TOPOLOGY

LEFSCHETZ

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BY

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PREFACE

Perhaps on no branch of mathematics did Poincaré lay his stamp more indelibly than on topology. To him we owe the basic notion of complex, the boundary relations and related numerical invariants, the first duality theorem. But Poincaré quickly centered his efforts upon the classification of manifolds and other questions, leaving the foundations in rather unstable equilibrium. It is largely to Veblen and Alexander that we owe the remedy for this state of affairs, and the present improved situation. A date marks the transition: 1922, when there appeared Veblen's excellent *Cambridge Colloquium Lectures: Analysis Situs*, which has deservedly become the standard work on the subject. The ground being thus well prepared, new developments came rapidly, and it is with the new phases of the subject that we shall be chiefly concerned here.

Our central topic is the theory of manifolds for which the first two chapters are the preparation. The manifolds are of Alexander's combinatorial type. While more general than the usual type their treatment is scarcely more difficult, and in addition it has been possible to prove that they are topologically invariant (van Kampen). They may be open or closed complexes, and in the former case no boundary conditions are imposed. The principal questions treated will include the general theory of the homology characters, their duality theorems, the intersection theory, the study of continuous transformations, their coincidences and fixed points, the applications to analytical and algebraic varieties. In all these directions we are taking up again questions with which we have dealt in recent papers (Transactions of the American Mathematical Society, 1926, 1927, Annals of Mathematics, 1927–1930), recasting and greatly extending our results. In the duality theory it will be recalled that two unrelated basic formulas, due to Poincaré and Alexander, were reduced to a single formula for relative invariants. This work of unification is carried still further, with new results, giving considerably greater reach to our formulas for coincidences and fixed points. We have succeeded, in fact, in extending them to arbitrary compact metric spaces, a fundamental result for real analysis.

Regarding the intersection theory, our first treatment was based on intersections of convex cells. This is an excellent procedure from the geometric standpoint and for the applications. Howewer, for the basic invariance proofs a procedure followed by Veblen-Weyl in the simplest case (isolated intersections) is to be preferred and we have adopted it here. Unfortunately, in their scheme the intersecting elements are in a very special mutual relationship, which makes it awkward in practise. We have therefore introduced a third and more general type of intersections, based on Brouwer's looping coefficients, which includes the other two types as special cases, and thus acts as the unifying element.

Side by side with general topological invariance, one may consider more strictly combinatorial invariance, or invariance under subdivision of the cells of a complex. In its treatment continuity should play no part, and its study constitutes combinatorial topology proper. While we have made a mild effort to separate the two theories, and to establish combinatorial invariance wherever readily possible, we have taken as our program the study of complexes and manifolds from any point of view whatever. In short, in the combinatorial theory, we have not adopted the abstract and postulational point of view of Dehn-Heegaard in their Encyklopädie article. We have taken configurations immersed in some Euclidean number space, and made free use of the elementary properties of the space as regards intersections of its linear subspaces. These properties, and the elementary properties of the rational number system, are all that we assume when we deal with the combinatorial theory.

Much of the topological part proper has been put in the seventh chapter, devoted to the infinite complex. It seems to be just what is wanted for the best treatment of that type of question. Infinite manifolds have already been studied by Kerékjártó, largely in the spirit of the classical theory of finite two dimensional manifolds. We are giving here however for the first time a general theory of infinite complexes and manifolds with their applications to metric spaces. Closely related recent contributions are the approximations to compact metric spaces by finite complexes introduced by Alexandroff, and the homology theory of these spaces due to Vietoris.

Perhaps the most novel feature of this book is the rôle played by the so-called "relative" concepts. It appears that almost all boundary relations and related homologies may be replaced by others in which we omit everything that is not on a certain configuration *A*, thus obtaining analogous relations relative *A*. For example it is advantageous to consider an open segment as a relative one-circuit, a plane polygonal region as a relative manifold. This is entirely in line with the "relative" concepts of point set theory as defined, say, by Hausdorff.

When the relative concepts are systematically introduced, it is found that a good part of the combinatorial theory of Poincaré, Veblen, Alexander, also our own results on continuous transformations, etc. have a greatly increased range of application. As a minor, but highly interesting feature, Brouwer's classical results on the invariance of dimensionality and regionality are found to be intimately tied up with the invariance of the homology characters.

References are indicated by the author's name, followed by a number in square brackets which corresponds to the bibliography at the end.

Valuable assistance was given me by several past or present young members of the Princeton group. Dr. D. A. Flanders was my collaborator in connection with the first two chapters and Dr. L. W. Cohen with Ch. VII § 4. The fairly complete bibliography is largely due to the efforts of Dr. A. B. Brown who also carefully read the proofs, and in this connection made many valuable suggestions. Messrs. W. W. Flexner, J. Singer, and A. W. Tucker helped me also in various ways, especially with the first three chapters, and in addition Mr. Singer made all the drawings. To one and all of them I wish to address here my sincere thanks.

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PREFACE TO THE SECOND EDITION

This second edition only differs from the first in the insertion of a collection of errata from a large list provided by Mr. R. S. Pieters. A couple of the corrections which it was too difficult to insert in the text have been relegated to a list of addenda at the end of the volume. These are marked in the text by the superscript [a]. The author wishes to express his deep appreciation to Mr. Pieters for his list, and likewise to the Chelsea Publishing Company for having undertaken the arduous task of preparing this second edition.

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INTRODUCTION

1. A rapid glance over the foundations of our subject, besides its intrinsic value, will also provide an excellent occasion for introducing various important concepts needed later. These foundations rest upon the theory of abstract spaces founded by Fréchet* [1]. On these questions the reader will consult with profit the interesting papers by Chittenden [1, 2].

2. *Topology* or *Analysis Situs* is usually defined as the study of properties of spaces or their configurations invariant under continuous transformations. But what are *spaces* and their *continuous* transformations?

Whatever a *space* may be it is difficult to reconcile it with anything conforming with our "spatial" intuition unless it is endowed with the following property, the most salient and primitive possessed by the familiar types: With each point there goes a portion of the space in which it is imbedded. This leads to the conception of an abstract space $\mathfrak{R}$ as a set of elements $\{x\}$, its *points*, together with an aggregate of subsets $\{N\}$, the *neighborhoods*, so chosen as to satisfy:

---

*The general symbolism of the theory of sets will be used throughout. A set of elements, any one of which is a, is designated by $\{a\}$. Let A, B be two sets. Then the set of all elements

in A or in B $= A + B =$ the *sum* of A and B;
in A but not in $B = A - B =$ the *complement* of B with respect to A;
in A and also in $B = A \cdot B =$ the *intersection* of A and B.

$A \supset B$, or $B \subset A$ mean that B is a subset of A. When B coincides with A we write $B = A$. Negation of a relation is designated by drawing a line through its symbol, as $A \not\subset B$, $A \neq B$, etc.

Axiom I. Between the sets $\{x\}$, $\{N\}$, there exists a correspondence such that to each x there corresponds one or more N's, the neighborhoods N^x of x. Every $N^x \supset x$.

This is the first of four axioms due to Hausdorff. The other three are given below and are of a more restrictive nature.

3. We keep $\mathfrak{R}$ fixed for the present and consider certain attributes of its subsets. It is surprising how far it is possible to go even on such a slender foundation as Axiom I.

Let A be a subset of $\mathfrak{R}$. The point x is an *inner* point of A whenever there is an $N^x \subset A$, which by Axiom I implies $x \subset A$.

A point x is a *boundary* point of A if every N^x includes points of both A and of its complement $\mathfrak{R} - A$.

The *boundary* $F(A)$ of A is the sum of the boundary points of A. The set A is *open* when $F(A) \subset \mathfrak{R} - A$, *closed* when $F(A) \subset A$. An open set consists solely of inner points. A closed set is the complement of an open set.

If $A \subset N \subset \mathfrak{R}$, A closed, N open, then N is called a *neighborhood* of A, and often designated by N^A, $\mathfrak{R}^A$, etc.

The sum $A + F(A)$ is called the *closure* of A and is designated by $\bar{A}$. It is a set function defined for *all* subsets of $\mathfrak{R}$ and will play a particularly important part in the sequel.

A set A is *dense* on $\mathfrak{R}$ whenever there are points of A on every N. An equivalent definition is that $\bar{A} = \mathfrak{R}$.

The point x is a *limit point* of A whenever A has points on every $N^x - x$. The closure $\bar{A} = A +$ all its limits points.

The infinite sequence $\{x_n\}$ is said to *converge to the limit* x whenever each N^x contains all but a finite number of points of the sequence.

Any given subset of $\mathfrak{R}$, say A, may be turned into a space by agreeing to choose as neighborhoods for A its intersections $N \cdot A$ with those of $\mathfrak{R}$. This convention will be adhered to in the sequel. It is then possible to introduce the sets open or closed *on* A, etc.; these are the well known *relative* concepts of point-set theory.

4. A *transformation* T of a space $\mathfrak{R}$ into another $\mathfrak{R}'$ is the assignment of a correspondence between their points such that to every point x of $\mathfrak{R}$ there corresponds at least one point $x' = T \cdot x$ of $\mathfrak{R}'$. The transform $T \cdot A$ of a subset A of $\mathfrak{R}$ is the set of all points $T \cdot x$ for $x \subset A$. The transformation is:

Single-valued whenever $x' = T \cdot x$ is unique for every x.

One-to-one when it assigns also a unique x to every x', thus pairing off in a unique way the points of $\mathfrak{R}$ and $\mathfrak{R}'$.

Continuous if whenever the transform $T \cdot A$ is relatively open for $T \cdot \mathfrak{R}$, then A is open for $\mathfrak{R}$.

Homeomorphic when it is one-to-one and continuous both ways.

We have now all the elements needed for a formal

DEFINITION OF TOPOLOGY: *It is the study of the properties of spaces that are preserved under homeomorphism.*

Any property of a space $\mathfrak{R}$ which belongs to every homeomorph of $\mathfrak{R}$ is said to be *topologically invariant.* A numerical function attached to $\mathfrak{R}$ which has the same value for all its homeomorphs is a *numerical topological invariant.* As an example, when $\mathfrak{R}$ is a sphere the fact that any simple closed curve on it decomposes it into two regions is topologically invariant, the number *two* is a numerical topological invariant.

The concepts introduced in No. 3 are topologically invariant: Thus when $\mathfrak{R}$ is homeomorphically transformed into $\mathfrak{R}'$, its closed (open) sets, etc. go over into similar sets for $\mathfrak{R}'$.

5. Given an abstract set $\mathfrak{R}$, it may be possible to convert it into a space by two distinct choices of subsets as neighborhoods, say $\{N\}$ and $\{N'\}$. We have thus two different spaces $\mathfrak{R}, \mathfrak{R}'$, built up out of the same abstract set of elements. The two spaces are considered as *identical* whenever the identical point-to-point transformation between them is homeomorphic. This means that any subset A of $\mathfrak{R}$ which is open when the N's are the fundamental neighborhoods remains open when they are replaced by the N''s and conversely. The two sets of neighborhoods are then called *equivalent.* The space $\mathfrak{R}$ is not so much characterized by a single ag-

gregate of neighborhoods as by a whole class of equivalent sets of them.

Since identical spaces are homeomorphic the concepts introduced in No. 3 are unchanged when a set of neighborhoods is replaced by an equivalent set. From the very general type of space considered so far we proceed to what might be termed "working spaces" by imposing restrictive axioms upon the neighborhoods. The set in widest use today goes back to Hausdorff [1] and characterizes what we shall call a *Hausdorff space.* It consists of Axiom I plus the following three:

Axiom II. If N, N' are any two neighborhoods of x, there exists a third neighborhood of x, $N'' \subset N \cdot N'$.

Axiom III. If the point $y \subset N^x$, there is an $N^y \subset N^x$.

Axiom IV. If $x \neq y$ there exist N^x, N^y without common points.

Among the Hausdorff spaces are included the usual simple types: projective, Euclidean, function, n-spaces, with simplexes (generalized triangles) as the neighborhoods.

A necessary and sufficient condition for the equivalence of two coincident Hausdorff spaces $\mathfrak{R}, \mathfrak{R}'$, defined by two distinct sets of neighborhoods $\{N\}$, $\{N'\}$ is that every N^x carry an N'^x and conversely.

6. A Hausdorff space $\mathfrak{R}$ is:

Regular whenever Axiom III may be replaced by the stronger condition: When $y \subset N^x$, there is an $\bar{N}^y \subset N^x$.

Separable (Fréchet) whenever it is identical with a space whose set of neighborhoods is enumerable.

Compact (Fréchet) whenever every infinite subset of $\mathfrak{R}$ has a limit point on $\mathfrak{R}$.

An important consequence of separability is the presence on $\mathfrak{R}$ of an enumerable dense subset: For if $\{N^i\}$ is the enumerable set of neighborhoods, we can take a point P^i on N^i and the sequence $\{P^i\}$ is dense on $\mathfrak{R}$.

The space whose points are all the real numbers with $\{N\}$ as the intervals whose end points are rational, is separable but not compact, since the set of all integers has no limit point.

7. The definition of a space by means of neighborhoods is, so to speak, purely qualitative. However, one of the most important properties of the more common spaces is that they can serve as a basis for making measurements, that a *metric* can be affixed to them. If the notion of neighborhood deserves logically the first place, nevertheless in all applications of mathematics the existence of a metric is the essential element.

If we take the metric as the basic factor the natural definition of a space $\mathfrak{R}$ assumes the following form: It is an abstract set of elements, its points, such that to their pairs x, y there may be attached a *distance function*, $d(x, y)$, or real function having the following properties of Euclidean distances:

I. $d(x, y) = 0$ when and only when $x = y$.

II. For any three points x, y, z of $\mathfrak{R}$ the *triangle* inequality holds, i. e.

$$d(y, z) \leqq d(x, y) + d(x, z).$$

From these two properties one may deduce, as observed by Lindenbaum [1], the other two usually ascribed to the distance:

III. $d(x, y) \geqq 0$.

It is obtained by making $z = y$ in II and taking into account I.

IV. $d(x, y) = d(y, x)$.

Making $z = x$ in II we have $d(y, x) \leqq d(x, y)$. Similarly $d(x, y) \leqq d(y, x)$, from which IV follows.

The *distance* $d(A, B)$, between two sets $A, B \subset \mathfrak{R}$, is the greatest lower bound of $d(x, y)$ for $x \subset A, y \subset B$. The *diameter* of A, $d(A)$, is the least upper bound of $d(x, y)$ for $x, y \subset A$. By *spheroid of center* A *and radius* ϱ, $\mathfrak{S}(A; \varrho)$, we understand the set of all points x such that $d(A; x) < \varrho$.

The spheroids $\mathfrak{S}(x, \varrho)$ *whose centers are the points* x *of* $\mathfrak{R}$ *constitute a set of neighborhoods which turn* $\mathfrak{R}$ *into a regular Hausdorff space.* The verification is instantaneous. The Hausdorff space thus defined is considered as identical with $\mathfrak{R}$, and continues to be designated by $\mathfrak{R}$. A practical result of importance is that *on a metric space the limit concept can be* *put in the same form as in the familiar ε, δ procedure of elementary analysis.*

For a metric space separability is equivalent to the existence of an enumerable dense subset: We have seen (No. 6), that such a set is present on every separable space. On the other hand, $\mathfrak{R}$ being metric, if $A = \{x_n\}$ is dense on it, the spheroids $\mathfrak{S}(x_n; \varrho_m)$, ϱ_m rational, are enumerable and their set is equivalent to $\{\mathfrak{S}(x; \varrho)\}$, x any point of $\mathfrak{R}$, ϱ arbitrary. Hence $\mathfrak{R}$ is separable.

The fundamental question that must be answered in order to bridge the two points of view: metric and qualitative, is then: Under what conditions is a Hausdorff space *metrisable*, (susceptible of having a distance function affixed to it)? In the light of the preceding remarks it is largely answered for all spaces that offer any interest by the following fundamental theorem due to Urysohn [1, 2], with an important complement regarding the condition of regularity by Tychonoff [1]:

In order that a separable Hausdorff space be metrisable, it is necessary and sufficient that it be regular.

As a rapid Corollary (Urysohn):

In order that a compact Hausdorff space be metrisable it is necessary and sufficient that it be separable.

We are then in possession today of a strictly qualitative characterization of metric spaces with an enumerable dense subset (separable metric spaces). These spaces include the most general type to be considered in any connection whatever in the present work. The preceding sketch is a sufficient description of their foundations to cover all our needs.

8. An important result obtained in recent years is the proof by Alexandroff [10] that compact metric spaces can be indefinitely approximated by certain polyhedral configurations, the *complexes.* The topological theory of complexes acquires thus a fundamental importance; it is in truth the necessary first step in a general study of metric spaces.