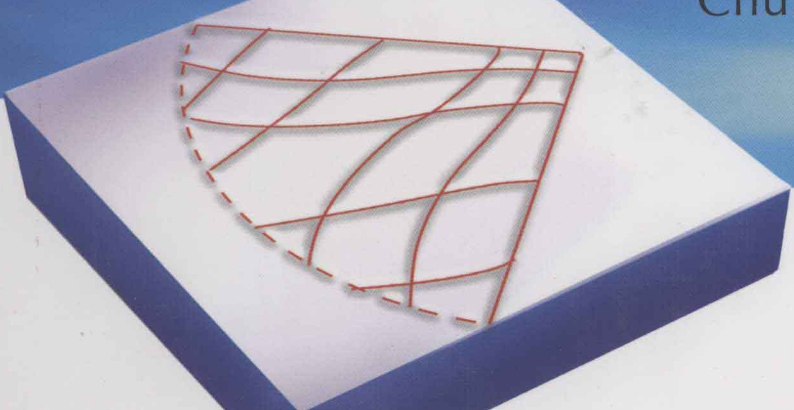
Series in Contemporary Applied Mathematics
CAM 13

Nonlinear Conservation Laws, Fluid Systems and Related Topics

非线性守恒律、流体力学方程组及相关主题

Gui-Qiang Chen
Ta-Tsien Li
Chun Liu
editors



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Pennsylvania State University, USA

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Preface

This book is a collection of lecture notes mainly from the short courses given in the 2007 Shanghai Summer School on Nonlinear Conservation Laws, Fluid Systems and Related Topics at Fudan University, July 5–August 4, 2007. There were more than 130 participants, including graduate students, postdoctors and junior faculty members from more than 30 universities in China and USA.

This summer school provided an occasion for a series of courses (25–26 hours each) by four distinguished contributors of this volume, Denis Serre (ENS-Lyon, France), Xiaoming Wang (Florida State University, USA), Tong Yang (CUHK, Hong Kong), and Yuxi Zheng (Penn State, USA), and a series of invited lectures by distinguished speakers including Jerry Bona (UIC, USA), Hongqiu Chen (The University of Memphis, USA), Emmanuele DiBenedetto (Vanderbilt University, USA), Willi Jäger (University of Heidelberg, Germany), Fanghua Lin (NYU, USA), Tai-Ping Liu (Stanford University, USA), Yuejun Peng (Université Blaise Pascal, France), Weike Wang (Shanghai Jiao Tong University, PRC), and Ping Zhang (Chinese Academy of Sciences, PRC), besides the editors of this volume.

This volume comprises five chapters, ranging from the mathematical theory and numerical approximation of both incompressible and compressible fluid flows, kinetic theory and conservation laws, to statistical theories for fluid systems, with expectation to lead the readers from the basics to the frontiers of the current research in these areas.

Chapter 1 is an introduction to the theory of incompressible inviscid flows with emphasis on classical results and recent developments. Chapter 2 is an introduction to one-dimensional hyperbolic systems of conservation laws with emphasis on theory, numerical approximation, and discrete shock profiles. Chapter 3 is an introduction to the kinetic theory, conservation laws and their intrinsic connections. Chapter 4 is an introduction to elementary statistical theories with applications to various fluid systems. Chapter 5 is an introduction to the Euler equations for compressible fluids in two space dimensions with emphasis on the self-similar isentropic irrotational case. These topics are naturally interrelated and represent a cross-section of the most significant recent advances and current trends in nonlinear conservation laws, fluid systems and related topics.

The editors would like to express their sincere thanks to all the authors in this volume for their contributions and to all the participants in the Summer School. Zhiqiang Wang and Chunlian Zhou deserve our special thanks for their prompt and effective assistance to make the Summer School run smoothly. The editors are grateful to Fudan University, the Mathematical Center of Ministry of Education of China, the National Natural Science Foundation of China (NSFC) and the Institut Sino-Francais de Mathématiques Appliquées (ISFMA) for their help and support. Finally, the editors wish to thank Tianfu Zhao (Senior Editor, Higher Education Press) for his patience and professional assistance.

Gui-Qiang Chen, Ta-Tsien Li, Chun Liu

March 2008

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Introduction to the Theory of Incompressible Inviscid Flows*

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Abstract

In this chapter, we consider the 3D incompressible Euler equations. We present classical and recent results on the issue of global existence/finite time singularity. We also introduce the theories of lower dimensional model equations of the 3D Euler equations and the vortex patch problem.

1 Introduction

The goal of these lecture notes is to introduce to the readers classical results as well as recent developments in the theory of 3D incompressible Euler equations. We will focus on the global existence/finite time singularity issue. We will start with the basic properties of the incompressible fluid flows, and then discuss the local and global well-posedness of the incompressible Euler equations. Of particular interest is the global existence or possible finite time blow-up of the 3D incompressible Euler equation. This is one of the most outstanding open problems in the past century. Here, we carefully examine the nature of the nonlinear vortex stretching term for the 3D Euler equation as well as several model problems for the 3D Euler equation. We put extra effort in taking into account the local geometrical properties and possible depletion of nonlinearity. By going through the nonlinear analysis of various fluid models,

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we can gain valuable insights into the fluid dynamic prolems being studied. Through the analysis, we can also learn how various functional analysis and PDE techniques are being used for realistic applications, and what are their strengths and limitations. We especially emphasize the interplay between the physical and geometric properties of the fluid flows and modern nonlinear PDE techniques. By going through these analyses systematically, we can have a good understanding of the state of the art of nonlinear PDE methods and their applications to fluid dynamics problems.

This chapter is organized as follows:

1. Introduction
2. Derivation and Exact Solutions
3. Local Well-posedness of the 3D Euler Equation
4. The BKM Blow-up Criterion
5. Recent Global Existence Results
6. Lower Dimensional Models for the 3D Euler Equation
7. Vortex Patch

2 Derivation and exact solutions

2.1 Derivation of the Euler equations

The equation that governs the evolution of inviscid and incompressible flow is the Euler equation. Here we first derive the 3D Euler equation briefly. For more detailed derivations, the readers should consult other textbooks in fluid mechanics, such as Chorin-Marsden [12], Lamb [31], Marchioro-Pulvirenti [36], or Lopes Filho-Nussenzveig Lopes-Zheng [33].

We consider a domain Ω which is filled with a fluid, such as water. In classical continuum mechanics, the fluid can be seen as consisting of infinitesimal particles. At each time t, each particle has a one-to-one correspondence to the coordinates $x = (x_1, x_2, x_3) \in \Omega$. The fluid can be described by its density ρ, velocity $\mathbf{u} = (u_1, u_2, u_3)$ and pressure p at each such point $x \in \Omega$. Under the above assumptions, we can denote the position of any particle at time t by $X(\alpha, t)$ which starts at the position $\alpha \in \Omega$ at $t = 0$. Its evolution is governed by the following differential equation:

$$\frac{dX(\alpha,t)}{dt} = \mathbf{u}(X(\alpha,t),t), \quad X(\alpha,0) = \alpha. \tag{2.1}$$

To study the dynamics of the fluid, we must establish relations between ρ, $\mathbf{u}$ and p. We do this by considering two basic mechanical rules: the conservation of mass, and the conservation of momentum.

The *conservation of mass* claims that, for any fixed region $W \subseteq \Omega$ which does not change with time,

$$\frac{d}{dt}\int_W \rho(x,t)\ dx = -\int_{\partial W} \rho(x,t)\mathbf{u}(x,t)\cdot\mathbf{n}(x,t)\ d\sigma \tag{2.2}$$

for all time t, where $\mathbf{n}(x,t)$ is the outer unit normal vector to ∂W, and $d\sigma$ is the area unit on ∂W. Using the Gauss theorem we arrive at

$$\frac{d}{dt}\int_W \rho(x,t)\ dx = -\int_W \nabla\cdot(\rho(x,t)\mathbf{u}(x,t))\ dx$$

which implies

$$\int_w (\rho_t + \nabla\cdot(\rho\mathbf{u}))\ dx = 0.$$

If we assume the continuity of the integrand $\rho_t + \nabla\cdot(\rho\mathbf{u})$, by the arbitrariness of W, we get

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0. \tag{2.3}$$

Since otherwise, there would be a point x_0 such that the integrand is not 0. Without loss of generality, we assume $(\rho_t + \nabla\cdot(\rho\mathbf{u}))(x_0) > 0$. Then by continuity, there is $r > 0$ such that $\rho_t + \nabla\cdot(\rho\mathbf{u}) > 0$ for any $x \in B(x_0, r)$. This leads to a contradiction by taking $W = B(x_0, r)$. Equation (2.3) is called the *continuity equation.*

Let J be the determinant of the Jacobian matrix, $\frac{\partial X}{\partial\alpha}$. It can ben proved by direct calculations (the reader should try to prove this as an exercise, see also Chorin-Marsden [12]) that

$$\frac{dJ}{dt} = (\nabla\cdot\mathbf{u})J, \quad J(0) = 1.$$

We assume that the flow is incompressible. Incompressibility implies that the flow is volume preserving. Using the above equation one can show that the velocity is divergence-free, i.e.

$$\nabla\cdot\mathbf{u} = 0. \tag{2.4}$$

In this case, we have the determinant of the Jacobian matrix, J, to be identically equal to one, i.e. $J \equiv 1$. If the initial density is constant, i.e. $\rho(x,0) \equiv \rho_0$, equation (2.3) implies that density is constant globally, i.e.

$$\rho(x,t) \equiv \rho_0.$$

Remark 2.1.

1. The above derivation of the mass conservation equation is under the assumption that ρ, $\mathbf{u}$ and ∂W are all smooth enough, e.g., C^1.
2. One can also derive (2.3) in a Lagrangian way, i.e., by considering an evolving region Ω_t that is a collection of particles. See e.g. Lopes Filho-Nussenzveig Lopes-Zheng [33].
3. Yet another way is through the variational formulation. See e.g. Marchioro-Pulvirenti [36].

The *conservation of momentum* means

$$\frac{d}{dt}\int_{\Omega_t} \rho\mathbf{u}\, dx = \mathbf{F}(\Omega_t), \tag{2.5}$$

where $\mathbf{F}(\Omega_t)$ is the force acting on Ω_t. Here $\Omega_t \equiv \cup_{\alpha\in\Omega_0} X(\alpha,t)$ for some $\Omega_0 \subseteq \Omega$ is a collection of particles that is carried by the flow. We first assume that the interaction in the fluid is local, i.e., all the forces between points inside Ω_t cancel each other by Newton's third law. This assumption implies

$$\mathbf{F}(\Omega_t) = \int_{\partial\Omega_t} \mathbf{f}\, d\sigma$$

for some $\mathbf{f}$. Our second assumption is that the fluid is ideal, which means that $\mathbf{f} = -p\mathbf{n}$, where $\mathbf{n}$ is the unit outer normal to $\partial\Omega_t$. Now the momentum relation becomes

$$\frac{d}{dt}\int_{\Omega_t} \rho\mathbf{u}\, dx = \int_{\partial\Omega_t} -p\mathbf{n}\, d\sigma = -\int_{\Omega_t} \nabla p\, dx,$$

where the second equality follows from the Gauss theorem

$$\int_{\Omega} \partial_i f\, dx = \int_{\partial\Omega} f n_i\, d\sigma.$$

To derive a pointwise equation similar to (2.3), we need to put the $\frac{d}{dt}$ inside the integration in the term

$$\frac{d}{dt}\int_{\Omega_t} \rho\mathbf{u}\, dx.$$

Note that since $\Omega_t = X(\Omega_0, t)$ depends on t, it is not the same as

$$\int_{\Omega_t} (\rho\mathbf{u})_t\, dx.$$

Instead of naïvely putting the differentiation inside, we proceed as follows. We first change variables from the Eulerian variable x to the Lagrangian variable α. Since the flow is incompressible, the determinant of the Jacobian matrix is equal to one, i.e., $det(X_\alpha) = 1$. Thus we have

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_t} \rho \mathbf{u}\, dx &= \frac{d}{dt}\int_{\Omega_0} \rho(X(\alpha,t),t)\mathbf{u}(X(\alpha,t),t)\, d\alpha \\
&= \int_{\Omega_0} \frac{d}{dt}\rho(X,t)\mathbf{u}(X,t) + \rho(X,t)\frac{d}{dt}\mathbf{u}(X,t)\, d\alpha \\
&= \int_{\Omega_0} (\rho_t + \mathbf{u}\cdot\nabla\rho)\,\mathbf{u} + \rho\,(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})\, d\alpha \\
&= \int_{\Omega_0} \rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})\, d\alpha \\
&= \int_{\Omega_t} \rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})\, dx,
\end{aligned}$$

where the first equality follows from the fact that the flow map $\alpha \mapsto X(\alpha, t)$ is one-to-one and has Jacobian 1, and the fourth equality follows from (2.3) and the incompressibility condition. Now we have

$$\int_{\Omega_t} \rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})\, dx = -\int_{\Omega_t} \nabla p\, dx.$$

Finally, by the arbitrariness of Ω_t, we get

$$\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p. \tag{2.6}$$

by an argument that is similar to the one leading to (2.3). (2.6) is the *balance of momentum.*

If we further assume that the flow has constant initial density, then we have $\rho(x,t) \equiv \rho_0$, and equation (2.6) is equivalent to:

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p,$$

where p is the "rescaled" pressure p/ρ_0.

Under these assumptions, we obatin the 3D Euler equation as follows:

$$\begin{aligned}
\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla p, \\
\nabla\cdot\mathbf{u} &= 0.
\end{aligned} \tag{2.7}$$

In the remaining part of this lecture note, we will focus on (2.7).

2.2 The Vorticity-Stream function formulation

2.2.1 Vorticity

We consider the Taylor expansion of the velocity $\mathbf{u}(x,t)$ at some point x.

$$\begin{aligned}\mathbf{u}(x+h,t) &= \mathbf{u}(x,t) + \nabla\mathbf{u}\cdot h + O(h^2)\\ &= \mathbf{u}(x,t) + \frac{\nabla\mathbf{u}+\nabla\mathbf{u}^t}{2}h + \frac{\nabla\mathbf{u}-\nabla\mathbf{u}^t}{2}h + O(h^2)\\ &\equiv \mathbf{u}(x,t) + S(x,t)h + \Omega(x,t)h + O(h^2),\end{aligned}$$

where S is symmetric and Ω is anti-symmetric. In 3D, it is easy to see that there is a vector ω such that

$$\Omega(x,t)h = \frac{1}{2}\omega(x,t)\times h.$$

This implies that locally, the flow is rotating around an axis $\xi(x,t) \equiv \frac{\omega(x,t)}{|\omega(x,t)|}$. The vector field $\omega(x,t)$ is called "vorticity". And it is easy to check that

$$\omega(x,t) = \nabla\times\mathbf{u}(x,t).$$

2.2.2 Vorticity-Stream function formulation

By taking $\nabla\times$ on both sides of the 3D Euler equation (2.7), we have

$$\omega_t + \mathbf{u}\cdot\nabla\omega = \omega\cdot\nabla\mathbf{u} = S\cdot\omega. \tag{2.8}$$

which is the vorticity formulation. The last equality follows from the fact that

$$\Omega\cdot\omega = \frac{1}{2}\omega\times\omega \equiv 0,$$

since by definition we have

$$\frac{1}{2}\omega\times h \equiv \Omega\cdot h$$

for any vector h. Now there are two unknowns ω and $\mathbf{u}$, so we have to find the relation between them to close the system. This relation is the so-called Biot-Savart law:

$$\mathbf{u}(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}\times\omega(y)\,dy. \tag{2.9}$$

Note that we need $u(x)$ to vanish at ∞ for the above formula to hold. To derive the Biot-Savart law, first define a vector valued function Ψ, called "stream function", such that

$$-\triangle\Psi = \omega.$$

Now it is easy to check that

$$\mathbf{u} = \nabla \times \Psi$$

satisfies

$$\nabla \times \mathbf{u} = \omega.$$

(Hint: Use the identity

$$-\nabla \times (\nabla \times) + \nabla(\nabla \cdot) = \triangle,$$

and then try to show

$$\|\nabla (\nabla \cdot \Psi)\|_{L^2}^2 = 0$$

using the same identity. Details are left as exercises. Or see Bertozzi-Majda [35]).

Now the Biot-Savart law (2.9) follows from the formula

$$\Psi = \frac{1}{4\pi} \int \frac{1}{|x-y|} \omega(y)\, dy,$$

where $\frac{1}{4\pi}\frac{1}{|x|}$ is the fundamental solution for the Poisson equation

$$-\triangle u = f$$

in 3D.

Besides (2.8), another important form of the vorticity evolution is the "stretching formula".

$$\omega(X(\alpha,t),t) = \nabla_\alpha X(\alpha,t)\omega_0(\alpha), \tag{2.10}$$

where $\omega_0(\alpha) = \omega(X(\alpha,0),0) = \omega(\alpha,0)$, and X is defined by (2.1). To prove it, just differentiate both sides with respect to time, which yields

$$\begin{aligned} \omega_t + \mathbf{u} \cdot \nabla \omega &= \nabla_\alpha \mathbf{u}(X(\alpha,t),t)\omega_0(\alpha) \\ &= \nabla \mathbf{u} \cdot (\nabla_\alpha X \cdot \omega_0) \\ &= \nabla \mathbf{u} \cdot \omega(x,t), \end{aligned}$$

which is just (2.8). One catch: this "proof" actually uses the uniqueness of the solution to the system (2.8), (2.9).

For the convenience of future references, we will denote the differentiation in time along the Lagrangian trajectory as $\frac{D}{Dt}$, which has the property:

$$\frac{D}{Dt} w = w_t + \mathbf{u} \cdot \nabla w.$$

$\frac{D}{Dt}$ is also called material derivative.

2.2.3 2D Euler equations

In some physical cases, such as the flow passing around a cylinder with infinite length, we can assume that $u_3 \equiv 0$ and $\mathbf{u}, p$ depend on x_1, x_2 only. In this case, the Euler equations (2.7) remains the same form, but the vorticity-stream function form reduces to

$$\omega_t + \mathbf{u} \cdot \nabla \omega = 0 \tag{2.11}$$

and

$$\mathbf{u}(x) = \frac{1}{2\pi} \int \frac{(x-y)^{\perp}}{|x-y|^2} \omega(y)\, dy, \tag{2.12}$$

where ω is a short-hand for ω_3.

One important difference between 2D and 3D Euler equations is that, the right hand side is 0 in (2.11), which means the vorticity is conserved along Lagrangian trajectory pathes. This point can be illustrated more clearly by looking at the "stretching formula" in 2D, which is

$$\omega(X(\alpha, t), t) = \omega_0(\alpha). \tag{2.13}$$

This difference plays an important role in the theory of 2D Euler equations, which is far more complete than its 3D counterpart.

2.3 Conserved quantities

2.3.1 Local conserved quantities

First we consider those quantities that are carried by a collection of flow particles.

Let C_0 be a closed curve in $\mathbb{R}^3$. We define

$$C_t = \cup_{\alpha \in C_0} X(\alpha, t)$$

and the circulation

$$\Gamma_{C_t} \equiv \oint_{C_t} \mathbf{u} \cdot ds.$$

Theorem 2.2 (Kelvin's Circulation Theorem). $\Gamma_{C_t} \equiv \Gamma_{C_0}$.

Proof. We first prove the following.

$$\frac{d}{dt} \int_{C_t} \mathbf{u} \cdot ds = \int_{C_t} \frac{D\mathbf{u}}{Dt} \cdot ds.$$

To prove it, let $\alpha(\beta)$ be a parametrization of the loop C_0, with $0 \leqslant \beta \leqslant 1$. Then C_t is parametrized as $X(\alpha(\beta), t)$. Thus

$$\begin{aligned}\frac{d}{dt}\int_{C_t} \mathbf{u}\cdot ds &= \frac{d}{dt}\int_0^1 \mathbf{u}(X(\alpha(\beta),t),t)\cdot\frac{\partial}{\partial\beta}X(\alpha(\beta),t)\ d\beta \\ &= \int_0^1 \frac{D\mathbf{u}}{Dt}(X(\alpha(\beta),t),t)\cdot\frac{\partial}{\partial\beta}X(\alpha(\beta),t)\ d\beta \\ &\quad + \int_0^1 \mathbf{u}(X(\alpha(\beta),t),t)\cdot\frac{\partial}{\partial\beta}\mathbf{u}(X(\alpha(\beta),t),t)\ d\beta,\end{aligned}$$

where we have used the relation

$$\frac{\partial X}{\partial t}(\alpha,t) = \mathbf{u}(X(\alpha,t),t).$$

Note that the first term is just

$$\int_{C_t} \frac{D\mathbf{u}}{Dt}\cdot ds,$$

we just need to show that the second term is 0. This is easy, since we have

$$\int_0^1 \mathbf{u}\cdot\frac{\partial}{\partial\beta}\mathbf{u}\ ds = \frac{1}{2}\int_0^1 \frac{\partial}{\partial\beta}(\mathbf{u}\cdot\mathbf{u})\ ds = 0,$$

which follows from the fact that C_t is a close loop.

Now we prove the circulation theorem. We have

$$\frac{d}{dt}\int_{C_t}\mathbf{u}\cdot ds = \int_{C_t}\frac{D\mathbf{u}}{Dt}\cdot ds = -\int_{C_t}\nabla p\cdot ds = -\int_{C_t} p_s ds = 0$$

since C_t is closed. This ends the proof. □

Next let C_0 be a general curve and $C_t = X(C_0, t)$. Then as long as the flow is still regular, C_t is still a curve in $\mathbb{R}^3$. C_t is called a vortex line if the following is satisfied

$$C_0 \text{ is tangent to } \omega_0(\alpha) \text{ at any } \alpha \in C_0. \tag{2.14}$$

One can verify that as long as (2.14) is satisfied, the same tangency condition is satisfied at every moment t, i.e.,

$$C_t \text{ is tangent to } \omega(x,t) \text{ at any } x \in C_t.$$

A collection of vortex lines is called a "vortex tube". One readily sees that vorticity is always tangent to the side surface of a vortex tube.

The above properties make vortex tube/line very important objects in the theories/numerical simulations/physical experiments of the 3D Euler equation, as we will reveal later in this lecture note.

2.3.2 Global conserved quantities

The most well-known global conserved quantities are the following (we will indicate the dimension and region/manifold, $\mathbb{T}^d$ stands for d-dimensional periodic torus):

1. The integral of velocity ($\mathbb{R}^d$ and $\mathbb{T}^d$, $d = 2, 3$).
$$\frac{d}{dt}\int \mathbf{u}\, dx = 0.$$
2. Kinetic energy ($\mathbb{R}^d$, $\mathbb{T}^d$, smooth bounded domain, $d = 2, 3$).
$$\frac{d}{dt}\int |\mathbf{u}|^2 \, dx = 0.$$

Remark 2.3. In the $\mathbb{R}^d$ case, caution must be taken. We actually need that the kinetic energy $\int |\mathbf{u}|^2 \, dx$ to be finite. In 3D this requirement is reasonable, while in 2D it is not.

3. Center of vorticity ($\mathbb{R}^2$, if $\mathbf{u}\omega$ decays fast enough at ∞).
$$\bar{x} = \int_{\mathbb{R}^2} x\omega \, dx = \text{const}.$$
4. Moment of inertia ($\mathbb{R}^2$, if $\mathbf{u}\omega$ decays fast enough at ∞).
$$I = \int_{\mathbb{R}^2} |x|^2 \omega \, dx = \text{const}.$$
5. Functions of vorticity ($d = 2$).
$$\int_{\Omega_t} f(\omega) \, dx = \int_{\Omega_0} f(\omega_0) \, d\alpha$$
for any measurable f and material domain Ω_t. In particular, we see that the L^p norm of ω is conserved for $1 \leqslant p \leqslant \infty$.
6. Other quantities.
$$\int_{\mathbb{R}^3} x \times \omega \, dx,$$
$$\int_{\mathbb{R}^3} x \times (x \times \omega) \, dx;$$
helicity
$$\int_{\mathbb{R}^3} \mathbf{u} \cdot \omega \, dx;$$
and spirality
$$\omega \cdot \gamma,$$

where $\gamma = \mathbf{u} + \nabla\phi$ with ϕ solving

$$\frac{D}{Dt}\phi = -|\mathbf{u}|^2/2 + p.$$

This quantity is conserved along particle trajectories.

2.4 Special flows

2.4.1 Axisymmetric flow

In this subsection we introduce the axisymmetric flow, i.e., when written in cylindrical coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and $x_3 = z$, the velocity u and the pressure p depend only on r and z. Unlike the 2D Euler equations, this particular flow retains some 3D characters and is often referred to as the $2\frac{1}{2}$-D equations.

We introduce the cylindrical frame of reference:

$$\begin{aligned} e_r &= (\cos\theta, \sin\theta, 0), \\ e_\theta &= (-\sin\theta, \cos\theta, 0), \\ e_z &= (0, 0, 1), \end{aligned}$$

and can easily rewrite the 3D Euler equations in the new frame, with $\mathbf{u} = \mathbf{u}(r, z)$ and $p = p(r, z)$, as

$$\mathbf{u}_t + (\mathbf{u} \cdot \tilde{\nabla})\mathbf{u} + B = -\tilde{\nabla} p, \tag{2.15}$$

where

$$\tilde{\nabla} = (\partial_r, 0, \partial_z)$$

and

$$B = \frac{u^\theta}{r}(-u^\theta, u^r, 0).$$

We leave the details (which can be found in e.g. Lopes Filho-Nussenzveig Lopes-Zheng [33]) for this system to the reader as exercises.

1. Derive equations (2.15).
2. Prove that, in the moving frame (e_r, e_θ, e_z), we have

$$\begin{aligned} \omega &= \omega^r e_r + \omega^\theta e_\theta + \omega^z e_z \\ &\equiv \left(-\partial_z u^\theta\right) e_r + \left(\partial_z u^r - \partial_r u^z\right) e_\theta + \left(\partial_r u^\theta + \frac{u^\theta}{r}\right) e_z. \end{aligned}$$

3. When $u^\theta \equiv 0$, (2.15) becomes axisymmetric flows without swirl. Prove that the equations are

$$\begin{aligned} \left(\partial_t + \mathbf{u} \cdot \tilde{\nabla}\right) \mathbf{u} &= -\tilde{\nabla} p, \\ \tilde{\nabla} \cdot (r\mathbf{u}) &= 0. \end{aligned} \tag{2.16}$$