Gerald B. Folland

Harmonic Analysis in Phase Space

相空间中的调和分析

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PREFACE

The phrase "harmonic analysis in phase space" is a concise if somewhat inadequate name for the area of analysis on $\mathbf{R}^n$ that involves the Heisenberg group, quantization, the Weyl operational calculus, the metaplectic representation, wave packets, and related concepts: it is meant to suggest analysis *on* the configuration space $\mathbf{R}^n$ done by working *in* the phase space $\mathbf{R}^n \times \mathbf{R}^n$. The ideas that fall under this rubric have originated in several different fields—Fourier analysis, partial differential equations, mathematical physics, representation theory, and number theory, among others. As a result, although these ideas are individually well known to workers in such fields, their close kinship and the cross-fertilization they can provide have often been insufficiently appreciated. One of the principal objectives of this monograph is to give a coherent account of this material, comprising not just an efficient tour of the major avenues but also an exploration of some picturesque byways.

Here is a brief guide to the main features of the book. Readers should begin by perusing the Prologue and perhaps refreshing their knowledge about Gaussian integrals by glancing at Appendix A.

Chapter 1 is devoted to the description of the representations of the Heisenberg group and various integral transforms and special functions associated to them, with motivation from physics. The material in the first eight sections is the foundation for all that follows, although readers who wish to proceed quickly to pseudodifferential operators can skip Sections 1.5–1.7.

The main point of Chapter 2 is the development of the Weyl calculus of pseudodifferential operators. As a tool for studying differential equations, the Weyl calculus is essentially equivalent to the standard Kohn-Nirenberg calculus—in fact, this equivalence is the principal result of Section 2.2—but it is somewhat more elegant and more natural from the point of view of harmonic analysis. Its close connection with the Heisenberg group yields some insights which are useful in the proofs of the Calderón-Vaillancourt (0,0) estimate and the sharp Gårding inequality in Sections 2.5 and 2.6 and in the arguments of Section 3.1. Since my aim is to provide a reasonably accessible introduction rather than to develop a general theory (in contrast to Hörmander [70]), I mainly restrict attention to the standard symbol classes $S^m_{\rho,\delta}$. Moreover, I assume that the relevant estimates on symbols and their derivatives hold uniformly on all of $\mathbf{R}^n$ rather than on compact sets. This simplification makes the theory cleaner without restricting its generality in an essential way, as the study of localized symbols can generally be reduced to the study of global ones by standard tricks involving cutoff functions.

Chapter 3 grew out of my attempt to understand the Córdoba-Fefferman paper [33] on wave packet transforms in the context of the Weyl calculus. What has resulted, in Sections 3.1 and 3.2, is a new approach to their results which shows that, for some purposes, their Gaussian wave packets can be replaced by arbitrary (nonzero) even Schwartz class functions.

Chapter 4 is devoted to the metaplectic representation. It is more comprehensive than most other accounts in the literature, but it is still only an introduction. An exhaustive discussion of the many facets and applications of this beautiful representation and its siblings and children would require a book by itself.

Finally, Chapter 5 is my own retelling of some recent work of R. Howe [76], whom I wish to thank for permission to include his results. It ties together a number of strands from the previous chapters and provides, in my opinion, a satisfying conclusion to the book.

One problem with writing a book like this is deciding what background to expect of the readers. Basically, I take for granted a knowledge of real analysis, Fourier analysis, and basic functional analysis such as can be found in the first eight chapters of my text [50]; on this foundation, plus a few additional facts from functional analysis and Lie theory that are needed here and there, the book is pretty much self-contained. However, the material in it impinges on a number of subjects, including partial differential equations, spectral theory and the analysis of self-adjoint operators, Hamiltonian mechanics, quantum mechanics, Lie groups, and representation theory. Readers who are acquainted with these subjects will find their appreciation enhanced thereby; those who are not will find a few places where the path is hard to follow, but they are urged to forge resolutely ahead.

Another problem is deciding where to stop. I have allowed the selection of topics treated in detail to be governed by personal taste, while providing an extensive list of references to related material. These references, together with the references *they* contain, should be enough to keep anyone busy for quite a while. Nonetheless, many readers will undoubtedly ask at some point or other, "Why didn't you mention topic X or the work of Y?" In some cases I can plead the necessity of keeping the scope of the book within reasonable bounds, but in others the answer must be—as Samuel Johnson said when asked why he had given an erroneous definition in his Dictionary—"Ignorance, madam, sheer ignorance."

The seeds in my mind from which this book grew began to germinate after I attended the conference on Harmonic Analysis and Schrödinger Equations at the University of Colorado in March 1986. The book took shape during my sabbatical in the following academic year. In particular, I gave a series

of lectures at the Bangalore Centre of the Indian Statistical Institute in the winter of 1987 that constituted a sort of first draft of Chapters 1, 2, and 4. I am grateful to the Indian Statistical Institute, the University of New South Wales, the Australian National University, and especially my friends on their faculties, for providing me with extremely congenial environments in which to work during the first half of 1987.

Gerald B. Folland

Seattle, Washington
June, 1988

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HARMONIC ANALYSIS IN PHASE SPACE

PROLOGUE.
SOME MATTERS OF NOTATION

Readers are urged to examine this material before proceeding further.

Integrals. The integral of a function f on $\mathbf{R}^n$ or $\mathbf{C}^n$ with respect to Lebesgue measure will be denoted by $\int f(\xi)\,d\xi$ where ξ is any convenient dummy variable, be it real or complex. Thus, if z is a (scalar) complex variable, dz denotes the element of area on $\mathbf{C}$ and not the holomorphic differential used in defining contour integrals.

Functions and Distributions. We use the standard notation of [50] for function spaces on $\mathbf{R}^n$ or $\mathbf{C}^n$. For example, $L^p(\mathbf{R}^n)$ is the L^p space with respect to Lebesgue measure, $\|f\|_p$ is the L^p norm of f, and $C_c^\infty(\mathbf{R}^n)$ is the space of C^∞ functions with compact support. The inner product on L^2 is denoted by

$$(1) \qquad \langle f, g\rangle = \int f(x)\overline{g(x)}\,dx.$$

Inner products on other Hilbert spaces will also be denoted by $\langle\cdot,\cdot\rangle$ or by $\langle\cdot,\cdot\rangle_*$ where $*$ is a subscript to label the Hilbert space in question.

$\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{S}'(\mathbf{R}^n)$ are the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions, respectively. "Convergence in $\mathcal{S}'$" always means convergence in the weak $*$ topology on $\mathcal{S}'$. The pairing between $\mathcal{S}$ and $\mathcal{S}'$ will be denoted either by integrals or by pointed brackets, in a manner consistent with equation (1). Thus, if $f \in \mathcal{S}'$ and $\phi \in \mathcal{S}$ we write

$$\int \phi(x)f(x)\,dx = \lim \int \phi(x)f_j(x)\,dx,$$

$$\langle \phi, f\rangle = \overline{\langle f, \phi\rangle} = \lim \int \phi(x)\overline{f_j(x)}\,dx,$$

where $\{f_j\}$ is any sequence of smooth functions that converges in $\mathcal{S}'$ to f. In general, we shall be quite cavalier about writing distributions as if they were functions, especially under integral signs. δ will denote the Dirac distribution defined by $\langle \delta, f\rangle = f(0)$.

Of fundamental importance for us are the operators X_j and D_j on distributions on $\mathbf{R}^n$ defined by

$$(2) \qquad (X_j f)(x) = x_j f(x), \qquad D_j f = \frac{1}{2\pi i}\frac{\partial f}{\partial x_j}.$$

We generally regard X_j and D_j as continuous operators on $\mathcal{S}$ or $\mathcal{S}'$. When we regard them as unbounded operators on L^2, their domains are the obvious maximal ones: the domain of X_j is the set of $f \in L^2$ such that $X_j f \in L^2$, and likewise for D_j.

Matrices and Vectors. We denote by $M_n(\mathbf{R})$ and $M_n(\mathbf{C})$ the space of $n \times n$ matrices over $\mathbf{R}$ and $\mathbf{C}$, respectively. We identify linear endomorphisms of $\mathbf{R}^n$ and $\mathbf{C}^n$ with their matrices with respect to the canonical basis, and hence think of elements of $M_n(\mathbf{R})$ and $M_n(\mathbf{C})$ as either matrices or linear maps, according to context. The transpose and Hermitian adjoint of a matrix A are denoted by $A^\dagger$ and A^*; for real matrices, when these two notions coincide, we generally use the notation A^*. We employ the standard notation for the classical groups of invertible matrices: $GL(n,\mathbf{R})$, $U(n)$, $SO(n)$, etc. The $n \times n$ identity matrix is denoted by I_n when precision is needed, but more often simply by I. For powers of determinants, we use the convention that is common for trig functions: $\det^\alpha A = (\det A)^\alpha$.

Except in a few instances where clarity demands otherwise, we denote the dot product of two vectors in $\mathbf{R}^n$ or $\mathbf{C}^n$ by simple juxtaposition:

$$xy = \sum_1^n x_j y_j \qquad (x, y \in \mathbf{R}^n \text{ or } \mathbf{C}^n).$$

Thus, the Hermitian inner product of $z, w \in \mathbf{C}^n$ is $z\overline{w}$. We also set

$$x^2 = xx = \sum x_j^2 \qquad (x \in \mathbf{R}^n \text{ or } \mathbf{C}^n),$$
$$|z|^2 = z\overline{z} = \sum |z_j|^2 \qquad (z \in \mathbf{C}^n).$$

When linear mappings intervene in such products, we shall generally take care to write them between the two vectors, thus:

$$xAy = yA^\dagger x = x \cdot Ay = \sum x_j A_{jk} y_k \qquad \big(x, y \in \mathbf{C}^n,\ A \in M_n(\mathbf{C})\big).$$

The notation xAy can be regarded as a shorthand for either the matrix product $x^\dagger Ay$ (where x and y are regarded as column vectors) or the physicists' bra–ket notation $\langle \overline{x}|A|y\rangle$. These "dotless products" may look a bit peculiar at first, but they are usually very efficient.

One other bit of notation for vectors will be frequently used in connection with pseudodifferential operators: if $\xi \in \mathbf{R}^n$,

$$\langle \xi \rangle = \big(1 + \xi^2\big)^{1/2}.$$

The Fourier Transform. In this book the Fourier transform and its inverse are defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int e^{-2\pi i x \xi} f(x)\, dx,$$

$$\mathcal{F}^{-1} f(x) = \int e^{2\pi i x \xi} f(\xi)\, d\xi,$$

for $f \in \mathcal{S}(\mathbf{R}^n)$. (Note the "dotless products," as discussed above, in the exponents.) $\mathcal{F}$ and $\mathcal{F}^{-1}$, of course, extend uniquely to linear automorphisms of $\mathcal{S}'(\mathbf{R}^n)$. The placement of the 2π's in the exponent is uncommon in partial differential equations but almost mandatory in harmonic analysis, for it is the only way, short of renormalizing Lebesgue measure, to make $\mathcal{F}$ both an isometry on L^2 and an algebra homomorphism on L^1:

$$\|\widehat{f}\|_2 = \|f\|_2 \qquad \text{and} \qquad (f * g)\widehat{\ } = \widehat{f}\widehat{g},$$

where

$$f * g(x) = \int f(x-y) g(y)\, dy = \int f(y) g(x-y)\, dy.$$

From the physical point of view, this convention regarding the 2π's amounts to setting Planck's constant h, rather than the more common $\hbar = h/2\pi$, equal to 1. It is the reason for the 2π in the definition of D_j (formula (2) above). It also has the effect that our definition of Hermite functions is not quite the standard one; see Section 1.7.

Incidentally, the Fourier inversion formula

$$\iint e^{2\pi i (u-v)\xi} f(v)\, dv\, d\xi = f(u)$$

can be expressed neatly in the language of distributions as

$$\int e^{2\pi i x \xi} d\xi = \delta(x). \tag{3}$$

Sometimes the most perspicuous way of evaluating an iterated integral involving exponentials is to pretend that the integral (3) is absolutely convergent and interchange the order of integration. This trick is used several times in the text; in each instance the reader may verify that it is an application of the Fourier inversion theorem.

Phase Space. There seems to be no system of terminology for the various objects associated with phase space that is consistent with itself as well as with the traditions of classical mechanics and differential equations and that leads to the most elegant formulas in all situations. The system used in this book was not adopted without considerable thought, but it sometimes leads to formulations that readers (including the author) may find discordant. The following paragraphs are in the nature of an apology for this state of affairs.

In the first place, although the functorially correct definition of phase space is $(\mathbf{R}^n)^* \times \mathbf{R}^n$ (or, in some contexts, the cotangent bundle of $\mathbf{R}^n$), I have abandoned any attempt to distinguish between $\mathbf{R}^n$ and $(\mathbf{R}^n)^*$. Maintaining this distinction seems to be more trouble than it is worth, especially when (as frequently happens) we have to consider both $(\mathbf{R}^n)^* \times \mathbf{R}^n$ and *its* dual space. So, in this book, phase space is just $\mathbf{R}^n \times \mathbf{R}^n$, or $\mathbf{R}^{2n}$ for short.

Next, there is the question of what to call points in phase space, or the coordinate functions on phase space. In classical mechanics the usual choice is (p, q), where $p \in \mathbf{R}^n$ denotes momentum and $q \in \mathbf{R}^n$ denotes position. On the other hand, in the literature of partial differential equations the same variables are usually denoted by ξ and x. I have found it convenient to employ both of these sets of labels: (ξ, x) for one copy of $\mathbf{R}^{2n}$ on which the symbols of pseudodifferential operators live, and (p, q) for another copy of $\mathbf{R}^{2n}$ (actually, its dual) on which their Fourier transforms live. The resulting usage of the letters p and q is sometimes, but not always, consistent with their interpretation as momentum and position.

I chose the ordering (p, q) in order to make certain formulas involving the Heisenberg group come out naturally (essentially, to avoid making Planck's constant negative). To prevent massive confusion, I was then forced to order the dual variables as (ξ, x) rather than (x, ξ). Consequently, in this book pseudodifferential operators are written as $\sigma(D, X)$, in flagrant disregard of the custom of writing them as $\sigma(X, D)$ or $\sigma(x, D)$. This, however, may serve the useful purpose of reminding readers conversant with pseudodifferential operators that $\sigma(D, X)$ is defined here by the Weyl calculus instead of the Kohn-Nirenberg calculus.

There are two canonical symplectic forms on $\mathbf{R}^n \times \mathbf{R}^n$, differing from each other by a factor of -1. One must simply make a choice; the symplectic form used here is denoted by square brackets and defined by

$$[(p, q), (p', q')] = pq' - qp' = \sum_1^n (p_j q_j' - q_j p_j'),$$

or

$$[X, Y] = X\mathcal{J}Y \qquad \text{where} \qquad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The point that I found most troublesome is the question of whether to use the Euclidean Fourier transform or the symplectic Fourier transform on $\mathbf{R}^{2n}$,

$$\mathcal{F}f(\xi,x)=\iint e^{-2\pi i(p\xi+qx)}f(p,q)\,dp\,dq=\iint e^{-2\pi i(p,q)\cdot(\xi,x)}f(p,q)\,dp\,dq$$

or

$$\mathcal{F}_{\text{symp}}f(\xi,x)=\iint e^{2\pi i(px-q\xi)}f(p,q)\,dp\,dq=\iint e^{2\pi i[(p,q),(\xi,x)]}f(p,q)\,dp\,dq,$$

and correspondingly, whether to parametrize the Schrödinger representation by

$$\rho(p,q)=e^{2\pi i(pD+qX)} \qquad \text{or} \qquad \rho'(p,q)=e^{2\pi i(pX-qD)},$$

where

$$e^{2\pi i(pD+qX)}f(x)=e^{\pi ipq+2\pi iqx}f(x+p),$$
$$e^{2\pi i(pX-qD)}f(x)=e^{-\pi ipq+2\pi ipx}f(x-q).$$

Since the symplectic structure of $\mathbf{R}^{2n}$ is of fundamental importance, the symplectic Fourier transform is in some ways more appropriate. Moreover, the operator $e^{2\pi i(pX-qD)}$ is geometrically more natural than $e^{2\pi i(pD+qX)}$, because it transforms wave packets whose mean momentum and position are a and b to wave packets whose mean momentum and position are $a+p$ and $b+q$, rather than $a+q$ and $b-p$. However, I was persuaded to use the Euclidean Fourier transform and the operators $e^{2\pi i(pD+qX)}$ by the following three considerations.

1. I find the symplectic Fourier transform confusing to use in performing specific calculations.
2. Occasionally we need to view $\mathbf{R}^{2n}$ both as the phase space of $\mathbf{R}^n$ and as a configuration space in its own right (i.e., as $\mathbf{R}^{n'}$ where n' happens to be $2n$), and consistency then demands the use of the Euclidean Fourier transform.
3. The parametrization $e^{2\pi i(pX-qD)}$ leads to some unsightly factors of $-i$ in the correspondence between the Schrödinger and Fock models.

This dilemma is in any event not of earthshaking importance, because the symplectic Fourier transform is simply the composition of the Euclidean Fourier transform with the map $(\xi,x)\to(x,-\xi)$, which belongs to both $SO(2n)$ and $Sp(n,\mathbf{R})$; and the operators $e^{2\pi i(pD+qX)}$ and $e^{2\pi i(pX-qD)}$ are intertwined by the Fourier transform on $\mathbf{R}^n$:

$$e^{2\pi i(pX-qD)}=\mathcal{F}e^{2\pi i(pD+qX)}\mathcal{F}^{-1}.$$

It is therefore a simple matter to translate formulas from one scheme to the other.