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Applied Mathematical Sciences 1

Partial Differential Equations

Third Edition

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Preface to the Third Edition

The book has been completely rewritten for this new edition. While most of the material found in the earlier editions has been retained, though in changed form, there are considerable additions, in which extensive use is made of Fourier transform techniques, Hilbert space, and finite difference methods.

A condensed version of the present work was presented in a series of lectures as part of the *Tata Institute of Fundamental Research — Indian Institute of Science* Mathematics Programme in Bangalore in 1977. I am indebted to Professor K. G. Ramanathan for the opportunity to participate in this exciting educational venture, and to Professor K. Balagangadharan for his ever ready help and advice and many stimulating discussions. Very special thanks are due to N. Sivaramakrishnan and R. Mythili, who ably and cheerfully prepared notes of my lectures which I was able to use as the nucleus of the present edition.

A word about the choice of material. The constraints imposed by a partial differential equation on its solutions (like those imposed by the environment on a living organism) have an infinite variety of consequences, local and global, identities and inequalities. Theories of such equations usually attempt to analyse the structure of individual solutions and of the whole manifold of solutions by testing the compatibility of the differential equation with various types of additional constraints. The problems arising in this way have challenged the ingenuity of mathematicians for centuries. It is good to keep in mind that there is no single "central" problem; new applications commonly lead to new questions never envisioned before. In this book emphasis is put on discovering significant features of a differential equation, and on exploring them as far as possible with a limited amount of machinery from mathematical analysis. Entanglement in a mass of technical details has been avoided, even when this resulted in less general or less complete results.

New Rochelle, N.Y. Fritz John

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1. Introduction

A partial differential equation (henceforth abbreviated as P.D.E.) for a function $u(x,y,\ldots)$ is a relation of the form

$$F(x,y,\ldots,u,u_x,u_y,\ldots,u_{xx},u_{xy},\ldots)=0, \tag{1.1}$$

where F is a given function of the independent variables $x,y,\ldots$, and of the "unknown" function u and of a *finite* number of its partial derivatives. We call u a *solution* of (1.1) if after substitution of $u(x,y,\ldots)$ and its partial derivatives (1.1) is satisfied identically in $x,y,\ldots$ in some region Ω in the space of these independent variables. Unless the contrary is stated we require that $x,y,\ldots$ are real and that u and the derivatives of u occurring in (1.1) are continuous functions of $x,y,\ldots$ in the real domain Ω.† Several P.D.E.s involving one or more unknown functions and their derivatives constitute a *system*.

The *order* of a P.D.E. or of a system is the order of the highest derivative that occurs. A P.D.E. is said to be *linear* if it is linear in the unknown functions and their derivatives, with coefficients depending on the independent variables $x,y,\ldots$. The P.D.E. of order m is called *quasi-linear* if it is linear in the derivatives of order m with coefficients that depend on $x,y,\ldots$ and the derivatives of order $<m$.

* ([7], [13], [26])

† For simplicity we shall often dispense with an explicit description of the domain Ω. Statements made then apply "locally," in a suitably restricted neighborhood of a point of $xy\ldots$-space.

2. Examples

Partial differential equations occur throughout mathematics. In this section we give some examples. In many instances one of the independent variables is the time, usually denoted by t, while the others, denoted by $x_1, x_2, \ldots, x_n$ (or by x, y, z when $n \leqslant 3$) give position in an n-dimensional space. The space differentiations often occur in the particular combination

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{2.1}$$

known as the *Laplace operator*. This operator has the special property of being invariant under rigid motions or equivalently of not being affected by transitions to other cartesian coordinate systems. It occurs naturally in expressing physical laws that do not depend on a special position.

(i) The *Laplace equation* in n dimensions for a function $u(x_1, \ldots, x_n)$ is the linear second-order equation

$$\Delta u = u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n} = 0. \tag{2.2}$$

This is probably the most important individual P.D.E. with the widest range of applications. Solutions u are called *potential* functions or *harmonic* functions. For $n=2$, $x_1 = x$, $x_2 = y$, we can associate with a harmonic function $u(x,y)$ a "conjugate" harmonic function $v(x,y)$ such that the first-order system of *Cauchy–Riemann* equations

$$u_x = v_y, \qquad u_y = -v_x \tag{2.3}$$

is satisfied. A real solution (u, v) of (2.3) gives rise to the *analytic* function

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \tag{2.4}$$

of the complex argument $z = x + iy$. We can also interpret $(u(x,y), -v(x,y))$ as the velocity field of an irrotational, incompressible flow. For $n=3$ equation (2.2) is satisfied by the velocity potential of an irrotational incompressible flow, by gravitational and electrostatic fields (outside the attracting masses or charges), and by temperatures in thermal equilibrium.

(ii) The *wave equation* in n dimensions for $u = u(x_1, \ldots, x_n, t)$ is

$$u_{tt} = c^2 \Delta u \tag{2.5}$$

($c = \text{const.} > 0$). It represents vibrations of strings or propagation of sound waves in tubes for $n=1$, waves on the surface of shallow water for $n=2$, acoustic or light waves for $n=3$.

(iii) *Maxwell's equation* in vacuum for the electric vector $E = (E_1, E_2, E_3)$ and magnetic vector $H = (H_1, H_2, H_3)$ form a linear system of essentially 6 first-order equations

$$\varepsilon E_t = \operatorname{curl} H, \qquad \mu H_t = -\operatorname{curl} E \tag{2.6a}$$

$$\operatorname{div} E = \operatorname{div} H = 0 \tag{2.6b}$$

with constants ε, μ. (If relations (2.6b) hold for $t=0$, they hold for all t as a consequence of relations (2.6a)). Here each component E_j, H_k satisfies the wave equation (2.5) with $c^2 = 1/\varepsilon\mu$.

(iv) *Elastic waves* are described classically by the linear system

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \mu \Delta u_i + (\lambda + \mu) \frac{\partial}{\partial x_i} (\operatorname{div} u) \tag{2.7}$$

($i = 1,2,3$), where the $u_i(x_1, x_2, x_3, t)$ are the components of the displacement vector u, and ρ is the density and λ, μ the Lamé constants of the elastic material. Each u_i satisfies the fourth-order equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \Delta \right) \left(\frac{\partial^2}{\partial t^2} - \frac{\mu}{\rho} \Delta \right) u_i = 0, \tag{2.8}$$

formed from two different wave operators. For *elastic* equilibrium ($u_t = 0$) we obtain the *biharmonic* equation

$$\Delta^2 u = 0. \tag{2.9}$$

(v) The equation of *heat conduction* ("heat equation")

$$u_t = k \Delta u \tag{2.10}$$

($k = \text{const.} > 0$) is satisfied by the temperature of a body conducting heat, when the density and specific heat are constant.

(vi) *Schrödinger's wave equation* ($n = 3$) for a single particle of mass m moving in a field of potential energy $V(x,y,z)$ is

$$i\hbar \psi_t = -\frac{\hbar^2}{2m} \Delta \psi + V \psi, \tag{2.11}$$

where $h = 2\pi\hbar$ is Planck's constant.

The equations in the preceding examples were all linear. Nonlinear equations occur just as frequently, but are inherently more difficult, hence in practice they are often approximated by linear ones. Some examples of nonlinear equations follow.

(vii) A *minimal surface* $z = u(x,y)$ (i.e., a surface having least area for a given contour) satisfies the second-order quasi-linear equation

$$(1 + u_y^2) u_{xx} - 2 u_x u_y u_{xy} + (1 + u_x^2) u_{yy} = 0. \tag{2.12}$$

(viii) The *velocity potential* $\phi(x,y)$ (for velocity components ϕ_x, ϕ_y) of a two-dimensional steady, adiabatic, irrotational, isentropic flow of density ρ satisfies

$$(1 - c^{-2} \phi_x^2) \phi_{xx} - 2 c^{-2} \phi_x \phi_y \phi_{xy} + (1 - c^{-2} \phi_y^2) \phi_{yy} = 0, \tag{2.13}$$

where c is a known function of the speed $q = \sqrt{\phi_x^2 + \phi_y^2}$. For example

$$c^2 = 1 - \frac{\gamma - 1}{2} q^2 \tag{2.14}$$

for a polytropic gas with equation of state

$$p = A \rho^\gamma. \tag{2.15}$$

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(ix) The *Navier–Stokes* equations for the viscous flow of an incompressible liquid connect the velocity components u_k and the pressure p:

$$\frac{\partial u_i}{\partial t}+\sum_k \frac{\partial u_i}{\partial x_k}u_k=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\gamma\Delta u_i, \tag{2.16a}$$

$$\sum_k \frac{\partial u_k}{\partial x_k}=0, \tag{2.16b}$$

where ρ is the constant density and γ the kinematic viscosity.

(x) An example of a third-order nonlinear equation for a function $u(x,t)$ is furnished by the *Korteweg–de Vries* equation

$$u_t+cuu_x+u_{xxx}=0 \tag{2.17}$$

first encountered in the study of water waves.

In general we shall try to describe the *manifold of solutions* of a P.D.E. The results differ widely for different classes of equations. Meaningful "well-posed" problems associated with a P.D.E. often are suggested by particular physical interpretations and applications.

3. Analytic Solution and Approximation Methods in a Simple Example*

We illustrate some of the notions that will play an important role in what follows by considering one of the simplest of all equations

$$u_t+cu_x=0 \tag{3.1}$$

for a function $u=u(x,t)$, where $c=\text{const.}>0$. Along a line of the family

$$x-ct=\text{const.}=\xi \tag{3.2}$$

("characteristic line" in the xt-plane) we have for a solution u of (3.1)

$$\frac{du}{dt}=\frac{d}{dt}u(ct+\xi,t)=cu_x+u_t=0.$$

Hence u is constant along such a line, and depends only on the parameter ξ which distinguishes different lines. The general solution of (3.1) then has the form

$$u(x,t)=f(\xi)=f(x-ct). \tag{3.3}$$

Formula (3.3) represents the general solution u uniquely in terms of its *initial values*

$$u(x,0)=f(x). \tag{3.4}$$

Conversely every u of the form (3.3) is a solution of (3.1) with initial values f provided f is of class $C^1(\mathbb{R})$. We notice that the value of u at any point (x,t) depends only on the initial value f at the single argument $\xi=x-ct$, the abscissa of the point of intersection of the characteristic line through

*([16], [18], [25])

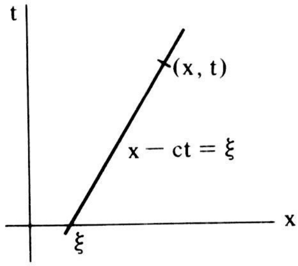

Figure 1.1

(x,t) with the initial line, the x-axis. The *domain of dependence* of $u(x,t)$ on the initial values is represented by the single point ξ. The *influence* of the initial values at a particular point ξ on the solution $u(x,t)$ is felt just in the points of the characteristic line (3.2). (Fig. 1.1)

If for each fixed t the function u is represented by its graph in the ut-plane, we find that the graph at the time $t=T$ is obtained by translating the graph at the time $t=0$ parallel to the x-axis by the amount cT:

$$u(x,0)=u(x+cT,T)=f(x).$$

The graph of the solution represents a *wave* propagating to the right with velocity c without changing shape. (Fig. 1.2)

We use this example with its explicit solution to bring out some of the notions connected with the numerical solution of a P.D.E by the *method of finite differences*. One covers the xt-plane by a rectangular grid with mesh size h in the x-direction and k in the t-direction. In other words one considers only points (x,t) for which x is a multiple of h and t a multiple of k. It would seem natural for purposes of numerical approximation to replace the P.D.E. (3.1) by the difference equation

$$\frac{v(x,t+k)-v(x,t)}{k}+c\frac{v(x+h,t)-v(x,t)}{h}=0. \tag{3.5}$$

Formally this equation goes over into $v_t+cv_x=0$ as $h,k\to 0$. We ask to

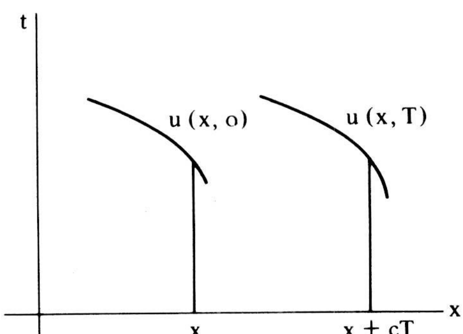

Figure 1.2

what extent a solution v of (3.5) in the grid points with initial values

$$v(x,0)=f(x) \tag{3.6}$$

approximates for small h,k the solution of the initial-value problem (3.1), (3.4).

Setting $\lambda=k/h$, we write (3.5) as a recursion formula

$$v(x,t+k)=(1+\lambda c)v(x,t)-\lambda c v(x+h,t) \tag{3.7}$$

expressing v at the time $t+k$ in terms of v at the time t. Introducing the *shift operator E* defined by

$$Ef(x)=f(x+h), \tag{3.8}$$

(3.7) becomes

$$v(x,t+k)=((1+\lambda c)-\lambda cE)v(x,t) \tag{3.8a}$$

for $t=nk$ this immediately leads by iteration to the solution of the initial-value problem for (3.5):

$$\begin{aligned} v(x,t)&=v(x,nk)=((1+\lambda c)-\lambda cE)^n v(x,0)\\ &=\sum_{m=0}^{n}\binom{n}{m}(1+\lambda c)^m(-\lambda cE)^{n-m}f(x)\\ &=\sum_{m=0}^{n}\binom{n}{m}(1+\lambda c)^m(-\lambda c)^{n-m}f(x+(n-m)h). \end{aligned} \tag{3.9}$$

Clearly the domain of dependence for $v(x,t)=v(x,nk)$ consists of the set of points

$$x,\ x+h,\ x+2h,\ \ldots,\ x+nh \tag{3.10}$$

on the x-axis, all of which lie between x and $x+nh$. The domain of the differential equation solution consists of the point $\xi=x-ct=x-\lambda nh$, which lies completely outside the interval $(x,x+nh)$. It is clear that v for $h,k\to 0$ cannot be expected to converge to the correct solution u of the differential equation, since in forming $v(x,t)$ we do not make use of any information on the value of $f(\xi)$, which is vital for determining $u(x,t)$, but only of more and more information on f in the interval $(x,x+(t/\lambda))$ which is irrelevant. The difference scheme fails the *Courant–Friedrichs–Lewy* test, which requires that the limit of the domain of dependence for the difference equation contains the domain of dependence for the differential equation.

That the scheme (3.5) is inappropriate also is indicated by its high degree of *instability*. In applied problems the data f are never known with perfect accuracy. Moreover, in numerical computations we cannot easily use the exact values but commit small round-off errors at every step. Now it is clear from (3.9) that errors in f of absolute value ε with the proper (alternating) sign can lead to a resulting error in $v(x,t)=v(x,nk)$ of size

$$\varepsilon\sum_{m=0}^{n}\binom{n}{m}(1+\lambda c)^m(\lambda c)^{n-m}=(1+2\lambda c)^n\varepsilon. \tag{3.11}$$

Thus for a fixed mesh ratio λ the possible resulting error in v grows exponentially with the number n of steps in the t-direction.

A more appropriate difference scheme uses "backward" difference quotients:

$$\frac{v(x,t+k)-v(x,t)}{k}+c\frac{v(x,t)-v(x-h,t)}{h}=0 \tag{3.12}$$

or symbolically

$$v(x,t+k)=((1-\lambda c)+\lambda cE^{-1})v(x,t). \tag{3.13}$$

The solution of the initial-value problem for (3.13) becomes

$$v(x,t)=v(x,nk)=\sum_{m=0}^{n}\binom{n}{m}(1-\lambda c)^m(\lambda c)^{n-m}f(x-(n-m)h). \tag{3.14}$$

In this scheme the domain of dependence for $v(x,t)$ on f consists of the points

$$x,\ x-h,\ x-2h,\ \ldots,\ x-nh=x-\frac{t}{\lambda} \tag{3.15}$$

Letting $h,k\to 0$ in such a way that the mesh ratio λ is held fixed, the set (3.15) has as its limit points the interval $[x-(t/\lambda),x]$ on the x-axis. The Courant–Friedrichs–Lewy test is satisfied, when this interval contains the point $\xi=x-ct$, that is when the mesh ratio λ satisfies

$$\lambda c\leqslant 1. \tag{3.16}$$

Stability of the scheme under the condition (3.16) is indicated by the fact that by (3.14) a maximum error of size ε in the initial function f results in a maximum possible error in the value of $v(x,t)=v(x,nk)$ of size

$$\varepsilon\sum_{m=0}^{n}\binom{n}{m}(1-\lambda c)^m(\lambda c)^{n-m}=\varepsilon((1-\lambda c)+\lambda c)^n=\varepsilon. \tag{3.17}$$

We can prove that the v represented by (3.14) actually converges to $u(x,t)=f(x-ct)$ for $h,k\to 0$ with $k/h=\lambda$ fixed, provided the stability criterion (3.16) holds and f has uniformly bounded second derivatives. For that purpose we observe that $u(x,t)$ satisfies

$$\begin{aligned}&|u(x,t+k)-(1-\lambda c)u(x,t)-\lambda cu(x-h,t)|\\&=|f(x-ct-ck)-(1-\lambda c)f(x-ct)-\lambda cf(x-ct-h)|\leqslant Kh^2,\end{aligned} \tag{3.18}$$

where

$$K=\tfrac{1}{2}(c^2\lambda^2+\lambda c)\sup|f''|, \tag{3.19}$$

as is seen by expanding f about the point $x-ct$. Thus, setting $w=u-v$ we have

$$|w(x,t+k)-(1-\lambda c)w(x,t)-\lambda cw(x-h,t)|\leqslant Kh^2$$

and hence

$$\sup_x |w(x,t+k)| \leqslant (1-\lambda c)\sup_x |w(x,t)| + \lambda c \sup_x |w(x-h,t)| + Kh^2$$
$$= \sup_x |w(x,t)| + Kh^2. \tag{3.20}$$

Applying (3.20) repeatedly it follows for $t=nk$ that

$$|u(x,t)-v(x,t)| \leqslant \sup_x |w(x,nk)|$$
$$\leqslant \sup_x |w(x,0)| + nKh^2 = \frac{Kth}{\lambda},$$

since $w(x,0)=0$. Consequently $w(x,t)\to 0$ as $h\to 0$, that is, the solution v of the difference scheme (3.12) converges to the solution u of the differential equation.

PROBLEMS

1. Show that the solution v of (3.12) with initial data f converges to u for $h\to 0$ and a fixed $\lambda \leqslant 1/c$, under the sole assumption that f is continuous. (Hint: the fact that both u and v change by at most ε when we change f by at most ε.)

2. To take into account possible round-off errors we assume that instead of (3.13) v satisfies an inequality
$$|v(x,t+k)-(1-\lambda c)v(x,t)-\lambda c v(x-h,t)| < \delta.$$
Show that for a prescribed δ and for K given by (3.19) we have the estimate
$$|u(x,t)-v(x,t)| \leqslant \frac{Kth}{\lambda} + \frac{t}{\lambda h}\delta \tag{3.21}$$
assuming that (3.16) holds and that $v(x,0)=f(x)$. Find values for λ and h based on this formula that will guarantee the smallest maximum error in computing $u(x,t)$.

3. Instability of a difference scheme under small perturbations does not exclude the possibility that in special cases the scheme converges towards the correct function, if no errors are permitted in the data or the computation. In particular let $f(x)=e^{\alpha x}$ with a complex constant α. Show that for fixed x, t and any fixed positive $\lambda = k/h$ whatsoever both the expressions (3.9) and (3.14) converge for $n\to\infty$ towards the correct limit $e^{\alpha(x-ct)}$. (This is consistent with the Courant–Friedrichs–Lewy test, since for an *analytic* f the values of f in any interval determine those at the point ξ uniquely.)

4. Quasi-linear Equations

The general first-order equation for a function $u=u(x,y,\ldots,z)$ has the form

$$f(x,y,\ldots,u,u_x,u_y,\ldots,u_z)=0. \tag{4.1}$$

Equations of this type occur naturally in the calculus of variations, in particle mechanics, and in geometrical optics. The main result is the fact that the general solution of an equation of type (4.1) can be obtained by solving systems of Ordinary Differential Equations (O.D.E.s for short). This is not true for higher-order equations or for systems of first-order equations. In what follows we shall mostly limit ourselves to the case of two independent variables x,y. The theory can be extended to more independent variables without any essential change.

We first consider the somewhat simpler case of a quasi-linear equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u). \tag{4.2}$$

We represent the function $u(x,y)$ by a surface $z=u(x,y)$ in xyz-space. Surfaces corresponding to solutions of a P.D.E. are called *integral surfaces* of the P.D.E. The prescribed functions $a(x,y,z), b(x,y,z), c(x,y,z)$ define a field of vectors in xyz-space (or in a portion Ω of that space). Obviously only the direction of the vector, the *characteristic direction*, matters for the P.D.E. (4.2). Since $(u_x, u_y, -1)$ constitute direction numbers of the normal of the surface $z=u(x,y)$, we see that (4.2) is just the condition that the normal of an integral surface at any point is perpendicular to the direction of the vector (a,b,c) corresponding to that point. Thus integral surfaces are surfaces that at each point are tangent to the characteristic direction.

With the field of characteristic directions with direction numbers (a,b,c) we associate the family of *characteristic curves* which at each point are tangent to that direction field. Along a characteristic curve the relation

$$\frac{dx}{a(x,y,z)} = \frac{dy}{b(x,y,z)} = \frac{dz}{c(x,y,z)} \tag{4.3}$$

holds. Referring the curve to a suitable parameter t (or denoting the common ratio in (4.3) by dt) we can write the condition defining characteristic curves in the more familiar form of a system of ordinary differential equations

$$\frac{dx}{dt} = a(x,y,z), \qquad \frac{dy}{dt} = b(x,y,z), \qquad \frac{dz}{dt} = c(x,y,z). \tag{4.4}$$

The system is "autonomous" (the independent variable t does not appear explicitly). The choice of the parameter t in (4.4) is artificial. Using any other parameter along the curve amounts to replacing a,b,c by proportional quantities, which does not change the characteristic curve in xyz-space or the P.D.E. (4.2). Assuming that a,b,c are of class C^1 in a region Ω, we know from the theory of O.D.E.s that through each point of Ω there passes exactly one characteristic curve. There is a 2-parameter family of characteristic curves in xyz-space (but a 3-parameter family of solutions $(x(t),y(t),z(t))$ of (4.4), since replacing the independent variable t by $t+c$ with a constant c changes the solution (x,y,z), but not the characteristic curve, which is its range).

If a surface S: $z=u(x,y)$ is a union of characteristic curves, then S is an integral surface. For then through any point P of S there passes a characteristic curve γ contained in S. The tangent to γ at P necessarily lies in the tangent place of S at P. Since the tangent to γ has the characteristic

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