Kung-ching Chang

Infinite Dimensional Morse Theory and Multiple Solution Problems

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Kung-ching Chang
Department of Mathematics
Peking University
Beijing, 100871
People's Republic of China

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PREFACE

The book is based on my lecture notes "Infinite dimensional Morse theory and its applications", 1985, Montréal, and one semester of graduate lectures delivered at the University of Wisconsin, Madison, 1987. Since the aim of this monograph is to give a unified account of the topics in critical point theory, a considerable amount of new materials has been added. Some of them have never been published previously.

The book is of interest both to researchers following the development of new results, and to people seeking an introduction into this theory. The main results are designed to be as self-contained as possible. And for the reader's convenience, some preliminary background information has been organized.

The following people deserve special thanks for their direct roles in helping to prepare this book.

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Kung-Ching Chang
Mathematical Institute, Peking University, Beijing.

INTRODUCTION

This book deals with Morse theory as a way of studying multiple solutions of differential equations which arise in the calculus of variations. The theory consists of two aspects: the global one, in which existence, including the estimate of the number of solutions, is obtained by the relative homology groups of two certain level sets, and the local one, in which a sequence of groups, which we call critical groups, is attached to an isolated critical point (or orbit) to describe the local behavior of the functional. Morse relations link these two ideas.

In comparison with degree theory, which has proved very useful in nonlinear analysis in proving existence and in estimating the number of solutions to an operator equation, Morse theory has a great advantage if the equation is variational. Relative homology groups and critical groups are series of groups that provide both a finer structure and better estimate of the number of solutions than does the degree, which is only an integer. The relationship between the Leray-Schauder index and critical groups is established.

The minimax method is another important tool in critical point theory. In this volume it is treated in a unified manner from the Morse theoretic point of view. The mountain pass theorem, the saddle point theorem and multiple solution theorems, discussed in Ljusternik-Schnirelman theory, index theory and pseudo index theory, are studied by observing the relative homology groups for specific level sets. Critical groups for critical points are also estimated. The purpose of this treatment is to provide a unified framework which contains different theories so that various techniques are able to be combined in estimating the number of critical points.

Applications to semilinear elliptic boundary value problems, periodic solutions of Hamiltonian systems, and geometric variational problems are also emphasized. These problems are chosen for their own interest as well as for explaining how Morse theory is applied.

The book is organized into five chapters and an appendix. Chapter 1 is devoted to Morse theory. Sections 1 and 2 review the basic facts of algebraic topology and infinite dimensional manifolds, respectively. Two deformation theorems, which play a fundamental role in critical point theory, are proved in detail in Section 3. Morse relations and the Morse handle body theorem are studied in Section 4. Section 5 deals with Gromoll-Meyer theory and discusses the main properties of critical groups for isolated critical points, including homotopy invariance and a shifting lemma. The Marino-Prodi approximation theorem is also studied in this section. In the rest of the chapter, Morse theory is extended: in Section 6.1, to manifolds with boundaries together with certain boundary value conditions, and, in Section 6.2, to locally convex closed sets. The latter extension is motivated by variational inequalities. G-equivariant Morse theory is investigated in Section 7, where all the main results of Sections 4 and 5 are completely extended to invariant functions under a compact Lie group action.

Chapter 2 views critical point theory with respect to homology groups. Sections 1 through 4 are devoted to this study. The homological link, subordinate homology classes, and Cech-Alexander-Spanier cohomological rings are used to link up minimax principles with Morse theory. Morse index estimates in Minimax theorems are also presented. In Section 5, we give some abstract critical point theorems which will be applied in subsequent chapters. Two perturbation theories are studied in Section 6, one of which is concerned with the perturbation effect on a critical manifold, and the other with Uhlenbeck's perturbation theory.

Semilinear elliptic BVPs are considered to be models in the applications of critical point theory. The reader will find that there are many different and very interesting results presented in Chapter 3. Although some of them will be familiar, the proofs given here are new and are based on the above unified framework. Problems with superlinear, asymptotically linear and bounded nonlinear terms are studied by example in Sections 2–4. Variational inequalities are also discussed.

Chapter 4 deals with some topics on Hamiltonian systems. Since there are special books on this subject, we satisfy ourselves with introducing material that does not overlap. The following problems were selected: asymptotically linear systems, Hamiltonians with periodic nonlinearities, second order systems with singular potentials, the double pendulum equation, Arnold conjectures on symplectic fixed points and on Lagrangian intersections. Our treatment of these is limited to examples.

In the final chapter, we analyze two-dimensional harmonic maps and the Plateau problem for minimal surfaces as examples from geometric variational problems. Because of the lack of the Palais-Smale condition, Morse theory for harmonic maps is established by the heat flow. The Plateau problem is considered to be a function defined on a closed convex set in a Banach space. Extended Morse theory is applied to give a proof of the Morse-Tompkins-Shiffman theorem on unstable coboundary minimal surfaces.

In the appendix, Witten's proof of the Morse inequalities is presented in a self-contained way. Although the material is totally independent of the context of this book, we introduce Witten's idea because the proof is so beautiful and surprising; moreover, it is a good example of the interplay between analysis and topology.

This book is not intended to be complete, either as a systematic study of Morse theory or as the presentation of many applications. We do not deal with Conley theory [Con1], stratified Morse theory, and the beautiful applications in the study of closed geodesics. (For an overview of the literature, the reader is referred to the book by Klingenberg [Kli1]) as well as to the study of gauge theory [AtB1].

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CHAPTER I

Infinite Dimensional Morse Theory

The basic results in Morse theory are the Morse inequalities and the Morse handle body theorem. They are established on the Banach Finsler manifolds or on the Hilbert Riemannian manifolds in Section 4. The tool in this study is the deformation theorem, which is introduced in Section 3. Some preliminaries on algebraic topology and on infinite dimensional manifolds are reviewed in Sections 1 and 2 respectively. Readers who are familiar with the background material may skip over these two sections. Gromoll-Meyer theory on isolated critical points plays an important role in the applications of Morse theory because the nondegeneracy assumption in the handle body theorem might not hold for concrete problems. Section 5 is devoted to introducing Gromoll-Meyer theory systematically and examines the splitting lemma, the homotopy invariance theorem, the shifting theorem, and the Marino Prodi approximation theorem. The rest of the chapter consists of the extensions of the basic results of Morse theory in different directions: in Section 6.1, to the extension to manifolds with boundaries as well as to the functions satisfying certain boundary value conditions, in Section 6.2, to the extension from manifolds to the locally convex closed subsets; and, in Section 7, to functions with symmetry under a compact Lie group action.

1. A Review of Algebraic Topology

The idea of algebraic topology is to assign algebraic data to topological spaces so that topological problems may be translated into algebraic ones. The singular homology group is an example of algebraic data. It is constructed of the maps of geometric simplexes into arbitrary topological spaces so that it is applicable to infinite dimensional problems.

Let X be a topological space, and let

$$\Delta_q = \left\{ \sum_{j=0}^{q} \lambda_j e_j \mid \lambda_j \geq 0,\ \sum \lambda_j = 1 \right\}$$

be the standard q-simplex, $q = 0, 1, \ldots$ where

$$\begin{aligned} e_0 &= (0, 0, \ldots 0, \ldots) \\ e_1 &= (1, 0, \ldots 0, \ldots) \\ \ldots & \quad \ldots \\ e_q &= (0, 0, \ldots \underset{q^{th}}{1}, \ldots) \\ \ldots & \quad \ldots \end{aligned}$$

are vectors in $\mathbb{R}^\infty$.

A singular q-simplex is defined as a continuous map $\varphi : \Delta_q \longrightarrow X$. Also, let $\sum_q$ denote the set of all singular q-simplexes.

Given an Abelian group G, we define the formal linear combinations: $\sigma = \sum g_i \sigma_i$, $g_i \in G$, $\sigma_i \in \sum_q$. These sums are called singular q-chains. The set of all singular q-chains is denoted by $C_q(X, G)$.

Suppose that X, X' are two topological spaces, and that

$$f : X \longrightarrow X'$$

is continuous, then

$$f_{\#} : \sigma = \sum g_i \sigma_i \longrightarrow \sum g_i f(\sigma_i)$$

is a reduced homomorphism: $C_q(X, G) \longrightarrow C_q(X', G)$.

For each $\sigma \in \sum_q$, we define the boundary operator

$$\partial \sigma = \sum_{j=0}^{q} (-1)^j \sigma^{(j)}$$

where $\sigma^{(j)} = \varphi[e_0, e_1, \ldots, \hat{e}_j, \ldots, e_q]$, $[e_0, e_1, \ldots, \hat{e}_j, \ldots, e_q]$ denotes the $q-1$ simplex generated by the vectors $e_0, e_1, \ldots, e_q$ except $e_j, j = 0, 1, \ldots, q$. Then we extend the operator ∂ linearly onto $C_q(X, G)$, i.e.,

$$\partial \sum g_i \sigma_i = \sum g_i \, \partial \sigma_i .$$

It is not difficult to verify:

(1) $\partial : C_q(X, G) \longrightarrow C_{q-1}(X, G)$ is a homomorphism, $q = 1, 2, \ldots$.

(2) $\partial^2 c = \partial \partial c = 0 \; \forall \, c \in C_q(X, G)$.

A different boundary opertor $\partial^{\#}$ can be defined on 0-chains as follows:

$$\partial^{\#} \sum g_i \sigma_i = \sum g_i \qquad \forall \, \sigma_i \in C_0(X, G), \; \forall \, i.$$

The relation

$$\partial^{\#} \partial = 0$$

also holds.

Suppose that (X, Y) is a pair of topological spaces, with $Y \subset X$ (being a subspace of X). We call (X, Y) a topological pair.

For two topological pairs (X, Y) and (X', Y'), we say that a map $f : (X, Y) \longrightarrow (X', Y')$ is continuous if $f : X \longrightarrow X'$ is continuous with $f(Y) \subset Y'$.

Two maps $f, g : (X, Y) \longrightarrow (X', Y')$ are called homotopic if $\exists\, F : [0, 1] \times X \longrightarrow X'$, which is continuous and satisfies

$$F(0, \cdot) = f,\ F(1, \cdot) = g,$$

and

$$F : [0, 1] \times Y \longrightarrow Y'.$$

Let (X, Y) be a topological pair, since

$$\partial : C_q(X, G) \longrightarrow C_{q-1}(X, G)$$

implies

$$\partial : C_q(Y, G) \longrightarrow C_{q-1}(Y, G).$$

The boundary operator induces a homomorphism $\overline{\partial}$ which makes the diagram

$$\begin{array}{ccc} C_q(X,G) & \longrightarrow & C_q(X,G)/C_q(Y,G) \\ \partial \downarrow & & \overline{\partial} \downarrow \\ C_{q-1}(X,G) & \longrightarrow & C_{q-1}(X,G)/C_q(Y,G) \end{array}$$

commutative. Clearly $\overline{\partial}\,\overline{\partial} = 0$. We call

$$C_q(X, Y, G) = C_q(X, G)/C_q(Y, G)$$

the singular q-relative chain module. Then we define

$Z_q(X, Y, G) = \ker(\overline{\partial})$, the singular q-relative closed chain module,
$B_q(X, Y, G) = \mathrm{Im}(\overline{\partial})$, the singular q-relative boundary module, and
$H_q(X, Y, G) = Z_q(X, Y, G)/B_q(X, Y, G)$, the singular q-relative *homology* module. The rank of $H_q(X, Y, G)$ is called the singular q-Betti number.

In the case where $Y = \emptyset$, we write $H_q(X, Y, G) = H_q(X, G)$. For $q = 0$, $H_0^{\#}(X, G)$ is defined as the quotient of $\ker(\partial^{\#})$ by $\mathrm{Im}(\partial)$, and for $q > 0$, let $H_q^{\#}(X, G) = H_q(X, G)$. We call $H_q^{\#}(X, G)$ the q-reduced homology module. The 0-reduced relative homology module $H_0^{\#}(X, Y, G)$ is defined as $H_0(X, Y, G)$ if $Y \neq \emptyset$ and $H_0^{\#}(X)$ if $Y = \emptyset$. The basic properties of singular homology modules are summarized as follows. Their proofs can be found in the book of M. J. Greenberg [Gr 1].

1. Suppose that $f : (X,Y) \longrightarrow (X',Y')$ is continuous, then there is a reduced homomorphism

$$f_* : H_q(X,Y;G) \to H_q(X',Y';G) \ \forall\, q.$$

(a) If $f = \text{id}$, then $f_* = \text{id}$;
(b) If $g : (X',Y') \longrightarrow (X'',Y'')$ is another continuous map, then the reduced homomorphism g_* satisfies

$$(gf)_* = g_* f_*.$$

(c) $\overline{\partial} f_* = f_* \overline{\partial}$.

2. *Homotopy invariance*: If $f,g : (X,Y) \longrightarrow (X',Y')$ are homotopic, then $f_* = g_*$.

Two topological pairs (X,Y) and (X',Y') are called homotopically equivalent if there exist continuous maps

$$\phi : (X,Y) \longrightarrow (X',Y'),$$
$$\psi : (X',Y') \longrightarrow (X,Y),$$

satisfying

$$\psi \circ \phi = \text{id}_{(X,Y)}, \quad \phi \circ \psi \cong \text{id}_{(X',Y')}.$$

Thus, if (X,Y) and (X',Y') are homotopically equivalent, then

$$H_q(X,Y,G) \cong H_q(X',Y',G) \quad \forall\, q.$$

We say (X',Y') is a deformation retract of (X,Y) if $X' \subset X$, $Y' \subset Y$, and if $\exists\, \eta : [0,1] \times X \longrightarrow X$ satisfying

$$\eta(0,\cdot) = \text{id}_X,\ \eta(1,X) \subset X',\ \eta(1,Y) \subset Y',$$
$$\eta(t,Y) \subset Y \text{ and } \eta(t,\cdot)|_{X'} = id_{X'},\ \forall\, t \in [0,1].$$

Thus, if (X',Y') is a deformation retract of (X,Y), then

$$H_q(X',Y',G) \cong H_q(X,Y,G).$$

3. *Excision*: If $U \subset X$ satisfies $\overline{U} \subset \text{int}(Y)$, then

$$H_q(X \backslash U, Y \backslash U, G) \cong H_q(X,Y,G).$$

4. *Exactness*: If $Z \subset Y \subset X$ are three topological spaces, and we define the injections $i : (Y,Z) \longrightarrow (X,Z)$, and $j : (X,Z) \longrightarrow (X,Y)$, then we have the following exact sequence:

$$\cdots \to H_q(Y,Z,G) \xrightarrow{i_*} H_q(X,Z,G) \xrightarrow{j_*} H_q(X,Y,G)$$
$$\xrightarrow{\partial} H_{q-1}(Y,Z,G) \to \cdots.$$

In particular, since $H_q(X,G) = H_q(X,\emptyset,G)$, we have

$$\cdots \to H_q(Y,G) \xrightarrow{i_*} H_q(X,G) \xrightarrow{j_*} H_q(X,Y,G) \xrightarrow{\partial} H_{q-1}(Y,G) \to \cdots .$$

The same exact sequence also holds for reduced homology modules.

5. If X consists of a family of path-connected components $\{X_k\}$, then

$$H_q(X,Y;G) \cong \oplus \sum H_q(X_k, X_k \cap Y; G) \quad \forall\, q.$$

6. $H_q(X,X;G) \cong 0, \quad \forall\, q.$

7. $H_0(X,G)$ is a free group on as many generators as there are path components of X.

If $Y \neq \emptyset, Y \subset X$, and X is path-connected, then

$$H_0(X,Y;G) \cong 0.$$

8. *Künneth formula:* Let X_1 and X_2 be subspaces of the topological space X. Denote $i_\nu : X_\nu \longrightarrow X$ as the injection, $\nu = 1,2$.

(X_1, X_2) is said to be an excisive couple of subspaces if the inclusion chain map

$$C_q(X_1,G) + C_q(X_2,G) \longrightarrow C_q(X_1 \cup X_2, G)$$

induces an isomorphism of homology.

For given topological pairs (X,Y), (X',Y'), we define their product $(X,Y)\times(X',Y')$ to be the pair $(X\times X', X\times Y' \cup Y\times X')$.

If G is a field, and if $\{X\times Y', Y\times X'\}$ is an excisive couple in $X\times X'$, then the cross product is an isomorphism:

$$H_*(X,Y;G)\otimes H_*(X',Y';G) \cong H_*((X,Y)\times(X',Y');G),$$

i.e.,

$$H_q(X\times X', X\times Y' \cup Y\times X';G) \cong \bigoplus_{p=0}^{q} H_q(X,Y;G)H_{q-p}(X',Y';G),$$

$\forall\, q = 0,1,2,\ldots$.

In the case where $G =$ a field Q,

$$\text{rank } H_q(X,Y;Q) = \dim H_q(X,Y;Q),$$

we write

$$\chi(X,Y;Q) = \sum_{q=0}^{\infty}(-1)^q \dim H_q(X,Y;Q),$$

and call it the Euler characteristic of (X,Y).

The following homology groups are often used.

(1) $$H_q(S^n,G)\cong\begin{cases}0 & q\neq n,\text{ when } q,n\geq 1\\ G & q=n\geq 1,\text{ and } q=0,n\geq 1,\\ G^2 & q=n=0.\end{cases}$$

(2) $$H_q(B^n,S^{n-1},G)\cong\begin{cases}0 & q\neq n,\\ G & q=n,\end{cases}$$

where B^n is the n-ball, and $S^{n-1}=\partial B^n$.

(3) $$H_q(T^n,G)\cong\begin{cases}G^{C_q^n} & 0\leq q\leq n,\\ 0 & q>n,\end{cases}$$

where $T^n=S^1\times\cdots\times S^1$ is the n-torus.

(4) $$H_q(P^n,Z_2)\cong\begin{cases}0 & q>n\\ Z_2 & q\leq n,\end{cases}$$

where P^n is the real n-projective space.

(5) $$H_q(CP^n,G)\cong\begin{cases}0 & q>2n\text{ or } q\text{ odd},\\ G & q\text{ even such that } 0\leq q\leq 2n,\end{cases}$$

where CP^n is the complex n-projective space, and $G=Q$, the rational field, or $\mathbb{Z}$.

Now we turn our study to singular cohomology. The singular q-cochain is defined to be the homomorphism $c: C_q(X,G)\longrightarrow G$:

$$[\sigma_1+\sigma_2,c]=[\sigma_1,c]+[\sigma_2,c],\ \forall\,\sigma_1,\sigma_2\in C_q(X,G),$$
$$[g\cdot\sigma,c]=g\cdot[\sigma,c]\ \forall\,g\in G,\ \forall\,\sigma\in C_q(X,G).$$

The set of all singular q-cochains $\mathrm{Hom}(C_q(X,G),G)$ is denoted by $C^q(X,G)$. $C^q(X,G)$ is a module:

$$[\sigma,c_1+c_2]=[\sigma,c_1]+[\sigma,c_2]\ \forall\,c_1,c_2\in C^q(X,G),\ \forall\,\sigma\in C_q(X,G),$$
$$[\sigma,g\cdot c]=g\cdot[\sigma,c],\ \forall\,g\in G,\ \forall\,\sigma\in C_q(X,G),\ \forall\,c\in C^q(X,G).$$

Thus the duality $[\ ,\ ]$ is a bilinear form on $C_q(X,G)\times C^q(X,G)$.

The dual operator of the boundary operator ∂ with respect to $[\ ,\ ]$ is called the coboundary operator and is denoted by δ:

$$[\partial\sigma,c]=[\sigma,\delta c]\ \forall\,\sigma\in C_q(X,G),\ \forall\,c\in C^{q-1}(X,G).$$

Hence, $\delta: C^{q-1}(X,G) \longrightarrow C^q(X,G)$ is a homomorphism, and $\partial^2 = 0$ implies

$$\delta^2 c = 0 \;\forall\; c \in C^q(X,G).$$

Singular cohomology is defined as follows: For a topological pair (X,Y), let

$$\overline{C}^q(X,Y;G) = \operatorname{Hom}(C_q(X,G)/C_q(Y,G),G),$$

and let

$$\overline{\delta}: \overline{C}^{q-1}(X,Y) \longrightarrow \overline{C}^q(X,Y)$$

be the dual operator of the boundary operator $\overline{\partial}: C_q(X,Y;G) \longrightarrow C_{q-1}(X,Y;G)$. Then define

$$H^q(X,Y;G) = \ker(\overline{\delta}) / \operatorname{Im}(\overline{\delta}).$$

It is easily seen that $\overline{C}^q(X,Y;G)$ is isomorphic to

$$C^q(X,Y;G) = \{c \in C^q(X,G) \mid [\sigma,c] = 0 \qquad \forall\, \sigma \in C_q(Y,G)\}.$$

The isomorphism is realized by the dual homomorphism

$$P^*: \overline{C}^q(X,Y;G) \longrightarrow \overline{C}^q(X,G)$$

of the homomorphism

$$P: C_q(X,G) \longrightarrow C_q(X,Y;G).$$

Therefore

$$Z^q(X,Y;G) := \ker(\overline{\delta}) = \{c \in C^q(X,G) | [\sigma,c] = 0 \;\forall\; \sigma \in B_q(X,Y;G)\},$$
$$B^q(X,Y;G) := \operatorname{Im}(\overline{\delta}) = \{c \in C^q(X,G) | [\sigma,c] = 0 \;\forall\; \sigma \in Z_q(X,Y;G)\}.$$

In general, we have a canonical homomorphism:

$$\alpha: H^q(X,Y;G) \longrightarrow H_q(X,Y;G)^*.$$

In the case where G is a field, α is surjective.

The properties of cohomology are very similar to those of homology. The important difference is as follows: Singular homology is a covariant functor of topological pairs, but singular cohomology is a contravariant functor.

(1′) If $f: (X,Y) \longrightarrow (X',Y')$ is continuous, then

$$f^*: H^*(X',Y',G) \longrightarrow H^*(X,Y;G)$$

We have

(a) If $f = \mathrm{id}$, then $f^* = \mathrm{id}$.
(b) If $g : (X', Y') \longrightarrow (X'', Y'')$ is continuous, then $(gf)^* = f^* g^*$.
(c) $\overline{\delta} f^* = f^* \overline{\delta}$.

(2′) If, $f, g : (X, Y) \longrightarrow (X', Y')$ are homotopic, then $f^* = g^*$. If $(X, Y) \sim (X', Y')$, then $H^*(X, Y; G) \cong H^*(X', Y'; G)$.

(3′) (Excision) $H^*(X \backslash U, Y \backslash U; G) \cong H^*(X, Y; G)$, if $\overline{U} \subset \mathrm{int}(Y)$.

(4′) (Exactness) If $Z \subset Y \subset X$, then the sequence

$$\cdots \longleftarrow H^q(Y, Z; G) \xleftarrow{i^*} H^q(X, Z; G) \xleftarrow{j^*} H^q(X, Y; G) \xleftarrow{\overline{\delta}} H^{q-1}(Y, Z; G) \longleftarrow \cdots$$

is exact.

(5′) $H^q(\{p\}, G) = \begin{cases} G & q = 0 \\ 0 & q \neq 0. \end{cases}$

(6′) If $(X \times Y', Y \times X')$ is an excisive couple in $X \times X'$, and $H^*(X, Y; G)$ is of finite type, i.e., $H^q(X, Y; G)$ is finitely generated for each q, and G is a field, then

$$H^*(X, Y; G) \otimes H^*(X', Y'; G) \cong H^*((X, Y) \times (X', Y'); G).$$

We can define a product on singular cohomolgy groups such that the singular cohomology groups become graded algebras.

We denote $C^*(X, G) = \oplus_{q=0}^{\infty} C^q(X, G)$, and define a cup product as follows: $\forall\, c \in C^p(X, G)$, $\forall\, d \in C^q(X, G)$, $\forall\, \sigma \in C_{p+q}(X, G)$, we consider affine maps

$$\lambda_p : \Delta_p \longrightarrow \Delta_{p+q}$$
$$\rho_q : \Delta_q \longrightarrow \Delta_{p+q}$$

to be

$$\lambda_p = (e_0, \ldots, e_p), \quad \rho_q = (e_p, e_{p+1}, \ldots, e_{p+q}).$$

and then define

$$[\sigma, c \cup d] = [\sigma \lambda_p, c] \cdot [\sigma \rho_q, d].$$

The cup product is bilinear, associative, and possesses the unit, i.e., the 0-cochain 1, which is defined by $[x, 1] = e\ \forall\, x \in X$.

We may easily prove that

$$\delta(c \cup d) = \delta c \cup d + (-1)^p c \cup \delta d \quad \forall\, c \in C^p(X, G),\ \forall\, d \in C^q(X, G).$$

Hence, $Z^*(X, G)$ is a subalgebra of $C^*(X, G)$ and $B^*(X, G)$ is an ideal of $Z^*(X, G)$. The cup product $\cup$ is well defined on $H^*(X, G)$, and makes it a graded algebra. Furthermore, if $f : X \longrightarrow Y$ is continuous, then $f^* : H^*(Y) \longrightarrow H^*(X)$ is a ring homomorphism: $f^*(c \cup d) = f^*(c) \cup f^*(d)$, which satisfies $f^* \delta = \delta f^*$.

The cap product is defined as the dual operator of the cup product, i.e., $\cap : C_{p+q}(X) \times C^p(X) \longrightarrow C_q(X)$,

$$\forall\, c \in C^p(X),\ \forall\, d \in C^q(X),\ \forall\, \sigma \in C_{p+q}(X),$$
$$[\sigma \cap c, d] = [\sigma, c \cup d],$$

or, equivalently,

$$\sigma \cap c = [\sigma\lambda_p, c]\sigma\rho_q.$$

The boundary operators relate the cap product as follows:

$$\partial(\sigma \cap c) = (-1)^p[(\partial\sigma) \cap c - \sigma \cap \delta c].$$

$\forall\, \sigma \in C_{p+q}(X)$, $\forall\, c \in C^p(X)$.

If $f : X \longrightarrow Y$ is continuous, then we have

$$f_*[\sigma \cap f^*(c)] = f_*(\sigma) \cap c.$$

Since $\forall\ \sigma\ \in\ Z_{p+q}(X)$, $\forall\ c\ \in\ Z^p(X)$, we have $\sigma \cap c \in Z_q(X)$, and $\forall\, \sigma \in B_{p+q}(X)$, $\forall\, c \in Z^p(X)$, we have $\sigma \cap c \in B_q(X)$, the cap product is well-defined on homology groups:

$$\cap : H_{p+q}(X) \times H^p(X) \longrightarrow H_q(X).$$

The definition of cup product and cap product can be extended to topological pairs. In fact, we have

$$\cap : H_{p+q}(X, Y; G) \times H^p(X, Y; G) \longrightarrow H_q(X, G)$$
$$\cap : H_{p+q}(X, Y; G) \times H^p(X, G) \longrightarrow H_q(X, Y; G),$$

and

$$\cup : H^p(X, Y_1; G) \times H^q(X, Y_2 : G) \longrightarrow H^{p+q}(X, Y_1 \cup Y_2; G),$$

if (Y_1, Y_2) is an excisive couple in X.

The cup length of a topological space X is defined as

$$CL(X) = \max\{l \in Z_+ \mid \exists\, c_1, ..., c_l \in H^*(X, G),$$
$$\dim(c_i) > 0, i = 1, \ldots, l, \text{ such that } c_1 \cup \cdots \cup c_l \neq 0\}.$$

This is a topological invariant which is very useful in critical point theory.

More generally, we define the cup length for a topological pair (X, Y):

$$CL(X, Y) = \max\{l \in Z_+ \mid \exists\, c_0 \in H^*(X, Y),\ \exists\, c_1, c_2, \ldots, c_l\ \in H^*(X),$$

with $\dim(c_i) > 0$, $i = 1, 2, \ldots, l$, such that $c_0 \cup c_1 \cup \cdots \cup c_l \neq 0\}$.

In the case where $Y = \emptyset$, we just take $c_0 \in H^0(X)$. These two definitions are the same.

We may characterize $CL(X, Y)$ by its dual.

Definition 1.1. Let (X, Y) be a pair of topological spaces and $Y \subset X$. For two nontrivial singular homology classes $[\sigma_1], [\sigma_2] \in H_*(X, Y)$, we say that $[\sigma_1]$ is subordinate to $[\sigma_2]$, denoted by $[\sigma_1] < [\sigma_2]$, if there exists $c \in H^*(X)$, with $\dim c > 0$ such that

$$[\sigma_1] = [\sigma_2] \cap c,$$

where $\cap$ is the cap product.

Let us define

$$\begin{aligned} L(X, Y) = \max \{ l \in Z_+ \mid \exists \text{ nontrivial classes } [\sigma_j] \in H_*(X, Y), \\ 1 \le j \le l, \text{ such that } [\sigma_1] < [\sigma_2] < \cdots < [\sigma_l] \}. \end{aligned}$$

Theorem 1.1. $L(X, Y) = CL(X, Y) + 1$.

Proof. For $L(X, Y) = l + 1$ if and only if $\exists$ nontrivial classes $[\sigma_0] < [\sigma_1] < \cdots < [\sigma_l]$ in $H_*(X, Y)$, i.e., $\exists\ c_i \in H^*(X)$, dim $c_i > 0, 1 \le i \le l$, such that

$$[\sigma_{i+1}] = [\sigma_i] \cap c_i,\ i = 1, 2, \ldots, l.$$

However, $\exists\ c_0 \in H^*(X, Y)$ such that $[\,[\sigma_0], c_0\,] \neq 0$ is equivalent to the nontriviality of $[\sigma_0]$. And since

$$\begin{aligned} [[\sigma_l], c_l \cup c_{l-1} \cup \cdots \cup c_0] &= [[\sigma_{l-1}], c_{l-1} \cup c_{l-2} \cup \cdots \cup c_0] \\ &= \cdots = [[\sigma_0], c_0], \end{aligned}$$

$\therefore\ L(X, Y) = l + 1$ if and only if $CL(X, Y) = l$. □

The homotopy group is another important topological invariant. Let us recall some basic definitions and properties in homotopy theory.

Let X be a topological space and p be a point in X. We call (X, p) a pointed space with base point p. A topological pair (X, Y), in which Y is a subspace of X that contains p, is called a pointed pair (often written (X, Y, p)).

A map f from pointed space (X, p) to a pointed space (X', p'), $f : X \to X'$, with $f(p) = p'$, is called a pointed map. Similarly, we define a pointed pair map, pointed homotopy, pointed pair homotopy, and so forth.

Let I^n denote the n-dimensional unit cube, $n \ge 1$, $I^{n-1} \subset I^n$ the bottom space ($t = (t_1, t_2, \ldots, t_n) \in I^n$, if $0 \le t_i \le 1$, $i = 1, 2, \ldots, n$, and $t \in I^{n-1}$