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1. INTRODUCTION

The differential equations of Mathematical Physics are very often associated to variational principles which state that the solution of the corresponding equation, under given boundary conditions, makes a certain functional stationary on a certain space of functions.

Those methods which replace the problem of solving the equation by the equivalent problem of seeking the function which makes the functional stationary are called variational methods.

A classical variational technique is the Ritz method, which reduces the problem of the minimization of a given functional F on a given space C to the minimization of the same functional F on a finite-dimensional subspace C' of C.

The finite element method is also a variational method in which the elements of C' are piecewise defined on a given domain. It does not always coincide with the Ritz method, however, because the finite-dimensional set C', on which F is made stationary, is generally not contained in C.

It became thus necessary to generalize the old theory to cover the new situations. New convergence theorems were namely stated and demonstrated. The presentation of such theory, using the concepts of functional analysis is the aim of Chapter 3.

The theory supposes that an extremum principle exists. Other variational principles (even Galerkin's method) can be used for establishing sequences of approximations, but the convergence of such sequences cannot be proved, within the frame of the present theory, without the help of an extremum principle.

Although the new convergence theorems were established with the aim of being applied to the finite element method, the theory has a much more general scope. Structural and non-structural applications can indeed be considered and even the linear assumption is not necessary.

Although non-structural applications can be covered, one of the most interesting applications of the convergence theorems appears in the theory of structures as it is shown in Chapter 4.

Chapters 5 and 6 respectively introduce the three- and two-dimensional models of the theory of structures. The discrete model is finally considered in Chapter 7.

A short account of the evolution of the finite element theory and of the papers which the author has been publishing on the subject will be presented now.

The finite element method is a discretization technique for the solution of differential equations, the characteristic feature of which is the superposition of coordinate fields piecewise defined on the domain.

Three stages can be distinguished in the method:

a) subdivision of the domain into subdomains;
b) definition of the finite-dimensional set of fields allowed within each domain (discretization of the field within each subdomain);
c) definition of the interaction between the different subdomains, i.e. of the way in which the field within a given subdomain is connected with the fields of the contacting subdomains.

Different modalities can be considered according as the allowed field within and the interaction between the elements are defined.

In the oldest and most used modality, presented by Turner, Clough, Martin and Topp [1] in 1956, the allowed fields are defined through the displacements, and the interaction through compatibility conditions.

The second modality (hybrid elements), in which the fields are defined through the displacements, and the interaction through compatibility conditions, is due to Pian [2] (1964).

A third modality was presented by de Veubeke [3], also in 1964, in which the fields are also defined through the stresses, but the interaction is defined through equilibrium conditions.

Finally a fourth modality (mixed elements), in which the fields are defined partly through the stresses, partly through the displacements and the interaction partly through equilibrium, partly through compatibility conditions, was introduced by Herrmann [4] in 1965.

In all the four modalities the problem arises of which criteria must be followed for the discretization of the field inside the subdomains.

Such problem was first discussed in connection with the first modality and became critical when the finite element technique started to be applied in the analysis of transversely loaded plates [5].

Up to that time it was assumed, more or less consciously, that continuity of the displacement across the element boundaries was a necessary and sufficient condition for the success of the method. Such condition had been satisfied without difficulty while problems of plane elasticity were the only ones considered. It ceased to be so with plates because, then, the generalized displacements are the transverse displacements and the rotations of the normal to the middle plane, and the continuity of the rotations across the element boundaries proved very difficult to achieve.

Certain unsuccessful results obtained at first were ascribed to the violation of continuity. Nevertheless, as soon as complete compatibility was secured with rectangular elements [6], it was seen that such elements yielded unsatisfactory results, which where explained by the fact that the sequences of approximate solutions generated by systems of elements with decreasing size did not converge to the exact solution.

The condition of the convergence to the exact solution appeared to be of fundamental importance, particularly after having been remarked that certain corrections applied to triangular elements, to ensure compatibility, really decreased the speed of convergence, which, after all, was what was important to increase [7].

A convergence criterion for the first modality was presented in 1965 [8]. According to it, every possible state of uniform strain must be allowed within the element.

In plane elasticity, such criterion requires that the polynomial expression for each displacement component contains an arbitrary constant and two linear terms multiplied by coefficients also arbitrary. Applied to the theory of plates and accepting Kirchhoff's assumption, which implies that rotations be derivatives of the transverse displacement, the criterion requires that the expression of the later contains at least an arbitrary constant and all the linear and quadratic terms also multiplied by arbitrary constants.

The above criterion was based on heuristic considerations and was not, at first, properly speaking demonstrated. A demonstration followed almost immediately for the cases in which compatibility is not violated. Nothing else was necessary than noticing that the finite element method became, in such case, a particular case of the Ritz method [14], and then applying to the finite element method the well-known convergence theory of the latter.

As it is known, given a certain functional F defined on a linear space, C, Ritz method makes it possible to determine the element of C which minimizes functional F. For this purpose, a sequence of linear subspaces of C with a finite number of dimensions is considered, and the element which minimizes F in each subspace is determined. The sequence of such elements converges to the exact solution, i.e. to the one which minimizes F in C, if the sequence of linear subspaces is complete with respect to a subset of C which contains the exact solution, i.e. if, given an element e whatsoever of the subset, it is possible to obtain a sequence converging to e made up of elements of the successive linear subspaces. Such is the so-called completeness criterion.

Now, in the case of the first modality, the functional to be minimized is the total potential energy and the linear space is the set of all the compatible fields. The successive subspaces are families of compatible fields generated by the successive systems of elements, the number of dimensions of each subspace being equal to the total number of nodal displacements corresponding to each system.

What was demonstrated was that the convergence criterion presented for the finite element method, which will be, from now on, more properly called completeness criterion, was no more that the completeness condition with respect to the set of compatible fields whose strains have bounded and continuous first order derivatives within each subdomain.

The case of the first modality for which compatibility was violated, remained unsolved, not to speak of the order modalities.

Before proceeding further, it should be noted that convergence and completeness have no meaning unless a definition for the distance between two fields is introduced. Indeed, a sequence of fields is said to converge to a limit when the sequence of distances between each of its terms and the limit tends to zero.

Convergence is uniform when the distance between two fields is defined in terms of the maximum modulus of the difference between the values of the fields at each point of the domain.

Now, the concept of uniform convergence usually applied in the finite difference method, proves poor when applied to a method like the finite element one closely connected with energy concepts (°).

On the other hand, it is much stronger than needed, as in practice it is not necessary that two fields coincide at all points, in the limit. In other words, uniform convergence may be replaced by energy convergence.

According to the precedent remarks, the square root of the strain energy of the difference between two fields was taken as the measure of the distance between them.

Such is the concept of distance used in Mikhlin's study on convergence in the Ritz method presented in his famous book [11] on variational methods in mathematical physics. This and other basic mathematical concepts used on some papers on finite elements were indeed supplied by Mikhlin, although the finite element method itself is mentioned nowhere in the book.

A paper [13] was published by the author in

(°) It is important to remember, however, that it was proved by Johnson and McLay, [12] in the case of the first modality without compatibility being violated, that, although stresses converge in the mean, displacements convergence uniformly for some kinds of elements.

1968 which is an attempt to describe the finite element method (first modality) as a general method for the solution of the very general class of elliptical equations considered by Mikhlin in his book. Such class comprises namely Navier's equations of two and three-dimensional elasticity and Lagrange's equations of the theory of plates. Following Mikhlin's example, the concepts and language of functional analysis were used in the description.

For such description, it became necessary to introduce the concept of principal derivatives, i.e. the derivatives of the field which must be kept continuous, in order that the energy remains bounded. In the case of Navier's equations of two and three-dimensional elasticity, principal derivatives have order zero, as only the actual displacements are to be kept continuous. In the case of Lagrange's equation, which appears in the theory of plates if Kirchhoff's simplification is introduced, principal derivatives have orders zero and one, as both the transverse displacements and their first derivatives must be continuous (°).

(°) However, if the transverse shear deformations are not neglected, principal derivatives are again of order zero, since the rotations of the normal to the middle surface are displacements just as the transverse displacement.

It was then demonstrated that the completeness criterion, with respect to the set of fields presenting continuous and bounded second derivatives of the principal derivatives, is that the first order derivatives of the principal derivatives of the field can assume arbitrary constant values within each element.

The same paper contains an analysis of convergence in the general case of the continuity of the principal derivatives being violated across the element boundaries. Notice that the discontinuity of the principal derivatives means violation of compatibility if the structural theories are concerned.

The basic concepts of this analysis had been developed first in connexion with a theory of shells [14, 15] involving moments with order up to N , in an attempt to prove that the solution yielded by this theory tends to the solution supplied by three-dimensional elasticity as N tends to infinity. This demonstration was adapted without much difficulty to the finite element theory, and it was readily understood then that a general convergence theorem, applying to the passage from a general structural model to another was implicit.

The analysis showed that completeness alone does not necessarily imply convergence if compatibility, or, in general terms, if the continuity of the principal derivatives is not achieved across the element boundaries. A supplementary condition must hold, viz. that the second derivatives of the principal derivatives remain bounded inside the elements when their size decreases indefinitely.

This supplementary condition was at first considered as disappointing, since completeness was then believed to be a sufficient condition for convergence, whether compatibility was violated or not.

Previous work carried out by Zienkiewicz's team [7] was certainly known which seemed to indicate that the plate element developed by this team did not always converge. It was thought however that the small convergence errors observed would be explained in any other way.

It became clear however, that such uncomfortable results were really due to deficiency of convergence, so that the cause was ascribed to the supplementary condition which, it was thought, could be dispensed with.

This led to a comment [16] on the paper in reference, prepared in collaboration with Zienkiewicz and Irons, in which it was explained that, in the case of plates, no convergence can be obtained unless the third derivatives of the transverse displacement remain bounded within the elements when their size decreases indefinitely. This condition is fulfilled when the elements are arranged in such a way that the nodes are all of the same kind.

Analogous considerations were contained in a paper [17] presented at the second Dayton Conference on matrix methods where the analysis in reference was particularized to structural models.

The arrangement of the elements has no influence if compatibility is not violated across the element boundaries. On the other hand, in the case of plane or three-dimensional elasticity, the second derivatives of the principal derivatives (which are of order zero) are the second derivatives of the displacements, which remain bounded whenever the field within each element is equilibrated by body forces with bounded density. This is particularly the case if such density vanishes. Completeness then ensures convergence, even without compatibility between the elements being achieved.

At this point it was understood that it would be convenient to work at the level of the general theory of structures. Indeed, on one hand, the studies carried out on the finite element method suggested new bases for the synthetic formulation of the theory; on the other hand, it was hoped - correctly, as subsequent events confirmed - that an overall view could help to clarify some particular cases.

The scheme of the synthetic formulation of the theory of structures had already been approached by the author in 1966, [18] in an attempt to unify, by means of the variational theorems, the different continuous and discrete models or theories applied in the analysis of the different types of struc-