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INVARIANT MEANS ON TOPOLOGICAL GROUPS

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INVARIANT MEANS on Topological Groups and Their Applications

by

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PREFACE

Translation invariant Banach means on spaces of functions associated with a topological group have interested generations of mathematicians since the appearance of von Neumann's article [72], which deals with discrete groups. These invariant means are generally created by highly non-constructive methods—most often by invoking the Hahn-Banach Theorem—and have many strange properties, thus they are often thought of as mathematical curiosities. However, in recent years, some remarkably diverse properties of locally compact groups have been found to depend on the existence of a Banach invariant mean on an appropriate translation-invariant space of functions. One of the most striking results is the following.

Theorem: If G is a locally compact group, there exists a left invariant mean on $L^\infty(G)$ if and only if every irreducible unitary representation of G is weakly contained in the left regular representation.

We give a self-contained exposition, accessible to anyone with a modest understanding of functional analysis, of this and many other recent discoveries relating existence of invariant means to algebraic and geometric properties of a locally compact group.

In the past authors have considered invariant means on a number of spaces of functions; for example, Hulanicki [37] discusses left invariant means on L^∞ in studying the weak containment property above, Rickert [66] shows that the "fixed point property" for G is tied to the existence of a left invariant mean on the bounded right uniformly continuous functions on G (see section 3.3), and Glicksberg [21], Reiter [58] relate ergodic properties of G to existence of an invariant mean on the space $CB(G)$ of bounded continuous functions (see section 3.7). The connection between these various types of invariant means is not at all apparent. We shall prove (combining recent work of the author and Namioka) that these diverse notions of invariant mean are all equivalent for locally compact groups (Theorem 2.2.1); using this equivalence we shall unify many results in the literature and divest them of restrictive hypotheses.

In the past there have been several papers which recount the then current state of the literature: the articles by Dixmier [11], Day [8], and Hewitt-Ross [34] (sections 17-18) are quite helpful and are accessible to most mathematicians. All were written before the equivalence of invariant means was recognized (a very recent development) and before the most important applications had appeared in the literature; we present direct, self-contained accounts of these modern developments. Some of these results are difficult to extract from the scattered literature on invariant means, and many are presented with new proofs, simpler than those which appear in the literature.

These notes are based on lectures presented at Berkeley in the Spring and Fall 1966 quarters. They have benefitted greatly from the author's correspondence with A. Hulanicki, I. Namioka, E. Granirer and W. R. Emerson, and from numerous conversations with colleagues and visiting faculty at Berkeley.

Frederick P. Greenleaf

Special Symbols

$\mathbf{R}$	real numbers
$\mathbf{C}$	complex numbers
$\mathbf{Z}$	integers
${}_x f(s) = f(x^{-1}s)$	[f defined on a group]
$L_x f(s) = f(xs)$	[f defined on any semigroup or group]
$R_x f(s) = f(sx)$	[f defined on any semigroup or group]
Ref	real part of a function f
$A \backslash B$	difference of sets A, B
$A \Delta B$	$(A\backslash B) \cup (B\backslash A)$, the symmetric difference of A, B
$\|A\|$	measure (or cardinality) of a set A
χ_A	characteristic function of A
ϕ_A	$\frac{1}{\|A\|} \chi_A$, normalized characteristic function of A
δ_x	Dirac measure [point mass] at x

CONTENTS

SECTION 1

INVARIANT MEANS ON DISCRETE GROUPS AND SEMIGROUPS

§1.1. MEANS AND INVARIANT MEANS

Let G be any set and X a closed subspace of $B(G)$, the space of *all* bounded complex-valued functions on G equipped with the sup norm $\|f\|_\infty$. Assume X includes all constant functions and is also closed under complex conjugation. Then a linear functional m on X is a *mean* if

(1) $\quad m(\bar{f}) = \overline{m(f)} \quad$ all $f \in X$.

(2) $\quad \inf\{f(x)\} \le m(f) \le \sup\{f(x)\}$ for all real-valued $f \in X$.

The second condition is equivalent to

(2′) $\quad m(f) \ge 0$ if $f \ge 0$, and $m(1) = 1$.

Thus (2) insures that $m(1) = 1$ and $\|m\| = 1$ for any mean. The means on X form a weak*-compact convex set Σ in X^*. If $\ell^1(G)$ is the space of all bounded discrete measures on G with total variation norm, then $B(G) = (\ell^1)^*$; obviously every non-negative measure $\mu \in \ell^1$ with $\|\mu\| = 1$ gives a mean on X: $m_\mu(f) = \langle \mu, f \rangle$ and these form a convex subset $\Sigma_d \subset \Sigma$, the set of *discrete means* on X. Furthermore, every mean m on S is the weak $*$ limit of some net of discrete means; otherwise the Hahn-Banach theorem (as in [13] V. 2.10) insures we can find a $\delta > 0$ and some $f \in X$ such that $\mathrm{Re}\,(m(f)) \ge \delta + \mathrm{Re}\,(m'(f))$ for all discrete means $m' \in \Sigma_d$. But since Σ_d includes all point masses, and $\mathrm{Re}\,(m(f)) = m(\mathrm{Re} f)$, we see that

$$m(\mathrm{Re} f) > \sup\{m'(\mathrm{Re} f)\colon m' \in \Sigma_d\} \geq \sup\{\mathrm{Re} f(x)\colon x \in G\}$$

which contradicts the definition of m being a mean. A similar argument applies to show density of the *finite means*: Σ_{fin}, those arising from measures which are finite sums of point masses.

If G is a group and if the function space X is left invariant, so $f \in X \implies {}_xf \in X$, where ${}_xf(t) = f(x^{-1}t)$,[(1)] then a mean m is *left invariant* (m a LIM) if

$$(3) \qquad m({}_xf) = m(f) \quad \text{all } x \in G, \text{ all } f \in X.$$

Likewise we say m is a *right invariant* mean if $m(f_x) = m(f)$ for all $x \in G$, where we define $f_x(t) = f(tx)$, and we define two-sided invariance in the usual way, assuming of course X is invariant under right and left translations.

There is an interesting duality between right and left invariant means if G is a group. For $f \in X$ define $f^{\sim}(x) = f(x^{-1})$. In many cases of interest $X = X^{\sim}$; in any event we have:

Lemma 1.1.1. If G is a group, there is a left invariant mean on $X \iff$ there is a right invariant mean on $X^{\sim}$.

Proof. Given left invariant mean m on X, define $\overline{m}$ on $X^{\sim}$ so $\overline{m}(f) = m(f^{\sim})$. It is easily verified that

$$(f^{\sim})_x = ({}_xf)^{\sim}$$

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(1) If G is a semigroup there is a slightly different notion of left translation: $L_sf(t) = f(st)$. If X is left invariant, in the sense that $f \in X \implies L_sf \in X$, and if G is actually a group, it is also left invariant in the above sense because $G = G^{-1}$. For functorial reasons we take the above definition of ${}_xf$ when G is a group: this way the action of G on X becomes a group representation instead of an anti-representation

$$\bar{m}((f^{\sim})_x) = \bar{m}((_x f)^{\sim}) = m(_x f) = m(f) = \bar{m}(f^{\sim}) .$$

Example 1.1.2. For a semi-group, there is no analog of this result: let G be a non-empty set with product $xy = y$ for $x, y \in G$. Then if $f \in B(G)$ we have $L_x f(t) = f(xt) = f(t)$ so *every* mean on $B(G)$ is left-invariant. But $f_x(t) = f(tx) = f(x)\cdot 1$ so if m is a right invariant mean: $m(f) = m(f_x) = m(f(x)\cdot 1) = f(x)$ for all $x \in G$; if G has more than one element, this is clearly impossible.

In many cases existence of left and right invariant means m_ℓ and m_r insures existence of a two-sided invariant mean m. The general idea of the proof, which makes sense if $X = B(G)$ for example, is to take $f \in X$, define $F(x) = \langle m_\ell, f_x \rangle$, and set $m(f) = m_r(F)$. It is readily verified that m is two-sided invariant, which proves:

Lemma 1.1.3. If G is a semigroup with a left invariant mean and a right invariant mean on $X = B(G)$, then there exists a two-sided invariant mean on X.

However, this construction does not always make sense, for if m_ℓ and m_r are invariant means on $X = CB(G)$: the continuous bounded functions on a topological group G, then there is no assurance that $f \in CB(G) \Longrightarrow F(x) = \langle m_\ell, f_x \rangle$ is in $CB(G)$. On the other hand if X is a space of suitably uniformly continuous functions on G, there is some hope of making this construction work. A few constructions for two-sided invariant means are discussed in the introductory section of [36].

Our main interest is in left and right invariant means on *groups* and for applications it is only important to know whether there is at least one such invariant mean on X; the uniqueness of such means is not relevant. In view of the duality exhibited in 1.1.1, we shall generally discuss left invariant means when dealing with groups. It is fortunate that uniqueness of invariant means is not important in applications because they are usually not unique. If $X = B(G)$ and G is a finite group, or if X is a reasonable space of continuous functions and G is a compact group, then the normalized Haar measure on G gives a left invariant mean (LIM) on X and it is easily seen that this is the only LIM on X. The question of uniqueness has received a great deal of study; see Day [8], sections 6–7, and also Hewitt-Ross [34], section 17.21. Some recent results, especially those in Granirer [26], allow us to prove the following definitive result.

Theorem 1.1.5. Let G be any discrete group which admits a LIM on $B(G)$. Then $B(G)$ has a unique LIM $\Longleftrightarrow G$ is finite.

We prove this in Appendix 1. The situation is incompletely understood for invariant means on spaces of continuous functions on a non-discrete topological group, cf. [27]. In 2.4, once we have developed techniques for constructing invariant means, we shall give some direct constructions of distinct invariant means (the methods of Appendix 1 are probabilistic).

§1.2 CONSTRUCTION OF INVARIANT MEANS

A discrete semigroup G is *left (right) amenable* if there is a left (right) invariant mean on $X = B(G)$; if G is a group these conditions are the same and we say that G is *amenable*. Our first problem is to find reasonable conditions on G which enable us (modulo the Hahn-Banach Theorem) to construct invariant means on $B(G)$. Dixmier [11] shows, following ideas which first appear in von Neumann [72], that existence of a LIM on $B(G)$ is equivalent to the following property of G.

(D) If $\{f_1, \ldots, f_N\}$ are real-valued functions in $B(G)$ and if $\{x_1, \ldots, x_N\} \subset G$ then:

$$\inf\{\sum_{i=1}^{N} (f_i - L_{x_i} f_i)\} \leq 0 .$$

Let X be the closed (real) subspace generated by $\{f - L_x f: x \in G,\ f$ real-valued$\}$. If m is a LIM on $B(G)$ it must annihilate X, so $0 = m(\phi) \geq \inf(\phi)$, all $\phi \in X$. Conversely if $\inf(\phi) \leq 0$ for all $\phi \in X$ then in $B_r(G)$, the real-valued bounded functions, consider $K = \{\phi \in B_r(G): \inf(\phi) > 0\}$. This open convex set is disjoint from the subspace X so by one form of Hahn-Banach there is a bounded linear functional m on $B_r(G)$ such that $m(X) = 0$, $m(f) > 0$ for all $f \in K$ By scaling we can arrange that $m(1) = 1$; thus m is a LIM on $B_r(G)$. Extend it to a LIM on $B(G)$ by taking $\overline{m}(f + ig) = m(f) + i \cdot m(g)$.

There are obvious right-handed and two-sided versions of this result, whose proofs we omit. Using this criterion we prove a basic existence theorem (following [34]; there is a gap in the proof which appears in [11]).

Theorem 1.2.1. There is an invariant measure on $B(G)$ for any abelian semigroup G.

Proof. Let $\{f_1, \ldots, f_N\} \subset B(G)$ and $\{x_1, \ldots, x_N\} \subset G$ be given. Write $\Lambda_p = \{(\lambda_1, \ldots, \lambda_N): \lambda_k$ integers, $1 \leq \lambda_k \leq p\}$ for $p = 1, 2, \ldots$, so that Λ_p has cardinality $|\Lambda_p| = p^N$, and define $\tau(\lambda) = x_1^{\lambda_1} \cdots x_N^{\lambda_N} \in G$. In any sum of the form

$$\Sigma\{f_k(\tau(\lambda)) - f_k(x_k \cdot \tau(\lambda)): \lambda \in \Lambda_p\}$$

all terms cancel except possibly those $f_k(\tau(\lambda))$ with $\lambda_k = 1$ and those $f_k(x_k \cdot \tau(\lambda))$ with $\lambda_k = p$ (there are only $2p^{N-1}$ such λ in Λ_p for each k). But if $\phi(t) = \Sigma_{k=1}^{N} f_k(t) - f_k(x_k t)$ and $m = \max\{\|f_k\|_\infty: k = 1, 2, \ldots N\}$, then

$$\begin{aligned}|\Lambda_p| \inf\{\phi(t)\colon t \in G\} &= p^N \inf\{\phi\} \le \Sigma\{\phi(\tau(\lambda))\colon \lambda \in \Lambda_p\} \\ &= \sum_{k=1}^{N} \{\Sigma\{f_k(\tau(\lambda)) - f_k(x_k\tau(\lambda))\colon \lambda \in \Lambda_p\}\} \\ &\le \sum_{k=1}^{N} 2p^{N-1}\|f_k\|_\infty \le 2mN\,p^{N-1} \qquad p = 1,2\ldots .\end{aligned}$$

Thus $\inf\{\phi\} \le 0$ as required. Q.E.D.

If G is a finite group, then there is a (unique) LIM on $B(G)$ corresponding to Haar measure, but if G is a finite semigroup there may not be any LIM on $B(G)$ as 1.1.2 shows. Rosen [67] has characterized the finite semigroups which are two-sided amenable as follows—we will not prove this here.

Proposition 1.2.2. A finite semigroup G has a two-sided invariant mean on $B(G)$ $<=>$ G has unique minimal left and right ideals; then these minimal ideals coincide in a two-sided ideal which is a finite group G^*. The (unique) invariant mean m on $B(G)$ is given by

$$m(f) = \frac{1}{|G^*|} \Sigma\{f(t)\colon t \in G^*\}$$

where $|G^*|$ = cardinality of G^*.

Example 1.2.3. If m is a left invariant mean on $B(G)$ we may define a left-invariant finitely additive measure μ on the collection $\Omega(G)$ of all subsets: $\mu(E) = m(\chi_E)$, where χ_E = characteristic function of $E \in \Omega(G)$. If G is the free group on two generators a, b such a measure cannot exist, thus G is not amenable: divide G into disjoint sets $\{H_i\colon i \in \mathbf{Z}\}$ with $x \in H_i$ $<=>$ when expressed as a reduced word, $x = a^i b^{i_1} a^{i_2} \ldots$