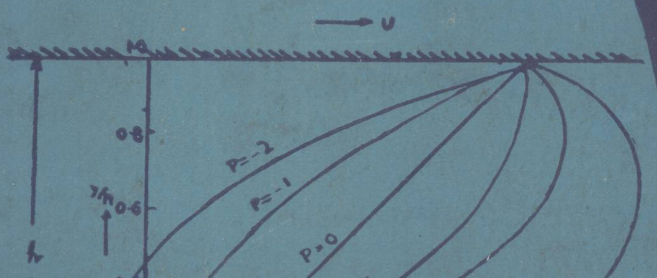
VISCOUS FLUID DYNAMICS

J.L. BANSAL



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VISCOUS FLUID DYNAMICS

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Dedicated

to

my Mother

PREFACE

The present book is the outcome of the lectures delivered by the author to the teachers in the Summer Institute, at all India level, held in Jaipur and to the graduate students and research scholars of the University of Jodhpur. The lectures have been received with considerable enthusiasm and his students' insisted to get the lecture notes published. It is because of students encouragement that the author ventures to publish them. These lectures have been revised and rewritten so as to present them in the form of a book. Mainly, the theory of laminar flow of a viscous incompressible fluid, which has reached to such a stage of perfection that a good portion of it can be taken up as a first course in the subject, has been presented here.

In the first chapter the basic concepts required in the development of the theory of viscous flow have been discussed and the constitutive equation for an isotropic Newtonian fluid is derived in Cartesian tensors. Chapter 2 deals with the fundamental equations, which govern the motion of a viscous compressible fluid. The derivation of the equations has been made simple and concise with the help of cartesian tensors. The governing equations, for compressible and incompressible fluids motion, in cartesian, cylindrical and spherical polar coordinates are given in tabular form for ready reference.

In chapter 3 the topics like Dynamical similarity and Dimensional analysis have been discussed and the physical importance of the non-dimensional parameters and coefficients, which play an important role in the study of the flow of viscous fluids, is given.

Since the Navier-Stokes' (N-S) equations of motion are nonlinear in character, there is no known general method to slove these equations. Only in few special cases exact solutions can be obtained by making certain assumptions about the state of the fluid and a simple configuration for the flow pattern. Some of these exact solutions have been given in Chapter 4.

The exact solutions of the N-S equationsdiscussed in Chapter 4 are valid for all ranges of viscosity but are few in number. However, there appear, at present, no such exact solution of the flow past bodies of finite size and, therefore, in order to study such flows the N-S equations have been simplified to mathematically tractable forms for two extreme cases (i) when viscosity is very large, which give rise to the theory of very slow motion developed by Stokes and Oseen and discussed in Chapter 5, and (ii) when viscosity is very small leading to the theory of Prandtl's laminar boundary layers treated in Chapter 6. In this book only the exact solutions of the two-demensional and axially-symmetrical steady incompressible boundary layers are discussed.

Chapter 7 deals with the integral methods for the approximate solution of the two-dimensional boundary layers based on Ka'rma'n momentum-integral equation. The energy integral equation has also been derived in order to show its connection with the simplified method of Walz and Thwaites.

The last Chapter, *i.e.* Chapter 8, is on thermal boundary layers where the exact solutions of both forced and free convection are discussed. Much emphasis is given on the Pohlhausen's problem of forced convection in boundary layer on a flat plate as it is of fundamental importance even in the study of compressible boundary layers. Analogous to the Ka'rma'n momentum-integral equation, the thermal energy integral equation is also derived and has been applied to the Pohlhausen's problems of both forced and free convection to obtain the respective approximate solutions.

It is hoped that the present book will serve as a text-book on the theory of viscous incompressible laminar flow to the students of various Universities and technological Institutes, who offer the subject as a special paper in their higher studies and an introductory book on laminar incompressible boundary layers to young research workers who may be new to the subject. The author is highly indebted to the authors of various reference books on the subject which he has consulted in the preparation of his lecture notes and trusts that the bibliography given at the end will serve as a partial acknowledgement of this debt.

The author would like to express his sincere thanks to Prof. P. L. Bhatnagar, Prof. J. N. Kapur, Dr. P. D. Verma and Dr. M. C. Gupta, with whom he had very useful discussions. His special thanks are due to Prof. R. S. Kushwaha, University of Jodhpur and Prof. G. C. Patni, University of Rajasthan, Jaipur who took pains to introduce a paper on the study of viscous flow at the graduate level in their respective Universities. It is his great fortune that he got an opportunity to study in the Prandtl Institute (AVA and, Max-Planck Institut für Strömungs forschung) at Göttingen under the able guidance of Prof. H. Schlichting and Prof. W. Wuest, whose works have always been a source of great inspiration to him and to them he owes much more than what he can express.

In conclusion, he is thankful to his students Dr. N. C. Jain and Mr. S. S. Tak, who have assisted in going through a part of the manuscript and to his wife Gaytree Bansal for her constant encouragement and full support during the preparation of the manuscript. Thanks are also due to Oxford and IBH for their excellent cooperation in producing this book.

University of Jodhpur
Jodhpur.
October, 1977

The Author

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CHAPTER 1

BASIC CONCEPTS

1.1 Fluid

All materials exihibit deformation under the action of forces. If the deformation in the material increases continually without limit under the action of shearing forces, however small, the material is called a "fluid". This continuous deformation under the action of forces is manifested in the tendency of fluids to flow.

Fluids are usually classified as liquids or gases. A liquid has interamolecular forces which hold it together so that it possesses volume but no definite shape. When it is poured into a container will fill the container upto the volume of the liquid regardless of the shape of the container. Liquids have but slight compressibility.* For most purposes it is, however, sufficient to regard liquids as "incompressible fluids." A gas, on the other hand, consists of molecules in motion which collide with each other tending to disperse it so that a gas has no set volume or shape. The interamolecular forces are extremely small in gases. A gas will fill any container into which it is placed and is therefore known as a (highly) "compressible fluid".

It is proper to remark here that for speeds which are not comparable with that of sound i.e. if the Mach number (the ratio of the velocity of flow to the velocity of sound) is small compared with unity the effect of compressibility on atmospheric air can be neglected and it may be considered to be a liquid, and

*The ability for changes in volume of a mass of fluid is known as "compressibility" in this sense it is called incompressible air. Taking the velocity of sound about 330 m/sec a flow velocity of 99 m/sec may be accepted as the upper limit when a gaseous flow can be considered incompressible.

1.2 Continuum hypothesis

In its most fundamental form, at the microscopic level, the description of the motion of a fluid involves a study of the behavior of all the discrete molecules which make up the fluid. However, when one is dealing with problems in which some characteristic length in the flow is very large compared with molecular distances, it is convenient to think of a lump of fluid sufficiently small from macroscopic point of view but large enough at the microscopic level so as to contain a large number of molecules (for instance, at normal temperature and pressure a volume of 10^{-12}cc of a gas contains about 2.7×10^{7} molecules) and to work with the average statistical properties of such large number of molecules. In such a case the detailed molecular structure is washed out completely and is replaced by a continuous model of matter having appropriate continuum properties so defined as to ensure that on the macroscopic scale the behavior of the model resembles with the behavior of the real fluid. When the characteristic length in the flow is not large compared with molecular distances, the continuum model is invalid and the flow must be analyzed on the molecular scale.

The smallest lump of fluid material having sufficiently large number of molecules to allow statistically of a continuum interpretation is here called a "fluid particle".

The material in this book will deal primarily with fluids obeying continuum hypothesis.

1.3 Viscosity

Viscosity of a fluid is that characteristic of real fluids which exhibits a certain resistance to alterations of form. Viscosity is also known as internal friction. All known fluids (gases or liquids) possess the property of viscosity in varying degrees. The nature of viscosity can be best illustrated with the aid of the following experiment of Newton :

Consider the motion of a fluid between two very long parallel plates at a distance, say, '*d*' apart, one of which is at rest and the other moving with a constant velocity U parallel to itself, as shown in Fig. 1.1. Because of viscosity, the fluid will also be in motion. Experiment teaches us that the fluid adheres to both walls* (no slip condition), so that its velocity at the lower plate, which is stationary, is zero and at the upper plate, which is moving, is equal to the velocity of the plate U. Experiments have shown that for a large class of fluids the tangential stress τ acting on either of the plates is proportional to the relative velocity between the plates and inversely proportional to the distance d. Thus we have

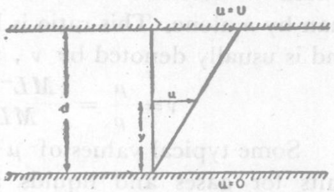

Fig. 1.1. Fluid motion between a stationary plate and a moving parallel plate.

$$\tau = \mu \frac{U}{d}, \tag{1.1}$$

where μ is a constant of proportionality, it is independent of U and d and depends only on the nature of the fluid. This constant is a measure of the viscosity of the fluid and is called the "coefficient of shear viscosity" or simply the "coefficient of viscosity".

For ordinary fluids, since there is no slip on the walls and the fluid is displaced in such a manner that the various layers of fluid slide uniformly over one another, the velocity u of a layer of the fluid at a distance y from the lower plate is then

$$u = U\frac{y}{d} \tag{1.2}$$

It may be seen from (1.2) that U/d in (1.1) may be replaced by the velocity gradient du/dy, hence

$$\tau = \mu \frac{du}{dy} \quad \text{(Newton's law of viscosity)} \tag{1.3}$$

Equation (1.3) may be regarded as the definition of viscosity.

Thus the ***coefficient of viscosity*** of a fluid may be defined as the tangential force required per unit area to maintain a unit velocity gradient, i.e., to maintain unit relative velocity between two layers unit distance apart.

* For a detailed study we refer to "Note on the conditions at the surface of contact of a fluid with a solid body" in the book "Modern developments in fluid dynamics" pp. 676-680 By S. Goldstein.

The dimensions of the coefficient of viscosity μ can be found as follows :

$$\mu=\frac{\text{shearing stress}}{\text{velocity gradient}}=\frac{\text{force/area}}{\text{velocity/length}}=ML^{-1}T^{-1} .$$

As we shall see later, the effect of viscosity on the motion of a fluid is determined by the ratio of μ to the density ρ rather than by μ alone. This ratio is known as the "kinematic viscosity" and is usually denoted by ν , thus

$$\nu=\frac{\mu}{\rho}=\frac{ML^{-1}T^{-1}}{ML^{-3}}=L^2T^{-1}.$$

Some typical values of μ and ν are given below in c.g.s. units for gases and liquids at 15°C and under atmospheric pressure.

	μ	ν
Gases		
Air	0.00018	0.15
Nitrogen	0.00017	0.15
Oxygen	0.0002	0.15
Hydrogen	0.00009	1.5
Helium	0.0002	0.12
Carbon dioxide	0.00014	0.077
Liquids		
Water	0.0114	0.0114
Mercury	0.016	0.0012
Paraffin oil	0.2	0.25
Glycerine	13	10
Castor oil	15	15
Pitch	10^{10}	10^{10}

For liquids the viscosity μ is nearly independent of pressure and decreases rapidly with increasing temperature. In the case of gases, to a first approximation, the viscosity can be taken to be independent of pressure but it increases with temperature. It is small for "thin" fluids, such as water or air, but large in the case of very viscous liquids, such as oil or glycerine.

It is appropriate to remark here that the example considered in Fig. 1.1 constitutes a particularly simple case of fluid motion. The generalization to the case of three-dimensional flow is contained in "Stokes' law of friction", the theory of which will now be developed.

STRAIN ANALYSIS

1.4 Most general motion of a fluid element

In general, the motion of a fluid element consists of three parts (i) a translation, (ii) a rotation, and (iii) a deformation. We shall show this by considering the relative motion between two neighbouring points of a fluid element. Let P and Q denote any two such points, at a time t, and let $\vec{r}$ and $\vec{r}+\vec{dr}$ be their position vectors referred to a fixed point O (Fig. 1.2). If $\vec{V}(\vec{r}, t)$ is the velocity at the point P at a time t, the velocity at Q at the same instant is expressed to a first order by

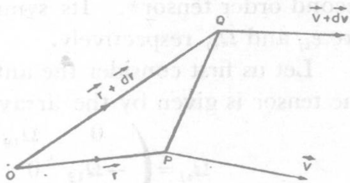

Fig. 1.2. Relative motion between two neighbouring points of a fluid element.

$$\vec{V}(\vec{r}+\vec{dr},t)=\vec{V}(\vec{r},t)+d\vec{V}$$

$$=\vec{V}(\vec{r},t)+(\vec{dr}.\,\text{grad})\vec{V}. \tag{1.4}$$

It is clear from above that in the velocity of the point Q, there is a part *viz.*, $\vec{V}(\vec{r}, t)$ which is the same as that of P. This part is the same for all points of the fluid element and, therefore, corresponds to a velocity of *translation* of the fluid element as a whole.

The component $d\vec{V}$ or $(\vec{dr}.\,\text{grad})\vec{V}$ is the relative velocity between P and Q, can be shown to be made up of a *rotation* and a *deformation*. We will carry out the proof in cartesian tensors.

Let us denote the cartesian components of $\vec{r}, \vec{dr}$ and $\vec{V}$ by x_i, dx_i and v_i $(i=1, 2, 3)$ respectively. The *jth* component of $(\vec{dr}.\,\text{grad})\,\vec{V}$ is then given by

$$dx_i\frac{\partial v_j}{\partial x_i}=dx_i\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}\right)+\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}-\frac{\partial v_i}{\partial x_j}\right)\right]. \tag{1.5}$$

Let

$$e_{ij}=\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j}\right), \tag{1.6}$$