Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

Subseries: Nankai Institute of Mathematics, Tianjin, P.R. China
vol. 6
Adviser: S.S. Chern, B.-j. Jiang

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The Topology of 4-Manifolds

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo Hong Kong

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Mathematics Subject Classification (1980): 57M99, 57N13

ISBN 3-540-51148-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-51148-2 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

PREFACE

In the late 1970's, Mike Freedman and I sketched an argument using immersion theory for showing that $\Omega_4^{SO} = Z$. In 1982–83, Iain Aitchison and I worked out new proofs and a reorganization of $\Omega_4^{\mathrm{spin}} = Z$, $p_1 = 3\sigma$, and Rohlin's theorem. In the last 5 years, further simplifications including a yet easier proof of $\Omega_4^{\mathrm{spin}} = Z$ have been found.

A first draft of Chapters XII and XIII was written at IMPA in Rio de Janeiro in fall 1982 and other bits at the University of Maryland in spring 1983, but the bulk of the writing was done at S.-S. Chern's suggestion at the Nankai Institute of Mathematics in May 1987. I was very ably assisted by Bao–zhen Yu, who found some gaps and corrected many errors, not all minor. I am indebted to Charles Livingston and the topology seminar at Indiana who found further gaffes in Fall 1987, and to Berkeley students, particularly Chris Herald, for checking the final version.

Recent work with Larry Taylor on Pin structures and non-orientable generalizations of Rohlin's Theorem has fed back into some further sharpenings of Chapter IV and the proof of Rohlin's Theorem.

Thanks to my collaborators, to IMPA, Maryland, and especially Nankai for their warm hospitality, to Faye Yeager for an excellent TeX manuscript, and to Deb Craig for help with the many figures.

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INTRODUCTION

When I began to think about 4-manifolds in 1973, the basic theorems included the Whitehead–Milnor theorem on homotopy type [**Wh**], [**Milnor1**], Rohlin's Theorem [**Rohlin**], $\Omega_4^{SO} = Z$, $\Omega_4^{\text{spin}} = Z$, the Hirzebruch index theorem $p_1 = 3\sigma$, and Wall's theorems on diffeomorphisms and h-cobordism [**Wall1**] and [**Wall2**]. These theorems were untranslated ([**Rohlin**]) or unreadable ([**Wh**]), or were special cases of big machines in algebraic topology ($\Omega_4^{SO} = Z$, $\Omega_4^{\text{spin}} = Z$, $p_1 = 3\sigma$), or, even though accessible, could, with hindsight, use streamlining (Wall's theorems).

In the early 1970's, Casson and Rohlin independently gave geometric proofs of Rohlin's Theorem and improvements followed ([**F-K**] and Y. Matsumoto and Guillou and Marin in [**G-M**]). Rohlin's proof of $\Omega_4^{SO} = Z$ was translated [**G-M**] and lectured on by Morgan and others, with the latest version in [**Melvin**]. But a geometric, low dimensional proof of Ω_4^{spin} was missing. The algebraic topological proofs are powerful, and beautiful mathematics in their own right, but there ought to be proofs of the fundamental 4-manifold theorems which belong to the field of 4-dimensions (or less), and prepare the student in the geometric side of the theory.

We give a geometric proof of $\Omega_4^{SO} = Z$ starting with an immersion of M^4 into R^6; it is different but not necessarily better than the proofs mentioned in the previous paragraph. It's unique virtue was that Iain Aitchison and I were able to make it work for Ω_4^{spin}, but not without some difficulties. Recently, a simple proof of $\Omega_4^{\text{spin}} = Z$ turned up, which only uses the fact (not the method of proof) that M^4 bounds if $p_1(M) = 0$. This work led to an improved proof of Rohlin's theorem using spin structures. These proofs are first presented here. Handlebody theory is also exploited to streamline some proofs, e.g., Wall's theorems, and a few new wrinkles are included here and there.

Chapters XII–XIII give a sketch of Casson's and Freedman's work on topologic handles and 4-manifolds. These chapters might profitably be read as an introduction to Freedman's fundamental paper [**Freedman1**] or concurrently with Casson's 1974 notes in [**G-M**]. Chapter XIV contains constructions of exotic smooth structures on R^4, a countable number which do not imbed in S^4 and one that does imbed in S^4.

A reader needs a good, intuitive understanding of smooth manifolds and bundles, knowledge of the simplest form of the immersion theorem (perhaps best read in [**H-P**]), and a decent understanding of characteristic classes as applied to low dimensions using the obstruction theory definition [**M-S**, chapter 12].

Framed links are used as the basic way of describing 4-manifolds; Chapter I covers this material. Homotopy type, intersection forms, characteristic classes and the index fall in Chapter II. Chapter III states classification theorems as of July 1987.

Spin structures are tricky fellows, especially over S^1 and surfaces, and they are presented carefully, I hope, in Chapter IV, with a fundamental example in V. Chapters VI–IX focus on the proofs that $\Omega_4^{SO} = Z$, $\Omega_4^{\text{spin}} = Z$, and $p_1 = 3\sigma$, beginning with the study of immersions and singular sets in VI. The remaining chapter titles are self explanatory.

II.3.1 refers to Theorem or Lemma 1 in §3 of Chapter II; 3.1 refers to Theorem or Lemma 1 in §3 of the same chapter. Similarly with figures. □ marks the end of a proof.

I. HANDLEBODIES AND FRAMED LINKS

§1. Handlebodies.

A handlebody decomposition of a compact manifold M^m is a sequence $B^m = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M$ where M_i is obtained from M_{i-1} by adding a k_i-handle, that is, $M_i = M_{i-1} \underset{f_i}{\cup} B^{k_i} \times B^{m-k_i}$ where $f_i : \partial B^{k_i} \times B^{m-k_i} \to \partial M_{i-1}$ is an imbedding which is called the attaching map (Figure 1.1). $M_0 = B^m = B^0 \times B^m$ is a zero-handle and there may be others. Handlebody decompositions exist for the categories TOP, PL and DIFF except for the case of 4-dimensional topological manifolds which are handlebodies iff they are smoothable (see [**K-S**] and [**Quinn**]). We are only interested in the smooth case where f_i has to be a smooth imbedding. Then M_i has "corners" where the k_i-handle was attached (Figure 1.1), but the phrase "corners can be smoothed" has been a phrase that I have heard for 30 years, and this is not the place to explain it.

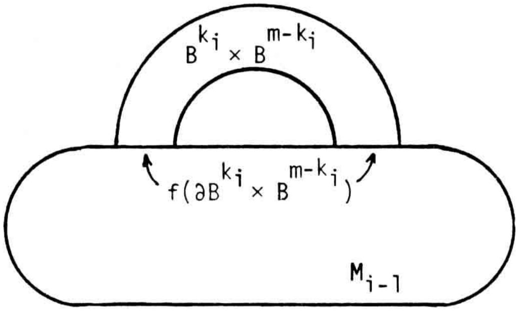


Figure 1.1

Smooth handlebody decompositions (handlebodies for short) correspond to Morse functions $h : M \to R$ (which have non-degenerate critical points at different levels). A critical point of h corresponds to $0 \times 0 \in B^{k_i} \times B^{m-k_i}$ and $B^{k_i} \times 0$ is the descending manifold and $0 \times B^{m-k_i}$ is the ascending manifold.

According to [**Cerf1**], any two Morse functions h_0, h_1 are homotopic by an arc h_t of functions, $t \in [0,1]$, which are Morse functions for all but finitely many t, at which h_t either has two critical points at the same level or a birth or a death occurs.

A death corresponds to a pair of handles cancelling and a birth to the creation of a pair, as is shown in Figure 1.2.

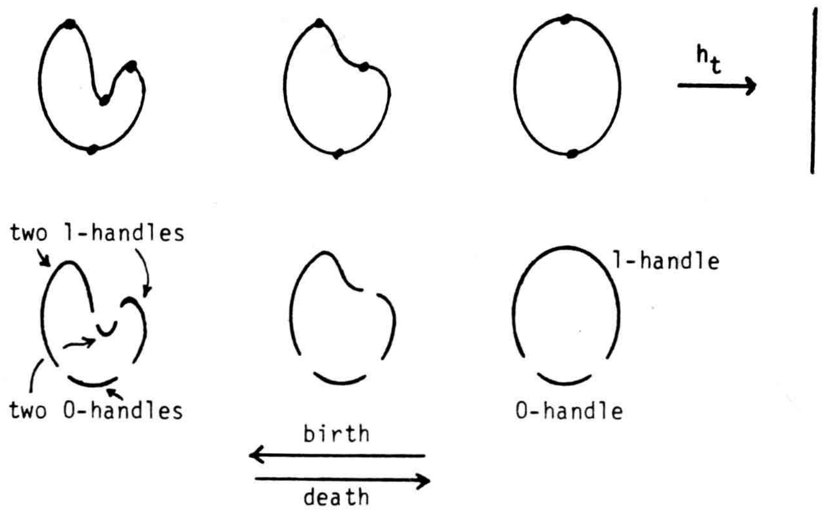


Figure 1.2

Thus, as we homotop h_0 to h_1, the h_t move through Morse functions, which correspond to isotopy of the attaching maps f_i, and whenever a birth or death is passed, a pair of handles are either created or cancelled.

We can summarize this by

THEOREM 1.1. *Any two smooth handlebody decompositions of M^m are related by isotopy of the attaching maps and creation or cancellation of handle pairs.*

It should be noted that handles can always be attached in the order of their indices. For if a $(k+1)$-handle $B^{k+1} \times B^{m-k-1}$ is attached first and then a k-handle $B^k \times B^{m-k}$, then by transversality the attaching sphere of the k-handle, $S^{k-1} \times 0$ misses the cosphere of the $(k+1)$-handle, $0 \times S^{m-k-2}$, (since $k-1+m-k-2 < m-1$) and hence can be isotoped off of the $(k+1)$-handle and added first. Moreover, the same argument shows that two k-handles can be attached in either order.

§2. Framed Links.

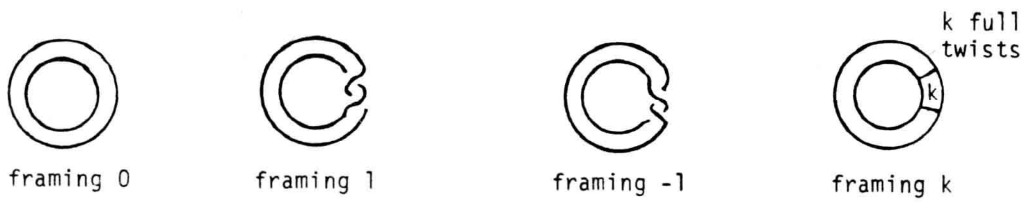
In dimension 4 we will visualize handlebodies by drawing their attaching maps, when possible, in $\partial M_0 = \partial B^4 = S^3$.

A 1-handle is attached by $S^0 \times B^3$, so we draw a pair of 3-balls in S^3 as in Figure 2.1. Often it will be convenient to denote a 1-handle by an unknotted circle with a "dot" on it. The circle bounds an obvious disk, and if we push that disk into B^4 (so that $(B^2, S^1) \to (B^4, S^3)$ is a proper imbedding) and remove a neighborhood of it, then the remainder is $S^1 \times B^3$, the result of adding a 1-handle to B^4. Thus arcs that go over the 1-handle should be drawn so as to go through the dotted circle.

Figure 2.1

We draw the attaching map of a 2-handle, $f(S^1 \times B^2)$, by drawing $f(S^1 \times 0)$, a knot in S^3, and labeling the knot with an integer, its framing. Let F^2 be a surface in B^4 which $f(S^1 \times 0)$ bounds. Then $f(S^1 \times B^2)$ corresponds to the zero framing of $f(S^1 \times 0)$ if it is the trivialization of the normal bundle of $f(S^1 \times 0)$ which extends to the normal bundle of F^2 in B^4. Equivalently, let F^2 be a Seifert surface for $f(S^1 \times 0)$ in S^3; then the zero-framing is the one for which $f(S^1 \times (1,0))$ is tangent to F^2. Framing k means that $f(S^1 \times B^2)$ differs from the zero framing by k full twists around $f(S^1 \times 0)$ (right-handed for $k > 0$, left for $k < 0$), that is, by $k \in Z = \pi_1(SO(2))$.

Figure 2.2 gives some examples where we have drawn $f(S^1 \times 0)$ and $f(S^1 \times e_1)$ for $e_1 = (1,0) \in B^2$.



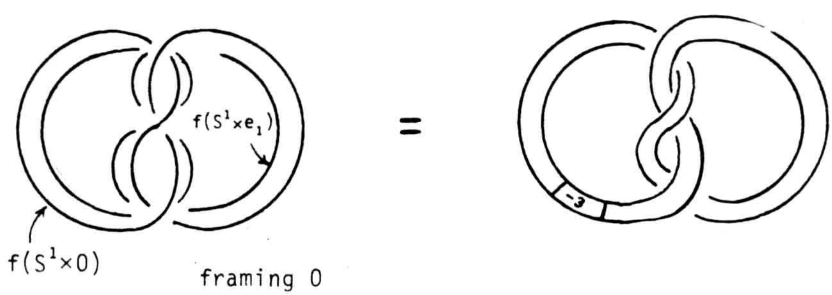


Figure 2.2

Sometimes the attaching circle of a 2-handle goes over a 1-handle; it is drawn as in Figure 2.3.

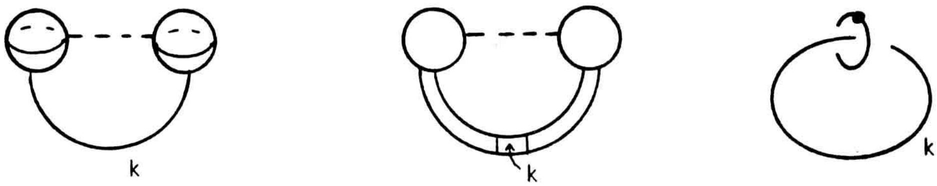


Figure 2.3

Then the attaching circle does not bound a Seifert surface in B^4, so to describe the framing we could draw $f(S^1 \times e_1)$. However, it is more convenient to fix a dotted line joining the two feet of the 1-handle and then to assume that $f(S^1 \times 0)$ goes parallel to the dotted line rather than over the 1-handle; now $f(S^1 \times 0)$ has a Seifert surface and a well defined zero framing. One has to be careful, when isotoping attaching maps, not to cross the dotted line, for that changes the zero-framing just as it would if we changed a crossing in $f(S^1 \times 0)$ (see Figure 2.4).

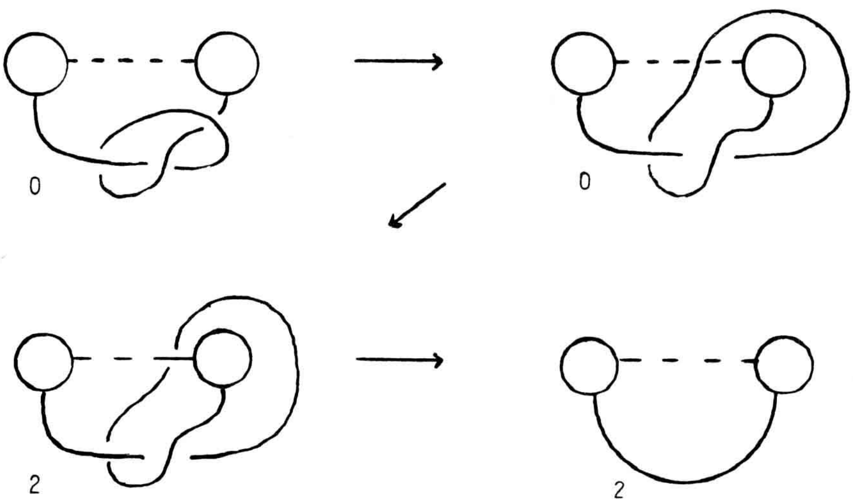


Figure 2.4

If $f(S^1 \times 0)$ goes algebraically zero times over the 1-handle, then it has a Seifert surface and the framing is defined without the use of a dotted line.

When we switch notation for the 1-handle to the circle with a dot, then we place the dotted circle so as to link the dotted line, and draw all the 2-handle attaching circles parallel to the dotted line through the dotted circle (Figure 2.3).

Adding a 1-handle to B^4 results in $S^1 \times B^3$ with boundary $S^1 \times S^2$. Adding a 2-handle to an unknot with zero framing gives $S^2 \times B^2$, also with boundary $S^1 \times S^2$. Handles which are attached later cannot tell what the $S^1 \times S^2$ is the boundary of. Switching the 1-handle to the 2-handle is the same as doing surgery on the obvious S^1 defined by the 1-handle with the trivial framing. The next lemma follows as an exercise from this discussion.

LEMMA 2.1. *Surgery on the S^1 defined by a 1-handle corresponds to removing the dot from the dotted circle and replacing it with a zero (if the trivial framing of the normal bundle of S^1 was used for surgery).*

If there are no 1-handles, then there is an obvious linking matrix A associated with the 2-handles: a_{ij} is the linking number of the i^{th} and j^{th} attaching circles which are oriented by the standard counterclockwise orientation of ∂B^2. a_{ii} is just the framing of the i^{th} handle. A is symmetric, and later will be seen to be the intersection matrix on the second homology of the 4-manifold (II, §1).

If there are 1-handles, we can draw them as dotted circles (oriented arbitrarily) and form an extended linking matrix A' where a_{ii} for a dotted circle is defined to be zero (as if surgery on the 1-handle was performed) and a_{ij} for a 1- and 2-handle is just the algebraic number of times the 2-handle goes over the 1-handle (or the linking between the dotted circle and the attaching circle). Two 1-handles must always be geometrically unlinked. So the extended linking matrix A' has the form

$$\begin{matrix}\text{1-handles }\{\\ \text{2-handles }\{\end{matrix}\quad \left(\begin{array}{c|c} 0 & * \\ \hline * & * \end{array}\right) = A'.$$

3-handles are attached by an imbedding $f : S^2 \times B^1 \to \partial M_i$. The framing is uninteresting, but 2-spheres are hard to draw, especially non-trivial ones. (A complicated one is drawn in **[H-K-K]**, §4.)

However, the 3-handles and 4-handle of a closed M^4 together are diffeomorphic to $\overset{k}{\natural} S^1 \times B^3$ (a 0-handle and k 1-handles), with boundary $\overset{k}{\sharp} S^1 \times S^2$. So the 3- and 4-handles are attached by a diffeomorphism of $\overset{k}{\sharp}(S^1 \times S^2)$. But any such diffeomorphism extends over $\overset{k}{\natural}(S^1 \times B^3)$ **[L-P]**, so it makes no difference how the 3- and 4-handles are attached. For the case $\partial M \neq \emptyset$, **[Trace2]** gives useful information on attaching 2-handles.

Given a framed link L, perhaps containing dotted circles, let M_L^4 denote the 4-manifold obtained by adding handles to the link L. This is a smooth 4-manifold with boundary ∂M_L. However, if ∂M_L is S^3 or $\overset{k}{\sharp} S^1 \times S^2$, then we can close up M_L by adding a 4-handle and perhaps 3-handles. In this case M_L may refer to either the manifold with boundary or the closed 4-manifold, according to context.

Given L, it is useful to know how to describe the double of M_L along ∂M_L. In this case, the 0-handle generates a 4-handle, and 1-handles generate 3-handles, and each 2-handle generates another 2-handle which is added to the co-circle, $0 \times \partial B^2$, of the generating 2-handle. This co-circle gets a framing from its neighboring co-circles $* \times \partial B^2$ which do not link ∂B^2, so the framing is zero. Thus we have shown

LEMMA 2.2. *Given M_L, the framed link for the double DM_L is obtained by adding unknotted circles, linking each 2-handle geometrically once, with framing zero as in Figure 2.5.*

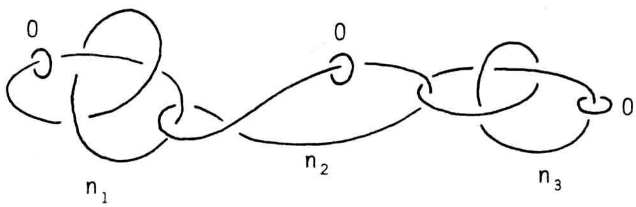


Figure 2.5

§3. Examples.

Figure 3.1

The 4-sphere is the empty link and $S^2 \times S^2$, $S^2 \tilde{\times} S^2$ and $\pm CP^2$ are drawn in Figure 3.1. To see these more clearly, note that the B^2-bundle ξ_k over S^2 with Euler class k (actually k times the generator of $H^2(S^2; Z)$) can be described by adding a 2-handle $B^2 \times B^2$ to an unknot in B^4 (thought of as $\partial B^2 \times 0$ in $B^2 \times B^2 = B^4$) with framing k; the S^2 is $B^2 \times 0 \cup B^2 \times 0$ and the framing gives the twist in ξ_k. Then $\pm CP^2$ is $\pm\xi_1 = \xi_{\pm 1}$ with a 4-handle attached to $\partial\xi_{\pm 1} = S^3$. The non-trivial S^2-bundle over S^2, $S^2 \tilde{\times} S^2$, has a fiber with trivial normal bundle and a section with non-trivial normal bundle (the left and right components of the link).

For a really non-trivial example of a 1-connected, closed M_L (it is trivial to draw non-closed examples—any link will do—but rarely is the boundary equal to $\overset{k}{\#} S^1 \times S^2$), we must turn to the Kummer surface. It is a complex surface with many definitions of the underlying 4-manifold, e.g., any nonsingular quartic in CP^3, say $x^4 + y^4 + z^4 + w^4 = 0$, (see **[H-K-K]**). Figure 3.2 shows a framed link for it with no 1- or 3-handles. It consists of a trefoil knot with framing zero and a small linking circle with framing -2. "On" a Seifert surface for the trefoil knot, draw twenty circles, weaving as drawn, all with framing -2. These twenty-two 2-handles, with a 0- and a 4-handle, describe the Kummer surface.

Our examples do not require 1- and 3-handles. It is not known whether a simply connected, closed 4-manifold needs 1 and/or 3-handles, but the Dolgachev surface (**[H-K-K]** §§3,4 and **[Don3]**) is a good candidate for needing them.

If $\partial M \neq \emptyset$, then simply connected 4-manifolds may require 1 or 3-handles; for example any contractible 4-manifold other than B^4 must have 1 or 3-handles. Casson gave a construction that produces contractible 4-manifolds that need 1-handles specifically (**[Kirby2]** Problem 4.18). Suppose that a contractible M^4 can be made without 1-handles. Then, inverting the Morse function, M^4 can be constructed from ∂M by adding the same number of 1-handles and 2-handles, and one 4-handle. It follows that $\pi_1(\partial M)$ can be killed by adding the same number of generators and relations. But a theorem of Gerstenhaber and Rothaus **[G-R]** states that a finitely presented group with

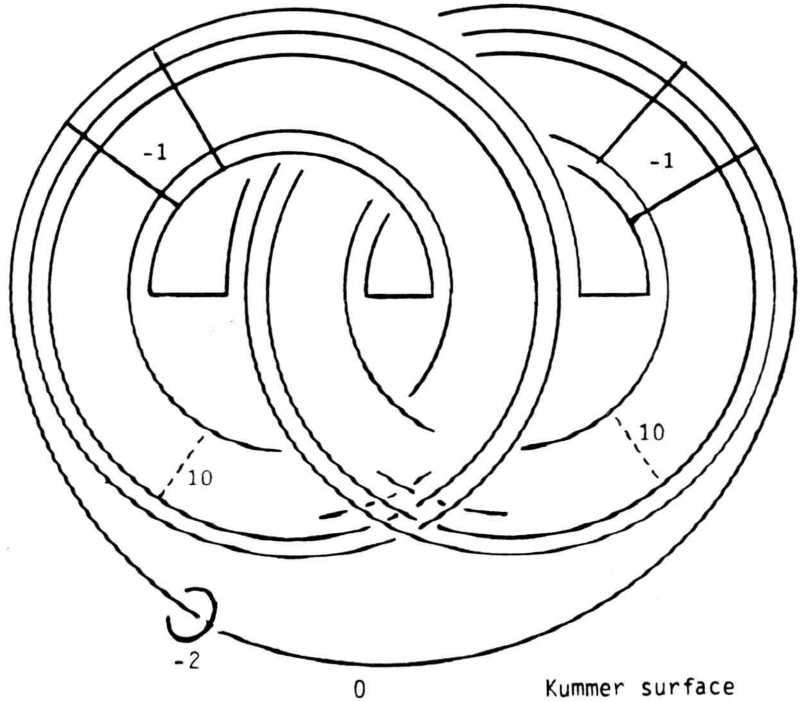


Figure 3.2

a representation into a linear group cannot be killed with an equal number of generators and relations. So any contractible M^4 whose $\pi_1(\partial M)$ has such a representation requires 1-handles. For a specific example, choose a Brieskorn homology 3-sphere which bounds a contractible 4-manifold, e.g. $\sum(2,3,13)$ (see [**A-K4**]) or $\sum(p, ps-1, ps+1)$ for p even, s odd (see [**C-H**] for other collections); note that $\pi_1(\sum(p,q,r))$ is a discrete subgroup of a compact, connected Lie group [**Milnor4**].

§4. Handle Slides.

According to Theorem 1.1 any two handle decompositions for M^4 are related by isotopy of attaching maps and births and deaths. In the language of framed links, a birth of a 1-2 handle pair or 2-3 pair is shown in Figure 4.1 by the sudden appearance, away from the rest of the link, of the indicated links. A death (or cancellation) is their disappearance.

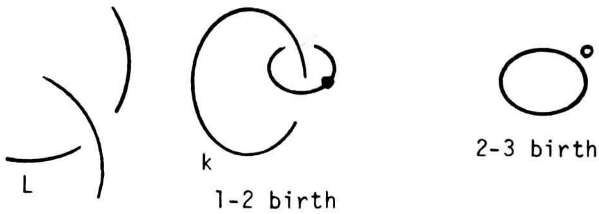


Figure 4.1

An isotopy of an attaching map becomes interesting when it goes "over" another handle rather than just moving about in $S^3 = \partial$(0-handle). The reader should picture an attaching circle which goes over the top of another 2-handle intersecting the critical point (the north pole) of the second 2-handle. If the attaching map is perturbed "left", it falls down to one side of the second attaching circle, if "right", then to the other side, Figure 4.2.

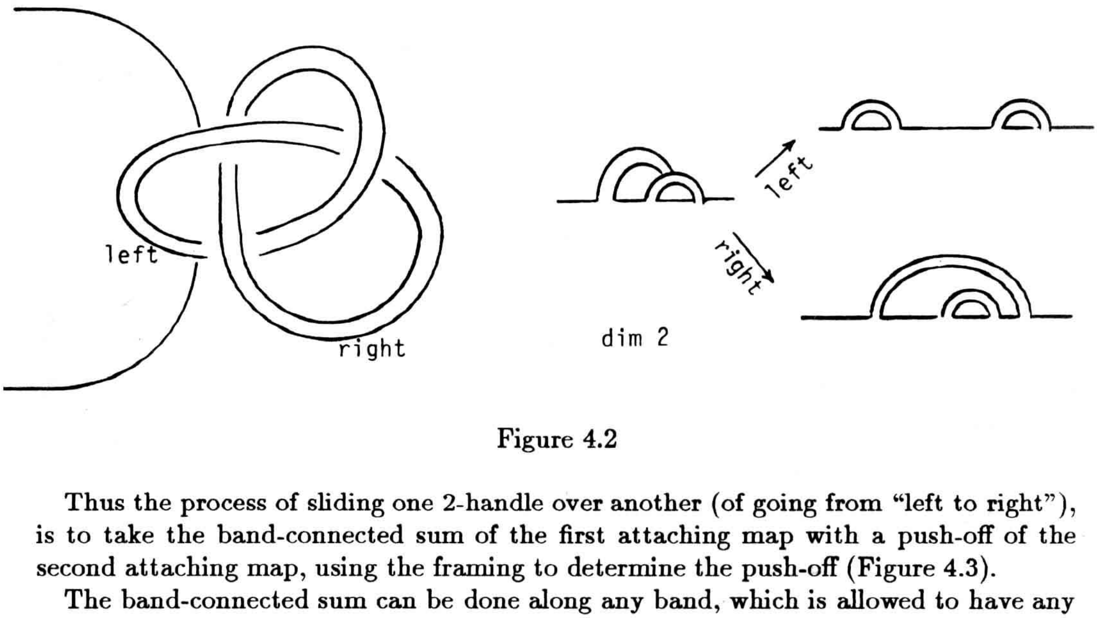


Figure 4.2

Thus the process of sliding one 2-handle over another (of going from "left to right"), is to take the band-connected sum of the first attaching map with a push-off of the second attaching map, using the framing to determine the push-off (Figure 4.3).

The band-connected sum can be done along any band, which is allowed to have any number of right or left half twists in it. The attaching circles should be oriented and then the band-connected sum will either "add" or "subtract" the push-off from the first circle.

The new framing can be computed from the linking matrix by the same process as a change of basis; if α slides over β, then the new basis should be $\alpha \pm \beta$ and β with framing and linking as in Figure 4.4. In Figure 4.3, $m = 0$. The reader can verify this by drawing $f(S^1 \times 0)$ and $f(S^1 \times e_1)$ for each handle, doing the band-connected sum, and computing the new linkings.

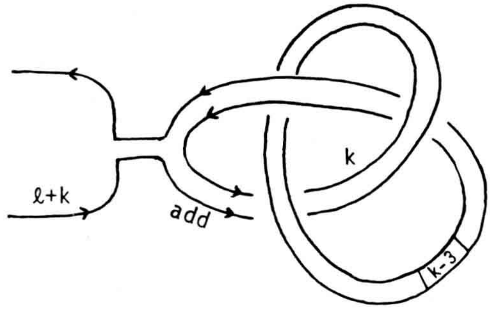


Figure 4.3

$$\begin{matrix}\alpha \\ \beta\end{matrix}\begin{pmatrix} \ell & m \\ m & k \end{pmatrix} \longrightarrow \begin{matrix}\alpha \pm \beta \\ \beta\end{matrix}\begin{pmatrix} \ell \pm 2m + k & m + k \\ m + k & k \end{pmatrix}$$

Figure 4.4

The same thing works if we "slide a 2-handle over a 1-handle"; we are thinking of the 1-handle as a dotted circle with a "framing" zero, and sliding a 2-handle over it (Figure 4.5) is the same as isotoping the attaching map of the 2-handle between the feet of the 1-handle (Figure 2.4). Note that the framing changes according to the change in the linking matrix when a 1-handle is added to a 2-handle, which corresponds to crossing the dotted line in Figure 2.4.

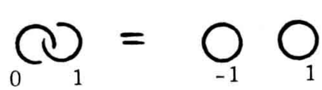
It is possible to make sense of sliding a 1-handle (dotted circle) over a 2-handle whose attaching circle is a slice knot ([**A-K3**], pg. 376); the knotted, dotted circle means remove the slice disk from B^4. But we won't pursue this notion, and from now on rule out the possibility of sliding a 1-handle over a 2-handle.

At this point there are a number of elementary examples that should be understood.

LEMMA 4.1. *An unknotted S^1 with framing ± 1 can always be moved away from the rest of the link L with the effect of giving all arcs going through S^1 a full ∓ 1 twist and changing the framings by adding ∓ 1 to each arc, assuming the arcs represent different components of L (in general they change according to change of basis in the linking matrix). See Figure 4.6.*

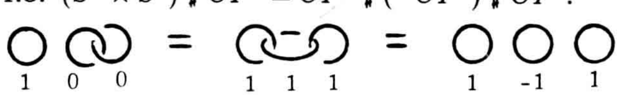
Proof: First do the case for one arc, $k = 1$, by sliding the arc once over the circle; we add if the linking between the oriented arc and circle is ∓ 1 compared to ± 1, and subtract otherwise. In general slide all arcs over the circle once. □

COROLLARY 4.2. $S^2 \tilde{\times} S^2 = CP^2 \sharp -\hat{C}P^2$.

Proof: 

□

COROLLARY 4.3. $(S^2 \times S^2) \sharp CP^2 = CP^2 \sharp (-CP^2) \sharp CP^2$.

Proof: 

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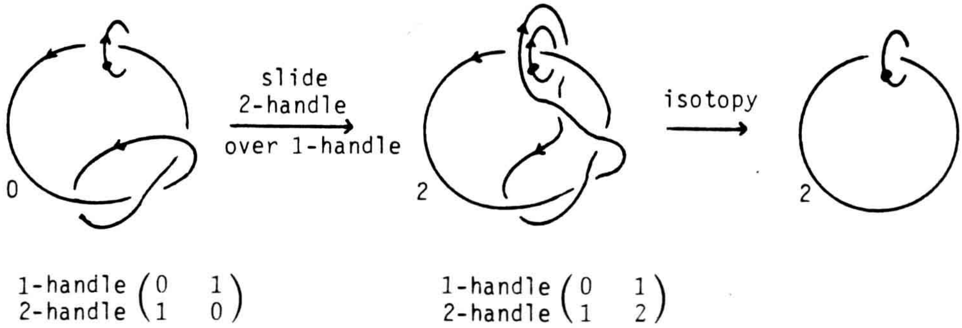

Figure 4.5

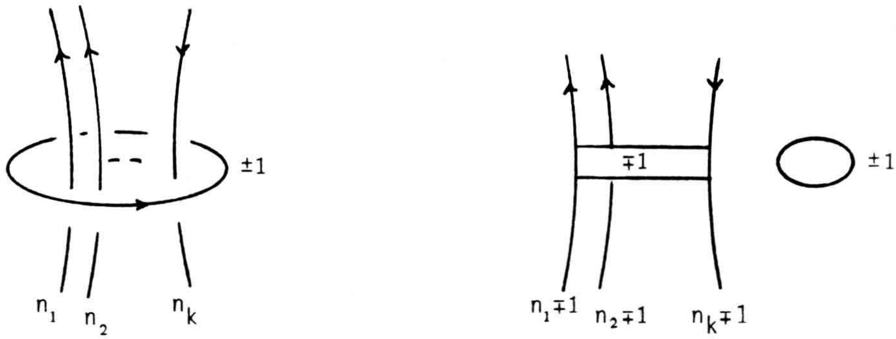

Figure 4.6

LEMMA 4.4.

Proof: Each time the left circle is slid over the right (with the proper band-connected sum), the framing changes by ± 2. □

LEMMA 4.5. *If in L (with no 1-handles) a component L_0 is an unknot with framing zero which links only one other component L_1 geometrically once, then $L_0 \cup L_1$ may be moved away from the rest of L without changing framings. Then L_1 can be unknotted*