac{1}{e^2}\frac{d^2z}{dt^2}=\frac{d^2z}{dr^2}+\frac{1}{r}\frac{dz}{dr}+\frac{1}{r^2}\frac{d^2z}{d\phi^2},$$

where z is the transverse displacement at time t at the point whose polar coordinates are (r, ϕ); and e is a constant depending on the density and tension of the membrane.

To obtain a normal solution he wrote

$$z=u\sin(\alpha t+A)\sin(\beta\phi+B),$$

where α, A, β, B are constants and u is a function of r; and the result of substitution of this value of z is the differential equation

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\left(\frac{\alpha^2}{e^2}-\frac{\beta^2}{r^2}\right)u=0.$$

The solution of this equation which is finite at the origin is given on p. 256 of Euler's memoir; it is

$$u=r^\beta\left\{1-\frac{\alpha^2r^2}{2(n+1)e^2}+\frac{\alpha^4r^4}{2\,.\,4(n+1)(n+3)e^4}-\ldots\right\},$$

where n has been written§ in place of $2\beta+1$.

This differential equation is now known as Bessel's equation for functions of order β; and β may have‖ any of the values 0, 1, 2, ...

Save for an omitted constant factor the series is now called a Bessel coefficient of order β and argument $\alpha r/e$. The periods of vibration, $2\pi/\alpha$, of a

* *Acta Acad. Petrop.* V. pars 1 (Mathematica), (1781), [published 1784], pp. 178—190.

† See also §§ 935, 936 (p. 187 *et seq.*) for the solution of an associated equation which will be discussed in § 3·52.

‡ *Novi Comm. Acad. Petrop.* X. (1764), [published 1766], pp. 243—260.

§ The reason why Euler made this change of notation is not obvious.

‖ If β were not an integer, the displacement would not be a one-valued function of position, in view of the factor $\sin(\beta\phi+B)$.

circular membrane of radius a with a fixed boundary* are to be determined from the consideration that u vanishes when $r = a$.

This investigation by Euler contains the earliest appearance in Analysis of a Bessel coefficient of general integral order.

1·4. *The researches of Lagrange, Carlini and Laplace.*

Only a few years after Euler had arrived at the general Bessel coefficient in his researches on vibrating membranes, the functions reappeared, in an astronomical problem. It was shewn by Lagrange† in 1770 that, in the elliptic motion of a planet about the sun at the focus attracting according to the law of the inverse square, the relations between the radius vector r, the mean anomaly M and the eccentric anomaly E, which assume the forms

$$M = E - \epsilon \sin E, \quad r = a(1 - \epsilon \cos E),$$

give rise to the expansions

$$E = M + \sum_{n=1}^{\infty} A_n \sin nM, \quad \frac{r}{a} = 1 + \tfrac{1}{2}\epsilon^2 + \sum_{n=1}^{\infty} B_n \cos nM,$$

in which a and ϵ are the semi-major axis and the eccentricity of the orbit, and

$$A_n = 2 \sum_{m=0}^{\infty} \frac{(-)^m n^{n+2m-1} \epsilon^{n+2m}}{2^{n+2m} m!(n+m)!}, \quad B_n = -2 \sum_{m=0}^{\infty} \frac{(-)^m (n+2m) . n^{n+2m-2} \epsilon^{n+2m}}{2^{n+2m} m!(n+m)!}.$$

Lagrange gave these expressions for $n = 1, 2, 3$. The object of the expansions is to obtain expressions for the eccentric anomaly and the radius vector in terms of the time.

In modern notation these formulae are written

$$A_n = 2J_n(n\epsilon)/n, \quad B_n = -2(\epsilon/n) J_n'(n\epsilon).$$

It was noted by Poisson, *Connaissance des Tems*, 1836 [published 1833], p. 6 that

$$B_n = -\frac{\epsilon}{n}\frac{dA_n}{d\epsilon};$$

a memoir by Lefort, *Journal de Math.* XI. (1846), pp. 142—152, in which an error made by Poisson is corrected, should also be consulted.

A remarkable investigation of the approximate value of A_n when n is large and $0 < \epsilon < 1$ is due to Carlini‡; though the analysis is not rigorous (and it would be difficult to make it rigorous) it is of sufficient interest for a brief account of it to be given here.

* Cf. Bourget, *Ann. Sci. de l'École norm. sup.* III. (1866), pp. 55—95, and Chree, *Quarterly Journal.* XXI. (1886), p. 298.

† *Hist. de l'Acad. R. des Sci. de Berlin*, XXV. (1769), [published 1771], pp. 204—233. [*Oeuvres*, III. (1869), pp. 113—138.]

‡ *Ricerche sulla convergenza della serie che serva alla soluzione del problema di Keplero* (Milan, 1817). This work was translated into German by Jacobi, *Astr. Nach.* XXX. (1850), col. 197—254 [*Werke*, VII. (1891), pp. 189—245]. See also two papers by Scheibner dated 1856, reprinted in *Math. Ann.* XVII. (1880), pp. 531—544, 545—560.

It is easy to shew that A_n is a solution of the differential equation

$$\epsilon^2 \frac{d^2 A_n}{d\epsilon^2} + \epsilon \frac{dA_n}{d\epsilon} - n^2(1-\epsilon^2) A_n = 0.$$

Define u by the formula $A_n = 2n^{n-1} e^{\int u d\epsilon}/n!$ and then

$$\epsilon^2 \left(\frac{du}{d\epsilon} + u^2\right) + \epsilon u - n^2(1-\epsilon^2) = 0.$$

Hence when n is large either u or u^2 or $du/d\epsilon$ must be large.

If $u = O(n^a)$ we should expect u^2 and $du/d\epsilon$ to be $O(n^{2a})$ and $O(n^a)$ respectively; and on considering the highest powers of n in the various terms of the last differential equation, we find that $a=1$. It is consequently *assumed* that u admits of an expansion in descending powers of n in the form

$$u = nu_0 + u_1 + u_2/n + \ldots,$$

where $u_0, u_1, u_2, \ldots$ are independent of n.

On substituting this series in the differential equation of the first order and equating to zero the coefficients of the various powers of n, we find that

$$u_0^2 = (1-\epsilon^2)/\epsilon^2, \quad \epsilon(u_0' + 2u_0 u_1) + u_0 = 0, \ldots$$

where $u_0' = du_0/d\epsilon$; so that $u_0 = \pm \frac{\sqrt{(1-\epsilon^2)}}{\epsilon}$, $u_1 = \frac{\frac{1}{2}\epsilon}{1-\epsilon^2}$, and therefore

$$\int u d\epsilon = n\left\{\log \frac{\epsilon}{1 \pm \sqrt{(1-\epsilon^2)}} \pm \sqrt{(1-\epsilon^2)} \mp 1\right\} - \tfrac{1}{4}\log(1-\epsilon^2) + \ldots,$$

and, since the value of A_n shews that $\int u d\epsilon \sim n \log \frac{1}{2}\epsilon$ when ϵ is small, the upper sign must be taken and no constant of integration is to be added.

From Stirling's formula it now follows at once that

$$A_n \sim \frac{\epsilon^n \exp\{n\sqrt{(1-\epsilon^2)}\}}{\sqrt{(\frac{1}{2}\pi)} \, . \, n^{\frac{3}{2}} (1-\epsilon^2)^{\frac{1}{4}} \{1+\sqrt{(1-\epsilon^2)}\}^n},$$

and this is the result obtained by Carlini. This method of approximation has been carried much further by Meissel (see § 8·11), while Cauchy* has also discussed approximate formulae for A_n in the case of comets moving in nearly parabolic orbits (see § 8·42), for which Carlini's approximation is obviously inadequate.

The investigation of which an account has just been given is much more plausible than the arguments employed by Laplace† to establish the corresponding approximation for B_n.

The investigation given by Laplace is quite rigorous and the method which he uses is of considerable importance when the value of B_n is modified by taking all the coefficients in the series to be positive—or, alternatively, by supposing that ϵ is a pure imaginary. But Laplace goes on to argue that an approximation established in the case of purely imaginary variables may be used 'sans crainte' in the case of real variables. To anyone who is acquainted with the modern theory of asymptotic series, the fallacious character of such reasoning will be evident.

* *Comptes Rendus*, XXXVIII. (1854), pp. 990—998.

† *Mécanique Céleste*, supplément, t. v. [first published 1827]. *Oeuvres*, v. (Paris, 1882), pp. 486—489.

The earlier portion of Laplace's investigation is based on the principle that, in the case of a series of positive terms in which the terms steadily increase up to a certain point and then steadily decrease, the order of magnitude of the sum of the series may frequently be obtained from a consideration of the order of magnitude of the greatest term of the series.

For other and more recent applications of this principle, see Stokes, *Proc. Camb. Phil. Soc.* VI. (1889), pp. 362—366 [*Math. and Phys. Papers*, V. (1905), pp. 221—225], and Hardy, *Proc. London Math. Soc.* (2) II. (1905), pp. 332—339; *Messenger*, XXXIV. (1905), pp. 97—101. A statement of the principle was given by Borel, *Acta Mathematica*, XX. (1897), pp. 393—394.

The following exposition of the principle applied to the example considered by Laplace may not be without interest:

The series considered is

$$B_n^{(1)} = 2 \sum_{m=0}^{\infty} \frac{(n+2m)\, n^{n+2m-2}\, \epsilon^{n+2m}}{2^{n+2m}\, m!\,(n+m)!},$$

in which n is large and ϵ has a fixed positive value. The greatest term is that for which $m=\mu$, where μ is the greatest integer such that

$$4\mu\,(n+\mu)\,(n+2\mu-2) \leqslant (n+2\mu)\, n^2\epsilon^2,$$

and so μ is approximately equal to

$$\tfrac{1}{2}n\,\{\surd(1+\epsilon^2)-1\} + \tfrac{1}{2}\epsilon^2/(1+\epsilon^2).$$

Now, if u_m denotes the general term in $B_n^{(1)}$, it is easy to verify by Stirling's theorem that, to a first approximation, $\dfrac{u_{\mu\pm t}}{u_\mu} \sim q^{t^2}$, where

$$\log q = -2\surd(1+\epsilon^2)/(n\epsilon^2).$$

Hence

$$B_n^{(1)} \sim u_\mu\,\{1+2q+2q^4+2q^9+\ldots\}$$
$$\sim 2u_\mu \surd\{\pi/(1-q)\},$$

since* q is nearly equal to 1.

Now, by Stirling's theorem,

$$u_\mu \sim \frac{\epsilon^{n-1}\exp\{n\surd(1+\epsilon^2)\}}{\pi n^2\,\{1+\surd(1+\epsilon^2)\}^n},$$

and so

$$B_n^{(1)} \sim \left\{\frac{2\surd(1+\epsilon^2)}{\pi n^3}\right\}^{\frac{1}{2}} \frac{\epsilon^n \exp\{n\surd(1+\epsilon^2)\}}{\{1+\surd(1+\epsilon^2)\}^n}.$$

The inference which Laplace drew from this result is that

$$B_n \sim -\left(\frac{2\surd(1-\epsilon^2)}{\pi n^3}\right)^{\frac{1}{2}} \frac{\epsilon^n \exp\{n\surd(1-\epsilon^2)\}}{\{1+\surd(1-\epsilon^2)\}^n}.$$

This approximate formula happens to be valid when $\epsilon < 1$ (though the reason for this restriction is not apparent, apart from the fact that it is obviously necessary), but it is difficult to prove it without using the methods of contour

* The formula $1+2\sum_{t=0}^{\infty} q^{t^2} \sim \surd\{\pi/(1-q)\}$ may be inferred from general theorems on series; cf. Bromwich, *Theory of Infinite Series*, § 51. It is also a consequence of Jacobi's transformation formula in the theory of elliptic functions,

$$\vartheta_3(0\,|\,\tau) = (-i\tau)^{-\frac{1}{2}}\,\vartheta_3(0\,|\,-\tau^{-1});$$

see *Modern Analysis*, § 21·51.

integration (cf. § 8·31). Laplace seems to have been dubious as to the validity of his inference because, immediately after his statement about real and imaginary variables, he mentioned, by way of confirmation, that he had another proof; but the latter proof does not appear to be extant.

1·5. *The researches of Fourier.*

In 1822 appeared the classical treatise by Fourier*, *La Théorie analytique de la Chaleur*; in this work Bessel functions of order zero occur in the discussion of the symmetrical motion of heat in a solid circular cylinder. It is shewn by Fourier (§§ 118—120) that the temperature v, at time t, at distance x from the axis of the cylinder, satisfies the equation

$$\frac{dv}{dt} = \frac{K}{CD}\left(\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx}\right),$$

where K, C, D denote respectively the Thermal Conductivity, Specific Heat and Density of the material of the cylinder; and he obtained the solution

$$v = e^{-mt}\left\{1 - \frac{gx^2}{2^2} + \frac{g^2x^4}{2^2.4^2} - \frac{g^3x^6}{2^2.4^2.6^2} + \dots\right\}$$

where $g = mCD/K$ and m has to be so chosen that

$$hv + K(dv/dx) = 0$$

at the boundary of the cylinder, where h is the External Conductivity.

Fourier proceeded to give a proof (§§ 307—309) by Rolle's theorem that the equation to determine the values of m has† an infinity of real roots and no complex roots. His proof is slightly incomplete because he assumes that certain theorems which have been proved for polynomials are true of integral functions; the defect is not difficult to remedy, and a memoir by Hurwitz‡ has the object of making Fourier's demonstration quite rigorous.

It should also be mentioned that Fourier discovered the continued fraction formula (§ 313) for the quotient of a Bessel function of order zero and its derivate; generalisations of this formula will be discussed in §§ 5·6, 9·65. Another formula given by Fourier, namely

$$1 - \frac{\alpha^2}{2^2} + \frac{\alpha^4}{2^2.4^2} - \frac{\alpha^6}{2^2.4^2.6^2} + \dots = \frac{1}{\pi}\int_0^\pi \cos(\alpha \sin x)\,dx,$$

had been proved some years earlier by Parseval§; it is a special case of what are now known as Bessel's and Poisson's integrals (§§ 2·2, 2·3).

* The greater part of Fourier's researches was contained in a memoir deposited in the archives of the French Institute on Sept. 28, 1811, and crowned on Jan. 6, 1812. This memoir is to be found in the *Mém. de l'Acad. des Sci.*, IV. (1819), [published 1824], pp. 185—555; V. (1820), [published 1826], pp. 153—246.

† This is a generalisation of Bernoulli's statement quoted in § 1·2.

‡ *Math. Ann.* XXXIII. (1889), pp. 246—266.

§ *Mém. des savans étrangers*, I. (1805), pp. 639—648. This paper also contains the formal statement of the theorem on Fourier constants which is sometimes called Parseval's theorem; another paper by this little known writer, *Mém. des savans étrangers*, I. (1805), pp. 379—398, contains a general solution of Laplace's equation in a form involving arbitrary functions.

The expansion of an arbitrary function into a series of Bessel functions of order zero was also examined by Fourier (§§ 314—320); he gave the formula for the general coefficient in the expansion as a definite integral.

The validity of Fourier's expansion was examined much more recently by Hankel, *Math. Ann.* VIII. (1875), pp. 471—494; Schläfli, *Math. Ann.* X. (1876), pp. 137—142; Dini, *Serie di Fourier*, I. (Pisa, 1880), pp. 246—269; Hobson, *Proc. London Math. Soc.* (2) VII. (1909), pp. 359—388; and Young, *Proc. London Math. Soc.* (2) XVIII. (1920), pp. 163—200. This expansion will be dealt with in Chapter XVIII.

1·6. *The researches of Poisson.*

The unsymmetrical motions of heat in a solid sphere and also in a solid cylinder were investigated by Poisson* in a lengthy memoir published in 1823.

In the problem of the sphere†, he obtained the equation

$$\frac{d^2R}{dr^2} - \frac{n(n+1)}{r^2}R = -\rho^2 R,$$

where r denotes the distance from the centre, ρ is a constant, n is a positive integer (zero included), and R is that factor of the temperature, in a normal mode, which is a function of the radius vector. It was shewn by Poisson that a solution of the equation is

$$r^{n+1}\int_0^\pi \cos(r\rho\cos\omega)\sin^{2n+1}\omega\, d\omega$$

and he discussed the cases $n=0, 1, 2$ in detail. It will appear subsequently (§ 3·3) that the definite integral is (save for a factor) a Bessel function of order $n+\frac{1}{2}$.

In the problem of the cylinder (*ibid.* p. 340 *et seq.*) the analogous integral is

$$\lambda^n\int_0^\pi \cos(h\lambda\cos\omega)\sin^{2n}\omega\, d\omega,$$

where $n=0, 1, 2, \ldots$ and λ is the distance from the axis of the cylinder. The integral is now known as Poisson's integral (§ 2·3).

In the case $n=0$, an important approximate formula for the last integral and its derivate was obtained by Poisson (*ibid.*, pp. 350—352) when the variable is large; the following is the substance of his investigation:

Let‡ $J_0(k)=\frac{1}{\pi}\int_0^\pi \cos(k\cos\omega)\,d\omega,\quad J_0'(k)=-\frac{1}{\pi}\int_0^\pi \cos\omega\sin(k\cos\omega)\,d\omega.$

Then $J_0(k)$ is a solution of the equation

$$\frac{d^2(y\sqrt{k})}{dk^2}+\left(1+\frac{1}{4k^2}\right)y\sqrt{k}=0.$$

* *Journal de l'École R. Polytechnique*, XII. (cahier 19), (1823), pp. 249—403.

† *Ibid.* p. 300 *et seq.* The equation was also studied by Plana, *Mem. della R. Accad. delle Sci. di Torino*, XXV. (1821), pp. 532—534, and has since been studied by numerous writers, some of whom are mentioned in § 4·3. See also Poisson, *La Théorie Mathématique de la Chaleur* (Paris, 1835), pp. 366, 369.

‡ See also Röhrs, *Proc. London Math. Soc.* V. (1874), pp. 136—137. The notation $J_0(k)$ was not used by Poisson.