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(by courtesy of Professor A. T. Fomenko of the Moscow State University)

Jean Jacod Albert N. Shiryaev

Limit Theorems for Stochastic Processes

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To our sons
Olivier, Vincent and Andrei

Introduction

The limit theorems in this book belong to the theory of weak convergence of probability measures on metric spaces.

More precisely, our main aim is to give a systematic exposition of the theory of convergence in law for those stochastic processes that are semimartingales.

The choice of the class of semimartingales as our chief object of study has two reasons. One is that this class is broad enough to accomodate most common processes: discrete-time processes, diffusions, many Markov processes, point processes, solutions of stochastic differential equations, ... Our second reason is that we have in our hands a very powerful tool for studying these processes, namely the stochastic calculus. Since the theory of semimartingales, and related topics as random measures, are not usually associated with limit theorems, we decided to write a rather complete account of that theory, which is covered in the first two chapters. In particular, we devote much space to a careful and detailed exposition of the notion of characteristics of a semimartingale, which extends the well-known "Lévy-Khintchine triplet" for processes with independent increments (drift term, variance of the Gaussian part, Lévy measure), and which plays a particularly important rôle in limit theorems.

The meaning of $X^n \xrightarrow{\mathscr{L}} X$ (that is, the sequence (X^n) of processes converges in law to the process X) is not completely straightforward. The first idea would be to use "finite-dimensional convergence", which says that for any choice $t_1, \dots, t_p$ of times, then $(X^n_{t_1}, \dots, X^n_{t_p})$ goes in law to $(X_{t_1}, \dots, X_{t_p})$. This is usually unsatisfactory because it does not ensure convergence in law of such simple functionals as $\inf(t\colon X^n_t > a)$ or $\sup_{s \leq 1} X^n_s$, etc... In fact, since the famous paper [199] of Prokhorov, the traditional mode of convergence is weak convergence of the laws of the processes, considered as random elements of some functional space. Because semimartingales are right-continuous and have left-hand limits, here the fundamental functional space will always be the "Skorokhod space" $\mathbb{D}$ introduced by Skorokhod in [223]: this space can be endowed with a complete separable metric topology, and $X^n \xrightarrow{\mathscr{L}} X$ will always mean weak convergence of the laws, relative to that topology.

How does one prove that $X^n \xrightarrow{\mathscr{L}} X$?, and in which terms is it suitable to express the conditions? The method proposed by Prokhorov goes as follows:

$$\overset{\text{(i)}}{\left|\begin{matrix}\text{Tightness of the}\\ \text{sequence } (X^n)\end{matrix}\right|} + \overset{\text{(ii)}}{\left|\begin{matrix}\text{Convergence of finite-}\\ \text{dimensional distributions}\end{matrix}\right|}$$

$$+ \overset{\text{(iii)}}{\left|\begin{matrix}\text{Characterization of } (X) \text{ by}\\ \text{finite-dimensional distributions}\end{matrix}\right|} \Rightarrow X^n \xrightarrow{\mathscr{L}} X$$

(as a matter of fact, this is even an equivalence; and of course (iii) is essentially trivial). Sometimes, we will make use of this method. However, it should be emphasized that very often step (ii) is a very difficult (or simply impossible) task to accomplish (with a notable exception concerning the case where the limiting process has independent increments). This fact has led to the development of other strategies; let us mention, for example, the method based upon the "embedding theorem" of Skorokhod, or the "approximation and σ-topological spaces methods" of Borovkov, which allows one to prove weak convergence for large classes of functionals and which are partly based upon (ii). Here we expound the strategy called "martingale method", initiated by Stroock and Varadhan, and which goes as follows:

$$\text{(i)} + \overset{\text{(ii}')}{\left|\begin{matrix}\text{Convergence of triplets}\\ \text{of characteristics}\end{matrix}\right|} + \overset{\text{(iii}')}{\left|\begin{matrix}\text{Characterization of } (X) \text{ by the}\\ \text{triplet of characteristics}\end{matrix}\right|} \Rightarrow X^n \xrightarrow{\mathscr{L}} X.$$

Here the difficult step is (iii′): we devote a large part of Chapter III to the explicit statement of the problem (called "martingale problem") and to some partial answers.

In both cases, we need step (i): in Chapter VI we develop several tightness criteria especially suited to semimartingales; we also use this opportunity to expose elementary—and less elementary—facts about the Skorokhod topology, in particular for processes indexed by the entire half-line $\mathbb{R}_+$.

The limit theorems themselves are presented in Chapters VII, VIII and IX (the reader can consult [166] for slightly different aspects of the same theory). Conditions insuring convergence always have a similar form, for simple situations (as convergence of processes with independent increments) as well as for more complicated ones (convergence of semimartingales to a semimartingale). Roughly speaking, they say that the triplets of characteristics of X^n converge to the triplet of characteristics of X. As a matter of fact, these conditions are more extensions of two sets of results that are apparently very far apart: those concerning convergence of rowwise independent triangular arrays, as in the book [65] of Gnedenko and Kolmogorov; and those concerning convergence of Markov processes (and especially of diffusion processes, in terms of their coefficients), as in the book [233] of Stroock and Varadhan.

Beside limit theorems, the reader will find apparently disconnected results, which concern absolute continuity for a pair of measures given on a filtered space,

and contiguity of sequences of such pairs. In fact, one of our motivations for including such material is that we wanted to give some statistically-oriented applications of our limit theorems (a second motivation is that we indeed find this subject interesting on its own): that is done in Chapter X, where we study convergence of likelihood ratio processes (in particular asymptotic normality) and the so-called "statistical invariance principle" which gives limit theorems under contiguous alternatives.

In order to prepare for these results, we need a rather deep study of contiguity: this is done in Chapter V, in which Hellinger integrals and what we call Hellinger processes are widely used. Hellinger processes are introduced in Chapter IV, which also contains necessary and sufficient conditions for absolute continuity and singularity in terms of the behaviour of those Hellinger processes. Finally, let us mention that some material about convergence in variation is also included in Chapter V.

Within each chapter, the numbering is as follows: 3.4 means statement number 4 in Section 3. When referring to a statement in a previous chapter, say Chapter II, we write II.3.4.

In addition to the usual indexes (Index of Symbols; Index of Terminology), the reader will find in the Index of Topics a reference to all the places in this book where we write about a specific subject: for example, a reader interested only in point processes should consult the Index of Topics first. Finally, all the conditions on the triplets of characteristics which appear in our limit theorems are listed in the Index of Conditions for Limit Theorems.

Parts of this work were performed while one or other author was enjoying the hospitality of the Steklov Mathematical Institute or the Université Pierre et Marie Curie, Paris VI. We are grateful for having had these opportunities.

Paris and Moscow,
June 1987

Jean Jacod
Albert N. Shiryaev

Basic Notation

$\mathbb{R} = (-\infty, +\infty)$ = the set of real numbers, $\mathbb{R}_+ = [0, \infty)$, $\bar{\mathbb{R}} = [-\infty, +\infty]$
$\bar{\mathbb{R}}_+ = [0, \infty]$
$\mathbb{Q}$ = the set of rational numbers, $\mathbb{Q}_+ = \mathbb{Q} \cap \mathbb{R}_+$
$\mathbb{N} = \{0, 1, 2, \ldots\}$ = the set of integers, $\mathbb{N}^* = \{1, 2, 3, \ldots\}$
$\mathbb{C}$ = the set of complex numbers
$\mathbb{R}^d$ = the Euclidian d-dimensional space
$|x|$ = the Euclidian norm of $x \in \mathbb{R}^d$, or the modulus of $x \in \mathbb{C}$
$x \cdot y$ = the scalar product of $x \in \mathbb{R}^d$ with $y \in \mathbb{R}^d$
$a \vee b = \sup(a, b)$, $a \wedge b = \inf(a, b)$
$x^+ = x \vee 0$, $x^- = (-x) \vee 0$ for $x \in \mathbb{R}$
1_A = the indicator function of the set A
A^c = the complement of the set A
ε_a = the Dirac measure sitting at point a
a. s. = almost surely
$\lim_{s\uparrow t} = \lim_{s\to t, s\le t}$, $\lim_{s\uparrow\uparrow t} = \lim_{s\to t, s<t}$
$\lim_{s\downarrow t} = \lim_{s\to t, s\ge t}$, $\lim_{s\downarrow\downarrow t} = \lim_{s\to t, s>t}$
$\otimes$ = tensor product (of spaces, of σ-fields)
$[x]$ = the integer part of $x \in \mathbb{R}_+$
$\mathrm{Re}(x)$, $\mathrm{Im}(x)$ = real and imaginary parts of $x \in \mathbb{C}$
$\ll$ absolute continuity between measures
$\sim$ equivalence between measures
$\perp$ singularity between measures
$\{\cdots\}$ denotes a set

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Chapter I. The General Theory of Stochastic Processes, Semimartingales and Stochastic Integrals

The "General Theory of Stochastic Processes", in spite of its name, encompasses the rather restrictive subject of stochastic processes indexed by $\mathbb{R}_+$. But, within this framework, it expounds deep properties related to the order structure of $\mathbb{R}_+$, and martingales play a central rôle.

By now, there exist several books that give more or less complete accounts on the theory: the basic book [33] of Dellacherie (which however does not deal with stochastic integrals at all), the very complete book [36] of Dellacherie and Meyer, or the book [180] of Métivier... But those may appear as a gigantic investment, for somebody who is not acquainted with the theory beforehand, as might presumably be many of the potential readers of this book. This is why we feel necessary to present a sort of "résumé" that brings out all the needed facts for limit theorems, along the quickest and (hopefully) most painless possible way (although this way is somehow old-fashioned, especially for the presentation of semimartingales and stochastic integrals).

As we wished this book to be as much self-contained as possible, we have provided below all the proofs, with a few exceptions concerning the theory of martingales (regularity of paths, Doob's inequality, Doob's optional theorem), and also two difficult but reasonably well-known results: the Doob-Meyer decomposition of submartingales, and the section theorem (for which we refer to [33] or [36]).

However, despite the fact that all proofs do appear, this chapter is written in the spirit of a résumé, not of a beginner's course: for instance there are almost no examples. So we rather advise the reader to go quickly through the statements (to refresh his mind about notation and definitions) and then to proceed directly to the next chapter.

1. Stochastic Basis, Stopping Times, Optional σ-Field, Martingales

Here are some standard notations to be used in all the book. If $(\Omega, \mathscr{F}, P)$ is a probability space, we denote by $E(X)$ the expectation of any integrable random variable X; if there is some ambiguity as to the measure P, we write $E_P(X)$.

$L^p = L^p(\Omega, \mathscr{F}, P)$, for $p \in [1, \infty)$, is the space of all real-valued random variables X such that $|X|^p$ is integrable, with the usual identification of any two a.s. (= almost surely) equal random variables. Similarly $L^\infty(\Omega, \mathscr{F}, P)$ is the set of all P-essentially bounded real-valued random variables. The corresponding norms are denoted by $\|X\|_{L^p}$.

If $\mathscr{G}$ is a sub-σ-field of $\mathscr{F}$, the conditional expectation of the variable X is well-defined whenever X is integrable or nonnegative or nonpositive, and we denote by $E(X|\mathscr{G})$ any version of it. As a matter of fact, it is also very convenient to use the notion of *generalized conditional expectation*, which is defined for *all* random variables by

1.1 $$E(X|\mathscr{G}) = \begin{cases} E(X^+|\mathscr{G}) - E(X^-|\mathscr{G}) & \text{on the set where } E(|X||\mathscr{G}) < \infty \\ +\infty & \text{elsewhere.} \end{cases}$$

In most cases, $X = Y$ (or $X \le Y$, etc...) stands for: "$X = Y$ a.s. (almost surely)" (or $X \le Y$ a.s., etc...).

§ 1a. Stochastic Basis

The reader will immediately notice that our main concern lies in stochastic processes indexed by $\mathbb{R}_+$, or perhaps an interval of $\mathbb{R}_+$. In this case, the theory is built upon what is commonly known as a "stochastic basis", to be recalled below. However, we will occasionally deal with discrete-time processes, that are indexed by $\mathbb{N}$. To help the reader to make the connexion between the two settings, at the end of every section of this chapter we provide an autonomous treatment for the "discrete time": for instance, § 1f of this section provides for the discrete version of what follows.

1.2 **Definition.** A *stochastic basis* is a probability space $(\Omega, \mathscr{F}, P)$ equipped with a *filtration* $\mathbf{F} = (\mathscr{F}_t)_{t \in \mathbb{R}_+}$; here, *filtration* means *increasing* and *right-continuous* family of sub-σ-fields of $\mathscr{F}$ (in other words, $\mathscr{F}_s \subset \mathscr{F}_t$ for $s \le t$ and $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$).
By convention, we set: $\mathscr{F}_\infty = \mathscr{F}$ and $\mathscr{F}_{\infty-} = \bigvee_{s \in \mathbb{R}_+} \mathscr{F}_s$. □

The stochastic basis $\mathscr{B} = (\Omega, \mathscr{F}, \mathbf{F}, P)$ is also called a *filtered probability space*. In many cases (but not always, as we shall see) it is possible to assume a further property, namely

1.3 **Definition.** The stochastic basis $(\Omega, \mathscr{F}, \mathbf{F}, P)$ is called *complete*, or equivalently is said to *satisfy the usual conditions* if the σ-field $\mathscr{F}$ is P-complete and if every $\mathscr{F}_t$ contains all P-null sets of $\mathscr{F}$. □

It is always possible to "complete" a given stochastic basis $(\Omega, \mathscr{F}, \mathbf{F}, P)$ as follows:

1.4 $\mathscr{F}^P$ denotes the P-completion of the σ-field $\mathscr{F}$; $\mathscr{N}^P$ denotes the set of all P-null sets of $\mathscr{F}^P$; $\mathscr{F}_t^P$ is the smallest σ-field that contains $\mathscr{F}_t$ and $\mathscr{N}^P$. It is very easy to check that $(\Omega, \mathscr{F}^P, \mathbf{F}^P = (\mathscr{F}_t^P)_{t \in \mathbb{R}_+}, P)$ is a new stochastic basis, called the *completion* of $(\Omega, \mathscr{F}, \mathbf{F}, P)$. □

Let us fix some terminology:

1.5 A *random set* is a subset of $\Omega \times \mathbb{R}_+$. □

1.6 A *process* (or, a *E-valued process*) is a family $X = (X_t)_{t \in \mathbb{R}_+}$ of mappings from Ω into some set E. Unless otherwise stated, E will be $\mathbb{R}^d$ for some $d \in \mathbb{N}^*$. □

A process may, and often will, be considered as a mapping from $\Omega \times \mathbb{R}_+$ into E, via

1.7 $$(\omega, t) \rightsquigarrow X(\omega, t) = X_t(\omega).$$

We shall say indifferently: the process "X", or "(X_t)", or "$(X_t)_{t \in \mathbb{R}_+}$". Each mapping: $t \rightsquigarrow X_t(\omega)$, for a fixed $\omega \in \Omega$, is called a *path*, or a *trajectory*, of the process X.

For example, the indicator function 1_A of a random set A is a process; its paths are the indicator functions of the $\mathbb{R}_+$-sections $\{t: (\omega, t) \in A\}$ of A.

A process X is called *càd* (resp. *càg*, resp. *càdlàg*), for "*c*ontinu *à d*roite" (resp. "*c*ontinu *à g*auche", resp. *c*ontinu *à d*roite avec des *l*imites *à g*auche") in French, if all its paths are right-continuous (resp. are left-continuous, resp. are right-continuous and admit left-hand limits). When X is càdlàg we define two other processes $X_- = (X_{t-})_{t \in \mathbb{R}_+}$ and $\Delta X = (\Delta X_t)_{t \in \mathbb{R}_+}$ by

1.8 $$\begin{cases} X_{0-} = X_0, \quad X_{t-} = \lim_{s \uparrow\uparrow t} X_s \quad \text{for } t > 0 \\ \Delta X_t = X_t - X_{t-} \end{cases}$$

(hence $\Delta X_0 = 0$, which differs from a convention that is sometimes used, as in [183]).

If X is a process and if T is a mapping: $\Omega \to \overline{\mathbb{R}}_+$, we define the "*process stopped at time T*", denoted by X^T, by

1.9 $$X_t^T = X_{T \wedge t}.$$

1.10 A random set A is called *evanescent* if the set $\{\omega: \exists t \in \mathbb{R}_+ \text{ with } (\omega, t) \in A\}$ is P-null; two E-valued processes X and Y are called *indistinguishable* if the random set $\{X \ne Y\} = \{(\omega, t): X_t(\omega) \ne Y_t(\omega)\}$ is evanescent, i.e. if almost all paths of X and Y are the same. □

Note that if X and Y are indistinguishable, one has $X_t = Y_t$ a.s. for all $t \in \mathbb{R}_+$, but the converse is not true. This converse is true, however, when both X and Y are càd, or are càg.

As for random variables, in most cases $X = Y$ (or $X \le Y$, etc...) for stochastic processes means "up to an evanescent set".

§ 1b. Stopping Times

Let $(\Omega, \mathscr{F}, \mathbf{F}, P)$ be a stochastic basis.

1.11 **Definitions.** a) A *stopping time* is a mapping $T: \Omega \to \bar{\mathbb{R}}_+$ such that $\{T \le t\} \in \mathscr{F}_t$ for all $t \in \mathbb{R}_+$.

b) If T is a stopping time, we denote by $\mathscr{F}_T$ the collection of all sets $A \in \mathscr{F}$ such that $A \cap \{T \le t\} \in \mathscr{F}_t$ for all $t \in \mathbb{R}_+$.

c) If T is a stopping time, we denote by $\mathscr{F}_{T-}$ the σ-field generated by $\mathscr{F}_0$ and all the sets of the form $A \cap \{t < T\}$, where $t \in \mathbb{R}_+$ and $A \in \mathscr{F}_t$. □

One readily checks that $\mathscr{F}_T$ is a σ-field. If $t \in \bar{\mathbb{R}}_+$ and $T(\omega) \equiv t$, then T is a stopping time and $\mathscr{F}_T = \mathscr{F}_t$ (recall that $\mathscr{F}_\infty = \mathscr{F}$ by 1.2); hence the notation $\mathscr{F}_T$ is not ambiguous. Similarly, for $T \equiv t$, one has $\mathscr{F}_{T-} = \mathscr{F}_0$ if $t = 0$, and $\mathscr{F}_{T-} = \bigvee_{s<t} \mathscr{F}_s$ is $t > 0$: hence the notation

1.12
$$\mathscr{F}_{t-} = \begin{cases} \mathscr{F}_0 & \text{if } t = 0 \\ \bigvee_{s<t} \mathscr{F}_s & \text{if } t \in (0, \infty] \text{ (recall 1.2 again for } \mathscr{F}_{\infty-}). \end{cases}$$

The σ-field $\mathscr{F}_t$ is usually interpreted as the set of events that occur before or at time t; then a stopping time is a random time T such that at each time t one may decide whether $T \le t$ or $T > t$ from what one knows up to time t; and $\mathscr{F}_T$ (resp. $\mathscr{F}_{T-}$) is interpreted as the set of events that occur before or at time T (resp. strictly before T).

Now we give a list of well-known and very useful properties of stopping times. All the proofs can be easily provided for by the reader, or may be found in any standard text-book.

1.13 If T is a stopping time and $t \in \mathbb{R}_+$, then $T + t$ is a stopping time. □

1.14 If T is a stopping time, then $\mathscr{F}_{T-} \subset \mathscr{F}_T$ and T is $\mathscr{F}_{T-}$-measurable. □

1.15 If T is a stopping time and if $A \in \mathscr{F}_T$, then

$$T_A(\omega) = \begin{cases} T(\omega) & \text{if } \omega \in A \\ +\infty & \text{if } \omega \notin A \end{cases}$$

is also a stopping time. □

1.16 A mapping $T: \Omega \to \bar{\mathbb{R}}_+$ is a stopping time if and only if $\{T < t\} \in \mathscr{F}_t$ for all $t \in \mathbb{R}_+$; in this case, a set $A \in \mathscr{F}$ belongs to $\mathscr{F}_T$ if and only if $A \cap \{T < t\} \in \mathscr{F}_t$ for all $t \in \mathbb{R}_+$ (the right-continuity of the filtration $\mathbf{F}$ is essential for this property). □

1.17 If S, T are two stopping times and if $A \in \mathscr{F}_S$, then $A \cap \{S \le T\} \in \mathscr{F}_T$, $A \cap \{S = T\} \in \mathscr{F}_T$, and $A \cap \{S < T\} \in \mathscr{F}_{T-}$. □

1.18 If (T_n) is a sequence of stopping times, then $S = \bigwedge T_n$ and $T = \bigvee T_n$ are two stopping times, and $\mathscr{F}_S = \bigcap \mathscr{F}_{T_n}$. □

1.19 **Lemma.** *Any stopping time T on the completed stochastic basis $(\Omega, \mathscr{F}^P, \mathbf{F}^P, P)$ is a. s. equal to a stopping time on $(\Omega, \mathscr{F}, \mathbf{F}, P)$.*

Proof. For each $t \in \mathbb{R}_+$ there exists $A_t \in \mathscr{F}_t$ such that $A_t = \{T < t\}$ a.s. (see 1.4). Then $T'(\omega) = \inf(s \in \mathbb{Q}_+ : \omega \in A_s)$ is an $\mathbf{F}$-stopping time (because $\{T' < t\} = \bigcup_{s \in \mathbb{Q}_+, s<t} A_s$ is in $\mathscr{F}_t$) and $T' = T$ a.s. (because $\{T < t\} = \bigcup_{s \in \mathbb{Q}_+, s<t} \{T < s\}$ is a. s. equal to $\{T' < t\}$, for all $t \in \mathbb{R}_+$). □

§ 1c. The Optional σ-Field

Here again, the stochastic basis $(\Omega, \mathscr{F}, \mathbf{F}, P)$ is fixed.

1.20 **Definition.** a) A process X is *adapted to the filtration* $\mathbf{F}$ (or, in short, *adapted*) if X_t is $\mathscr{F}_t$-measurable for every $t \in \mathbb{R}_+$.

b) The *optional σ-field* is the σ-field $\mathscr{O}$ on $\Omega \times \mathbb{R}_+$ that is generated by all càdlàg adapted processes (considered as mappings on $\Omega \times \mathbb{R}_+$). □

A process or a random set that is $\mathscr{O}$-measurable is called *optional.*

1.21 **Proposition.** *Let X be an optional process. When considered as a mapping on $\Omega \times \mathbb{R}_+$, it is $\mathscr{F} \otimes \mathscr{R}_+$-measurable. Moreover, if T is a stopping time, then*

a) $X_T 1_{\{T<\infty\}}$ *is $\mathscr{F}_T$-measurable* (hence, X is adapted).

b) *the stopped process X^T is also optional.*

Proof. The set of all processes that are $\mathscr{F} \otimes \mathscr{R}_+$-measurable and meet (a) and (b) for all stopping times is obviously a vector lattice and is stable under pointwise convergence. Thus, by Definition 1.20 and a monotone class argument, it is enough to prove that every càdlàg adapted process X satisfies the claimed properties.

If $n \in \mathbb{N}^*$ we define a new process X^n by putting $X^n_t = X_{k/2^n}$ for $t \in [(k-1)/2^n, k/2^n)$, where $k \in \mathbb{N}^*$. Since

$$\{X^n \in B\} = \bigcup_{k \in \mathbb{N}^*} \left[\{\omega: X_{k/2^n}(\omega) \in B\} \times \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) \right],$$

we have $\{X^n \in B\} \in \mathscr{F} \otimes \mathscr{R}_+$ for all Borel sets B, hence X^n is $\mathscr{F} \otimes \mathscr{R}_+$-measurable. Since X is càd, the sequence (X^n) converges pointwise to X, which therefore is also $\mathscr{F} \otimes \mathscr{R}_+$-measurable.

Let T be a stopping time, and put $T_n = \infty$ on the set $\{T = \infty\}$ and $T_n = k/2^n$ on the set $\{(k-1)/2^n \le T < k/2^n\}$. Each T_n is obviously a stopping time, and the sequence (T_n) decreases to T. Since

$$\{X_{T_n} \in B\} \cap \{T_n \le t\} = \bigcup_{k \in \mathbb{N}^*, k/2^n \le t} [\{X_{k/2^n} \in B\} \cap \{T_n = k/2^n\}]$$

is in $\mathscr{F}_t$, we obtain that $X_{T_n} 1_{\{T_n < \infty\}}$ is $\mathscr{F}_{T_n}$-measurable. Since X is càd, $X_{T_n} 1_{\{T_n < \infty\}}$ converges to $X_T 1_{\{T < \infty\}}$. Thus it follows from 1.18 that $X_T 1_{\{T < \infty\}}$ is $\mathscr{F}_T$-measurable and we have (a). Finally X^T is also càdlàg by construction, and since $X_t^T = X_t 1_{\{t < T\}} + X_T 1_{\{T \le t\}}$ it follows from what precedes that X^T is adapted: therefore it is optional, and we have (b). □

There exists a characterization of the optional σ-field that differs from the definition 1.20 and gives some insight for this notion. To this effect, let us first introduce what a stochastic interval is: if S, T are two stopping times, one may define four kinds of *stochastic intervals*, that are the following four random sets:

1.22
$$\begin{cases} [\![S, T]\!] = \{(\omega, t) : t \in \mathbb{R}_+, S(\omega) \le t \le T(\omega)\} \\ [\![S, T[\![= \{(\omega, t) : t \in \mathbb{R}_+, S(\omega) \le t < T(\omega)\} \\ ]\!]S, T]\!] = \{(\omega, t) : t \in \mathbb{R}_+, S(\omega) < t \le T(\omega)\} \\ ]\!]S, T[\![= \{(\omega, t) : t \in \mathbb{R}_+, S(\omega) < t < T(\omega)\}. \end{cases}$$

Instead of $[\![T, T]\!]$, we write $[\![T]\!]$: that is, $[\![T]\!]$ is the restriction of the graph of the mapping $T: \Omega \to \bar{\mathbb{R}}_+$, to the set $\Omega \times \mathbb{R}_+$, and we abuse the terminology by calling $[\![T]\!]$ *the graph of the stopping time T*.

The process $1_{[\![0, T[\![}$ is càdlàg, and is obviously adapted if and only if T is a stopping time; then by 1.20 we have $[\![0, T[\![\in \mathscr{O}$ for each stopping time T. More generally:

1.23 **Proposition.** *If S, T are two stopping times and if Y is an $\mathscr{F}_S$-measurable random variable, the four processes $Y 1_{[\![S,T]\!]}$, $Y 1_{[\![S,T[\![}$, $Y 1_{]\!]S,T]\!]}$, $Y 1_{]\!]S,T[\![}$ are optional.*

Proof. It is enough to prove the result when Y is the indicator function of a set $A \in \mathscr{F}_S$. Let us consider for example $X = 1_A 1_{]\!]S,T]\!]}$. Then X is the pointwise limit of $X^n = 1_A 1_{[\![S_n, T_n[\![}$ where $S_n = S + 1/n$ and $T_n = T + 1/n$. X^n is càdlàg by construction and, using 1.17 and the fact that $A \in \mathscr{F}_S \subset \mathscr{F}_{S_n}$, we check that X^n is adapted: hence X^n is optional, and thus so is X. The proof for the other kinds of stochastic intervals is the same. □

1.24 **Proposition.** *Every process X that is càg and adapted is optional.*

Proof. For each $n \in \mathbb{N}^*$ define a new process X^n by

$$X^n = \sum_{k \in \mathbb{N}} X_{k/2^n} 1_{[\![k/2^n, (k+1)/2^n[\![}.$$

Proposition 1.23 yields that X^n is optional. Since X is càg, the sequence (X^n) converges pointwise to X, hence X is optional. □

If X is a càdlàg adapted process, it is obvious that X_- is also adapted; hence it follows from 1.24 that:

1.25 **Corollary.** *If X is a càdlàg adapted process, the two processes X_- and ΔX are optional* (recall that $\Delta X = X - X_-$).

1.26 **Remark.** One may also prove the following, stronger, results, which will not be used in this book:

(a) Any càd adapted process is optional;

(b) the σ-field $\mathscr{O}$ is generated by the stochastic intervals $[\![0, T[\![$, where T is any stopping time. □

Next we study hitting times. Firstly, we have a fairly general (and difficult) result, due to Hunt; although it will not be used in this book, we recall it (without proof: see e.g. [33]) because of its theoretical importance.

1.27 **Theorem.** *If A is an optional random set, its début $T(\omega) = \inf(t: (\omega, t) \in A)$ is a stopping time relative to the completed filtration $\mathbf{F}^P$ introduced in* 1.4 (or, equivalently, is a. s. equal to a stopping time of the original filtration $\mathbf{F}$, by Lemma 1.19).

In particular if X is an $\mathbb{R}^d$-valued optional process and if B is a Borel subset of $\mathbb{R}^d$, then $T = \inf(t: X_t \in B)$ is a stopping time of the completed filtration $\mathbf{F}^P$ (apply 1.27 to the optional random set $A = \{X \in B\}$).

As for us, instead of using the full force of this result, we shall only use a very particular and easy case, namely

1.28 **Proposition.** a) *If X is an $\mathbb{R}^d$-valued adapted càd process and if B is an open subset of $\mathbb{R}^d$, then $T = \inf(t: X_t \in B)$ is a stopping time.*

b) *If X is an $\mathbb{R}$-valued adapted càd process with nondecreasing paths and if $a \in \bar{\mathbb{R}}$, then $T = \inf(t: X_t \ge a)$ is a stopping time.*

(Here there is no need to complete the filtration, unlike in 1.27).

Proof. a) Since B is open and X is càd, we have

$$\{T < t\} = \bigcup_{s \in \mathbb{Q}_+, s < t} \{X_s \in B\}$$

and since X is adapted, the right-hand side above is in $\mathscr{F}_t$, so the result follows from 1.16.

b) When X is non-decreasing and càd, then $\{T \le t\} = \{X_t \ge a\}$, which belongs to $\mathscr{F}_t$ because X is adapted: hence the result. □

We end this paragraph by some easy results on the structure of the jumps of a càdlàg adapted process.

1.30 **Definition.** A random set A is called *thin* if it is of the form $A = \bigcup [\![T_n]\!]$, where (T_n) is a sequence of stopping times; if moreover the sequence (T_n) satisfies $[\![T_n]\!] \cap [\![T_m]\!] = \varnothing$ for all $n \neq m$, it is called an *exhausting sequence* for A. □

Of course a thin set is optional and all its sections $\{t: (\omega, t) \in A\}$ are at most countable; conversely one may prove that any optional set whose sections are at most countable is thin in the sense of 1.30: this is a difficult result, that will not be used here (see [33]).

1.31 **Lemma.** *Any thin random set admits an exhausting sequence of stopping times.*

Proof. Let $A = \bigcup [\![T_n]\!]$, where $(T_n)_{n \in \mathbb{N}}$ is a sequence of stopping times. The set $C_n = \bigcap_{0 \leq m \leq n-1} \{T_m \neq T_n\}$ is in $\mathscr{F}_{T_n}$ by 1.17, hence 1.15 implies that $S_n = (T_n)_{C_n}$ is a stopping time: the sequence (S_n) is thus an exhausting sequence for A. □

1.32 **Proposition.** *If X is a càdlàg adapted process, the random set $\{\Delta X \neq 0\}$ is thin*; an exhausting sequence for this set $\{\Delta X \neq 0\}$ is called a sequence that *exhausts the jumps* of X.

Proof. Let $n \in \mathbb{N}^*$. Put $S(n,0) = 0$ and define by induction

$$S(n, p+1) = \inf(t > S(n,p): |X_t - X_{S(n,p)}| > 2^{-n}).$$

Then for n, p fixed we have $S(n, p+1) = \inf(t: |Y_t| > 2^{-n})$, where

$$Y = (X - X_{S(n,p)}) 1_{[\![S(n,p),\infty[\![}$$

and Y is a càdlàg adapted process (use 1.23). Hence we deduce from 1.28 that $S(n,p)$ is a stopping time. Moreover 1.21 and 1.25 yield that $A(n,p) = \{S(n,p) < \infty, \Delta X_{S(n,p)} \neq 0\}$ is in $\mathscr{F}_{S(n,p)}$, hence by 1.15 each $T(n,p) = S(n,p)_{A(n,p)}$ is also a stopping time. Now, since X is càdlàg we obviously have: $\lim_{p \uparrow \infty} \uparrow S(n,p) = \infty$ and it easily follows that $\{\Delta X \neq 0\} = \bigcup_{n,p \in \mathbb{N}^*} [\![T(n,p)]\!]$, hence the result. □

§ 1d. The Localization Procedure

In this short subsection, we describe a procedure that is used over and over.

1.33 **Definition.** If $\mathscr{C}$ is a class of processes, we denote by $\mathscr{C}_{\text{loc}}$ the *localized class*, defined as such: a process X belongs to $\mathscr{C}_{\text{loc}}$ if and only if there exists an increasing sequence (T_n) of stopping times (depending on X) such that $\lim_{(n)} T_n = \infty$ a. s. and that each stopped process X^{T_n} belongs to $\mathscr{C}$. The sequence (T_n) is called a *localizing sequence* for X (relative to $\mathscr{C}$). □

For instance, if $\mathscr{C}$ is the class of all *bounded processes*, $\mathscr{C}_{\text{loc}}$ is the class of the so-called *locally bounded processes*. If we may anticipate on the next paragraph, if $\mathscr{C}$ is the class of all submartingales, $\mathscr{C}_{\text{loc}}$ will be the class of the so-called *local* submartingales.

Of course, $\mathscr{C} \subset \mathscr{C}_{\text{loc}}$. The localization is most useful for the classes that satisfy the following property (*all* the classes of processes encountered in this book will satisfy the next property!)

1.34 **Definition.** A class $\mathscr{C}$ of processes is called *stable under stopping* if for any $X \in \mathscr{C}$ and any stopping time T, the stopped proced X^T belongs to $\mathscr{C}$. □

1.35 **Lemma.** *Let $\mathscr{C}$ and $\mathscr{C}'$ be two classes of processes, that are stable under stopping. Then*

(a) *$\mathscr{C}_{\text{loc}}$ is stable under stopping, and* $(\mathscr{C}_{\text{loc}})_{\text{loc}} = \mathscr{C}_{\text{loc}}$.

(b) $(\mathscr{C} \cap \mathscr{C}')_{\text{loc}} = \mathscr{C}_{\text{loc}} \cap \mathscr{C}'_{\text{loc}}$.

Proof. (a) That $\mathscr{C}_{\text{loc}}$ is stable under stopping is trivial. Let $X \in (\mathscr{C}_{\text{loc}})_{\text{loc}}$, and (T_n) be a localizing sequence such that $X^{T_n} \in \mathscr{C}_{\text{loc}}$. For each $n \in \mathbb{N}$ there exists a localizing sequence $(T(n,p))_{p \in \mathbb{N}}$ such that $(X^{T_n})^{T(n,p)} \in \mathscr{C}$, and there exists an integer p_n such that $P(T(n,p_n) < T_n \wedge n) \leq 2^{-n}$.

Put $S_n = T_n \wedge [\bigwedge_{m \geq n} T(m, p_n)]$. Each S_n is a stopping time and since the sequence (T_n) is increasing, then so is the sequence (S_n). One has:

$$P(S_n < T_n \wedge n) \leq \sum_{m \geq n} P(T(m, p_m) < T_n \wedge n)$$
$$\leq \sum_{m \geq n} P(T(m, p_m) < T_m \wedge m) \leq \sum_{m \geq n} 2^{-m} = 2^{-(n-1)}.$$

Because $\lim_{(n)} T_n = \infty$ a. s., it follows that $\lim_{(n)} S_n = \infty$ a. s., and (S_n) is a localizing sequence. Now,

$$X^{S_n} = ((X^{T_n})^{T(n,p_n)})^{S_n}$$

and, since $\mathscr{C}$ is stable under stopping, it follows that $X^{S_n} \in \mathscr{C}$. Hence $X \in \mathscr{C}_{\text{loc}}$.

(b) The inclusion $(\mathscr{C} \cap \mathscr{C}')_{\text{loc}} \subset \mathscr{C}_{\text{loc}} \cap \mathscr{C}'_{\text{loc}}$ is trivial. Conversely, let $X \in \mathscr{C}_{\text{loc}} \cap \mathscr{C}'_{\text{loc}}$, and let (T_n) and (T'_n) be two localizing sequences such that $X^{T_n} \in \mathscr{C}$ and that $X^{T'_n} \in \mathscr{C}'$. Put $S_n = T_n \wedge T'_n$. The sequence (S_n) is increasing and $\lim_{(n)} S_n = \infty$ a. s.; since $\mathscr{C}$ and $\mathscr{C}'$ are stable under stopping, $X^{S_n} = (X^{T_n})^{T'_n} \in \mathscr{C}$ and similarly $X^{S_n} \in \mathscr{C}'$. Therefore $X \in (\mathscr{C} \cap \mathscr{C}')_{\text{loc}}$. □

The first property above means that one cannot iterate the localization procedure and obtain larger and larger classes of processes. Typically, the previous lemma is set to work in the following sort of situation:

"Theorem". Let $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ be three classes that are stable under stopping; associate to each $X \in \mathscr{C}_{\text{loc}} \cap \mathscr{C}'_{\text{loc}}$ another process $Y = \alpha(X)$, with the property that $\alpha(X^T) =$

$(\alpha(X))^T$ for every stopping time. Then if $\alpha(X)\in\mathscr{C}''_{\text{loc}}$ for all $X\in\mathscr{C}\cap\mathscr{C}'$, we also have $\alpha(X)\in\mathscr{C}''_{\text{loc}}$ for all $X\in\mathscr{C}_{\text{loc}}\cap\mathscr{C}'_{\text{loc}}$.

"Method of Proof": Apply 1.35. In a "real" proof, when we encounter a situation of this type we write the ritual sentence: *by localization, we may assume that* $X\in\mathscr{C}\cap\mathscr{C}'$.

§ 1e. Martingales

In this subsection we review a number of properties of martingales, submartingales and supermartingales, that are essentially due to Doob. They are stated without proof (except for the final property), the proofs may be found in most standard books (see e.g. [33], [43]).

1.36 **Definition.** A *martingale* (resp. *submartingale*, resp. *supermartingale*) is an *adapted* process X on the basis $(\Omega,\mathscr{F},\mathbf{F},P)$, whose P-almost all paths are *càdlàg*, such that every X_t is integrable, and that for $s\le t$:

$$X_s=E(X_t|\mathscr{F}_s)\quad(\text{resp. } X_s\le E(X_t|\mathscr{F}_s),\text{ resp. } X_s\ge E(X_t|\mathscr{F}_s)).\qquad\square$$

1.37 **Remark.** We somehow depart from standard conventions in this definition; namely, the stochastic basis is *not* assumed to be complete. Nevertheless, the subsequent properties are true, as the reader will check by himself (it is very easy), thanks to the following: if X is a submartingale on the complete basis $(\Omega,\mathscr{F}^P,\mathbf{F}^P,P)$, there exists a process X', P-indistinguishable from X, adapted to the (uncomplete) filtration $\mathbf{F}$, and also an $\mathbf{F}$-stopping time T, such that for all ω the path $X_\cdot(\omega)$ is càd everywhere and làd everywhere except at $T(\omega)$, and moreover $P(T<\infty)=0$. □

1.38 We say that a process X admits a *terminal variable* X_∞ if X_t converges a. s. to a limit X_∞ as $t\uparrow\infty$; in such a case, the variable X_T is (a. s.) well defined for any stopping time T, with $X_T=X_\infty$ on $\{T=\infty\}$. □

1.39 **Theorem.** *Let X be a supermartingale such that there exists an integrable random variable Y with $X_t\ge E(Y|\mathscr{F}_t)$ for all $t\in\mathbb{R}_+$. Then*

a) (Doob's limit Theorem) *X_t converges a. s. to a finite limit X_∞.*

b) (Doob's stopping Theorem) *If S, T are two stopping times, the random variables X_S and X_T are integrable, and $X_S\ge E(X_T|\mathscr{F}_S)$ on the set $\{S\le T\}$. In particular, X^T is again a supermartingale.*

Now we introduce the two following classes of martingales:

1.40 **Definition.** We denote by $\mathscr{M}$ the class of all *uniformly integrable martingales*, that is of all martingales X such that the family of random variables $(X_t)_{t\in\mathbb{R}_+}$ is uniformly integrable. □

1.41 **Definition.** We denote by $\mathscr{H}^2$ the class of all *square-integrable martingales*, that is of all martingales X such that $\sup_{t\in\mathbb{R}_+}E(X_t^2)<\infty$. □

We obviously have $\mathscr{H}^2\subset\mathscr{M}$. The following theorem will imply that both $\mathscr{M}$ and $\mathscr{H}^2$ are *stable under stopping*.

1.42 **Theorem.** a) *If X is a uniformly integrable martingale, then X_t converges a. s. and in L^1 to a terminal variable X_∞, and $X_T=E(X_\infty|\mathscr{F}_T)$ for all stopping times T. Moreover, X is square-integrable if and only if X_∞ is square-integrable, in which case the convergence $X_t\to X_\infty$ also takes place in L^2.*

b) *If Y is an integrable random variable, there exists a uniformly integrable martingale X, and only one up to an evanescent set, such that $X_t=E(Y|\mathscr{F}_t)$ for all $t\in\mathbb{R}_+$; moreover, $X_\infty=E(Y|\mathscr{F}_{\infty-})$.*

(Observe that no completion of the filtration is needed here).

1.43 **Theorem** (Doob's inequality). *If X is a square-integrable martingale,*

$$E\left(\sup_{t\in\mathbb{R}_+}X_t^2\right)\le 4\sup_{t\in\mathbb{R}_+}E(X_t^2)=4E(X_\infty^2).$$

Here is another, very useful, characterization of the elements of $\mathscr{M}$:

1.44 **Lemma.** *Let X be an adapted càdlàg process, with a terminal random variable X_∞. Then X is a uniformly integrable martingale if and only if for each stopping time T, the variable X_T is integrable and satisfies $E(X_T)=E(X_0)$.*

Proof. The necessary condition comes immediately from 1.42. To prove the sufficient condition, we remark first that X_∞ is integrable by hypothesis. Then if $t\in\mathbb{R}_+$ and $A\in\mathscr{F}_t$, we define the stopping time T by $T=t$ on A and $T=\infty$ on the complement A^c. We have $E(X_T)=E(X_t1_A)+E(X_\infty 1_{A^c})$ and $E(X_\infty)=E(X_\infty 1_A)+E(X_\infty 1_{A^c})$. Our assumption implies that $E(X_T)=E(X_\infty)$, hence $E(X_t1_A)=E(X_\infty 1_A)$ by difference. This being true for all $A\in\mathscr{F}_t$, it follows that $X_t=E(X_\infty|\mathscr{F}_t)$. Then one easily deduce that $X\in\mathscr{M}$ from 1.42. □

1.45 **Definition.** A *local martingale* (resp. a *locally square-integrable martingale*) is a process that belongs to the localized class $\mathscr{M}_{\text{loc}}$ (resp. $\mathscr{H}^2_{\text{loc}}$) constructed from $\mathscr{M}$ (resp. $\mathscr{H}^2$) via 1.33. □

1.46 **Definition.** A process X is *of class* (D) if the set of random variables $\{X_T$: T finite-valued stopping time$\}$ is uniformly integrable. □

1.47 **Proposition.** a) *Each martingale is a local martingale* (hence, $\mathscr{M}_{\text{loc}}$ is also the localized class obtained via 1.33 from the class of martingales).

b) *Each uniformly integrable martingale is a process of class* (D).

c) *A local martingale is a uniformly integrable martingale if and only if it is a process of class* (D).

Proof. a) Let X be a martingale, and put $T_n = n$. Then $X_t^{T_n} = E(X_n|\mathscr{F}_t)$ for all $t \in \mathbb{R}_+$ and 1.42 implies that $X^{T_n} \in \mathscr{M}$.

b) The statement follows from 1.42 and from the well-known fact that if $Y \in L^1$, the set of random variables $\{E(Y|\mathscr{G}): \mathscr{G}$ any sub-σ-field of $\mathscr{F}\}$ is uniformly integrable.

c) Only the sufficient condition remains to be proved. Let $X \in \mathscr{M}_{\text{loc}}$ be of class (D), and let (T_n) be a localizing sequence for X. If $s \le t$,

$$(1) \qquad X_{s \wedge T_n} = X_s^{T_n} = E(X_t^{T_n}|\mathscr{F}_s) = E(X_{t \wedge T_n}|\mathscr{F}_s).$$

The two sequences $(X_{s \wedge T_n})_{n \in \mathbb{N}}$ and $(X_{t \wedge T_n})_{n \in \mathbb{N}}$ are uniformly integrable because X is of class (D), and they converge to X_s and X_t a.s. respectively because $\lim_{(n)} T_n = \infty$ a.s. Hence the convergence is also in L^1, and so it passes through the conditional expectation in (1), which thus yields $X_s = E(X_t|\mathscr{F}_s)$ and X is a martingale. At last, since X is of class (D), it is *a*-fortiori uniformly integrable. □

We end this paragraph by showing, through two examples, the differences between a uniformly integrable martingale, a martingale, and a local martingale.

1.48 **Example.** Let $(Z_n)_{n \in \mathbb{N}^*}$ be a sequence of i.i.d. random variables with $P(Z_n = 1) = P(Z_n = -1) = 1/2$. Put $\mathscr{F}_t = \sigma(Z_p: p \in \mathbb{N}^*, p \le t)$ and $X_t = \sum_{1 \le p \le [t]} Z_p$, where $[t]$ denotes the integer part of $t \in \mathbb{R}_+$. Then X is trivially a martingale, but by the central limit theorem X_t does not converge a.s. as $t \uparrow \infty$: hence X is not uniformly integrable. □

1.49 **Example.** Let $(A_n)_{n \in \mathbb{N}^*}$ be a measurable partition with $P(A_n) = 2^{-n}$ and $(Z_n)_{n \in \mathbb{N}^*}$ be a sequence of random variables that are independent of the A_n's and with $P(Z_n = 2^n) = P(Z_n = -2^n) = 1/2$. Put $\mathscr{F}_t = \sigma(A_n: n \in \mathbb{N}^*)$ if $t \in [0, 1)$ and $\mathscr{F}_t = \sigma(A_n, Z_n: n \in \mathbb{N}^*)$ if $t \in [1, \infty)$. Put

$$Y_n = \sum_{1 \le p \le n} Z_p 1_{A_p}$$

$$X_t = \begin{cases} 0 & \text{if } t \in [0, 1) \\ Y_\infty & \text{if } t \in [1, \infty) \end{cases}$$

$$T_n = \begin{cases} +\infty & \text{on the set } \bigcup_{1 \le p \le n} A_p \\ 0 & \text{elsewhere.} \end{cases}$$

(T_n) is clearly a sequence of stopping times that increases to $+\infty$. The process X^{T_n} is equal to 0 (resp. Y_n) on $[\![0, 1[\![$ (resp. $[\![1, \infty[\![$) and Y_n is bounded and independent from the σ-field $\mathscr{F}_{1-}$: hence $X^{T_n} \in \mathscr{M}$ and X is a local martingale. However, it is not a martingale, because $X_1 = Y_\infty$ is not integrable. □

§ 1f. The Discrete Case

When the time-set is not $\mathbb{R}_+$ but $\mathbb{N}$, we have a theory that is similar to the previous one, although much simpler. We will very briefly sketch this theory and show how it is connected to the "continuous-time" one.

1. Let us first define what a stochastic basis is, in this setting.

1.50 **Definition.** A *discrete stochastic basis* is a *probability space* $(\Omega, \mathscr{F}, P)$ equipped with a *filtration* $\mathbf{F} = (\mathscr{F}_n)_{n \in \mathbb{N}}$; here, a *filtration* means an *increasing* family of sub-σ-fields of $\mathscr{F}$ (i.e. $\mathscr{F}_n \subset \mathscr{F}_m$ if $n \le m$). Notice that the right-continuity has no meaning here. □

A *random set* is a subset of $\Omega \times \mathbb{N}$. A *process* is a family $X = (X_n)_{n \in \mathbb{N}}$ of mappings from Ω into some set E, and it can also be viewed as a mapping from $\Omega \times \mathbb{N}$ into E, via:

$$(\omega, n) \rightsquigarrow X(\omega, n) = X_n(\omega).$$

The notions of càd, càg, or càdlàg processes, have no signification here. However, analoguously to 1.8, to each process X we associate two other processes $X_- = (X_{n-})$ and $\Delta X = (\Delta X_n)$ by

$$1.51 \qquad \begin{cases} X_{0-} = X_0, \quad X_{n-} = X_{n-1} & \text{if } n \ge 1 \\ \Delta X_n = X_n - X_{n-} \end{cases}$$

If X is a process, and T a mapping: $\Omega \to \bar{\mathbb{N}}$, we define the process X^T "stopped at time T" by $X_n^T = X_{T \wedge n}$.

A *stopping time* T and its associated σ-fields $\mathscr{F}_T$ and $\mathscr{F}_{T-}$ are defined exactly like in 1.11, except that here T is a mapping: $\Omega \to \bar{\mathbb{N}}$ and that $\mathbb{R}_+$ is replaced by $\mathbb{N}$. The properties 1.13 to 1.18 are of course valid. Moreover, we have the obvious and useful additional property:

1.52 A mapping $T: \Omega \to \bar{\mathbb{N}}$ is a stopping time if and only if $\{T = n\} \in \mathscr{F}_n$ for each $n \in \mathbb{N}$; in this case, a set $A \in \mathscr{F}$ belongs to $\mathscr{F}_T$ if and only if $A \cap \{T = n\} \in \mathscr{F}_n$ for each $n \in \mathbb{N}$. □

The notion of optionality is rather trivial here:

1.53 **Definition.** The *optional σ-field* is the σ-field $\mathscr{O}$ on $\Omega \times \mathbb{N}$ that is generated by all *adapted* processes, that is all processes X such that X_n is $\mathscr{F}_n$-measurable for each $n \in \mathbb{N}$. □

Most of the results of § c have no interesting counterpart in the discrete case. Let us however mention 1.21 (which is much easier to prove than in the continuous case), 1.25 that is trivial, and 1.27 that is very easy to prove here:

1.54 **Theorem.** *If A is an optional random set, its début* $T(\omega) = \inf(n \in \mathbb{N}$: $(\omega, n) \in A)$ *is a stopping time* (no completion required here).

Proof. The hypothesis means that the process $X = 1_A$ is adapted; hence the result follows from the equality:

$$\{T \leq n\} = \bigcup_{0 \leq p \leq n} \{X_p = 1\}. \qquad \square$$

Finally, the notion of localization, and all the definitions and theorems of § e about martingales are valid without changes (except that $\mathbb{R}_+$ and $\bar{\mathbb{R}}_+$ are everywhere replaced by $\mathbb{N}$ and $\bar{\mathbb{N}}$, and of course we can drop the "càdlàg" assumption in Definition 1.36). Note that even in the discrete case we have the three notions of a uniformly integrable martingale, of a martingale, and of a local martingale, and the examples 1.48 and 1.49 may easily be translated into the discrete case to show that a martingale may not be uniformly integrable, or that a local martingale may not be a martingale (see also 1.64 below).

2. Now we wish to show that the discrete case actually reduces to a particular case of the general one. To this effect, we consider a *discrete stochastic basis* $\mathscr{B} = (\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_n)_{n \in \mathbb{N}}, P)$.

We associate to $\mathscr{B}$ a "continuous" stochastic basis $\mathscr{B}'$ as follows:

1.55 $\mathscr{B}' = (\Omega, \mathscr{F}, \mathbf{F}' = (\mathscr{F}'_t)_{t \in \mathbb{R}_+}, P)$, with $\mathscr{F}'_t = \mathscr{F}_n$ for $t \in [n, n+1)$.

In particular, we have:

1.56 $\mathscr{F}'_n = \mathscr{F}_n$ if $n \in \mathbb{N}$, $\mathscr{F}'_{n-} = \mathscr{F}'_{n-1} = \mathscr{F}_{n-1}$ if $n \in \mathbb{N}^*$.

1.57 **Lemma.** *Any $\mathscr{B}$-stopping time T is also a $\mathscr{B}'$-stopping time, and we have $\mathscr{F}'_T = \mathscr{F}_T$ and $\mathscr{F}'_{T-} = \mathscr{F}_{T-}$.*

Proof. For each $A \in \mathscr{F}$ we have $A \cap \{T \leq t\} = A \cap \{T \leq n\}$ when $t \in [n, n+1)$. Hence by 1.56, $A \cap \{T \leq t\} \in \mathscr{F}'_t$ for all $t \in \mathbb{R}_+$ if and only if $A \cap \{T \leq n\} \in \mathscr{F}_n$ for all $n \in \mathbb{N}$: this proves that a *$\mathscr{B}$-stopping time T* is a $\mathscr{B}'$-stopping time and that $\mathscr{F}'_T = \mathscr{F}_T$.

If $A \in \mathscr{F}'_t$ and $t \in [n, n+1)$, we have $A \in \mathscr{F}_n$ and $A \cap \{t < T\} = A \cap \{n < T\}$; since $\mathscr{F}'_0 = \mathscr{F}_0$, it proves that $\mathscr{F}'_{T-} \subset \mathscr{F}_{T-}$. The converse inclusion is proved similarly. $\square$

1.58 **Lemma.** *Let T' be a $\mathscr{B}'$-stopping time, and put $T = n$ if $n \leq T' < n+1$, $T = \infty$ if $T' = \infty$. Then T is a $\mathscr{B}$-stopping time and $\mathscr{F}'_{T'} = \mathscr{F}_T$, and $\mathscr{F}'_{T'-} \subset \mathscr{F}_{T-}$* (note that in general we do not have $\mathscr{F}'_{T'-} = \mathscr{F}_{T-}$; the proof, which is similar to that of 1.57, is left to the reader).

Now, let X be a process on $\mathscr{B}$. We associate to it a process X' on $\mathscr{B}'$ as follows:

1.59 $X'_t = X_n$ if $t \in [n, n+1)$.

Note that X' is càdlàg, and the following statements are obvious:

1.60 X is $\mathscr{B}$-adapted if and only if X' is $\mathscr{B}'$-adapted.

1.61 If T and T' are like in 1.58, we have $(X^T)' = X'^{T'}$.

1.62 The process X is of class (D) on $\mathscr{B}$ if and only if X' is of class (D) on $\mathscr{B}'$.

1.63 The process X is a martingale (resp. a supermartingale, resp. a uniformly integrable martingale, resp. a local martingale) on $\mathscr{B}$ if and only if X' is a martingale (resp. a supermartingale, resp. a uniformly integrable martingale, resp. a local martingale) on $\mathscr{B}'$.

These facts show why the discrete case is indeed "included" within the continuous one. For instance, 1.60 and the property of the process X' given by 1.59 to be automatically càdlàg explain why "optional" and "adapted" mean the same thing on the basis $\mathscr{B}'$.

3. Here is an exception to what we just wrote above: the following does not easily reduce to a property in continuous time.

1.64 **Proposition.** *Let X be an adapted process on $\mathscr{B}$. Then X is a local martingale, if and only if:*

(i) *X_0 is integrable, and*

(ii) *for all $n \in \mathbb{N}^*$, $E(|X_n| | \mathscr{F}_{n-1}) < \infty$ a.s. and $E(X_n | \mathscr{F}_{n-1}) = X_{n-1}$.*

Remember that $E(\cdot | \mathscr{F}_{n-1})$ is the "extended" conditional expectation. So in (ii) the fact that $E(X_n | \mathscr{F}_{n-1}) = X_{n-1}$ automatically implies that $E(|X_n| | \mathscr{F}_{n-1}) < \infty$ (for clarity, we prefer to explicitely state the two conditions). But $E(X_n | \mathscr{F}_{n-1}) = X_{n-1}$ does *not* imply that X_n is integrable; indeed, integrability for all X_n plus $E(X_n | \mathscr{F}_{n-1}) = X_{n-1}$ for $n \in \mathbb{N}^*$ are necessary and sufficient for X to be a *martingale*.

Proof. a) *Necessary condition*: Let (T_n) be a localizing sequence of stopping times for the local martingale X. Then $X_0 = X_0^{T_n}$ is integrable, and $E(X_p^{T_n} | \mathscr{F}_{p-1}) = X_{p-1}^{T_n}$ for all $p \in \mathbb{N}^*$: therefore $E(X_p | \mathscr{F}_{p-1}) = X_{p-1}$ on the $\mathscr{F}_{p-1}$-measurable set $\{T_n > p-1\}$, and since $\bigcup_n \{T_n > p-1\} = \Omega$ we obtain (ii).

b) *Sufficient condition*: Assume (i) and (ii) and set $T_n = \inf(p: \sum_{1 \leq k \leq p+1} E(|X_k| | \mathscr{F}_{k-1}) \geq n)$. Then $\{T_n \geq p\}$ clearly belongs to $\mathscr{F}_{p-1}$, hence T_n is a stopping time. Moreover,

$$E(|X_p^{T_n}|) = E(|X_{T_n \wedge p}|) \leq E(|X_0|) + n < \infty$$

and (ii) yields $E(X_p^{T_n} | \mathscr{F}_{p-1}) = X_{p-1}^{T_n}$ because $\{T_n \geq p\} \in \mathscr{F}_{p-1}$. Hence X^{T_n} is a martingale, and X is a local martingale because $T_n \uparrow \infty$ as $n \uparrow \infty$ by (ii) again. $\square$