Generalized Inverse Operators and Fredholm Boundary-Value Problems

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Utrecht • Boston, 2004

VSP
an imprint of Brill Academic Publishers
P.O. Box 346
3700 AH Zeist
The Netherlands

Tel: +31 30 692 5790
Fax: +31 30 693 2081
vsppub@brill.nl
www.brill.nl
www.vsppub.com

First published in 2004

ISBN 90-6764-407-2

A C.I.P. record for this book is available from the Library of Congress

Printed in The Netherlands by Ridderprint bv, Ridderkerk.

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NOTATION

B	Banach space of vector functions $z\colon [a,b] \to R^n$
$C[a,b]$	space of vector functions $x\colon [a,b] \to R^n$ continuous on an interval $[a,b]$
$C^1[a,b]$	space of vector functions $x\colon [a,b] \to R^n$ continuously differentiable on an interval $[a,b]$
$D_p^n[a,b]$	Banach space of vector functions $z\colon [a,b] \to R^n$ absolutely continuous on an interval $[a,b]$ with norm $\lvert\lvert z\rvert\rvert_{D_p^n} = \lvert\lvert \dot z\rvert\rvert_{L_p^n} + \lvert\lvert z(a)\rvert\rvert_{R^n}$
$\dim \operatorname{im} L$	dimension of the image of an operator L
$\dim \ker L$	dimension of the kernel of an operator L
$(G\cdot)(t)$	generalized Green operator for a semihomogeneous boundary-value problem
$G(t,\tau)$	generalized Green matrix
H	Hilbert space of vector functions $z\colon [a,b] \to R^n$
$\operatorname{im} L$ $(R(L))$	image of a linear operator L
$\ker L$ $(N(L))$	kernel (null space) of a linear operator L
$K(t,\tau)$	Cauchy matrix
l	linear vector functional
$lX(\cdot)$	$m \times n$ constant matrix that is the result of the action of an m-dimensional linear vector functional l on the columns of an $n \times n$ matrix $X(t)$
L^-, L^+	operator generalized inverse or pseudoinverse to an operator L
$L_{l(r)}^+$	left (right) pseudoinverse of an operator L
L^{-1}	inverse of an operator L
$L_{l(r)}^{-1}$	left (right) inverse of an operator L

$L_p^n[a,b]$	Banach space of vector functions $z\colon [a,b] \to R^n$ integrable to pth power $(1 < p < \infty)$ equipped with norm $\|\|z\|\|_{L_p^n} = \left(\int_a^b \sum_{i=1}^{n} \|z_i(t)\|^p \, dt \right)^{1/p}$
$\mathcal{L}(B_1, B_2)$	space of bounded linear operators L acting from a Banach space B_1 into a Banach space B_2
$\mathcal{P}_L$	linear operator (projector) projecting a Banach space B onto the null space of an operator L
$\mathcal{P}_{Y_L}$	linear operator (projector) projecting a Banach space B onto a subspace $Y \subset B$ isomorphic to the null space $N(L^*)$ of the operator adjoint to an operator L
$P_L = P_{N(L)}$	linear operator (orthoprojector) projecting a Hilbert space H onto the null space of an operator L
R^n	Euclidean vector space of constant vectors
$\operatorname{sign} t$	sign function
δ_{ij}	Kronecker symbol
$\chi_{[a,b]}(t)$	characteristic function of an interval $[a,b]$

PREFACE

The problems of development of constructive methods for the analysis of linear and weakly nonlinear boundary-value problems for a broad class of functional differential equations, including systems of ordinary differential and difference equations, systems of differential equations with delay, systems with pulse action, and integro-differential systems, traditionally occupy one of the central places in the qualitative theory of differential equations [5], [148], [51], [94], [118], [104]. This is explained, first of all, by the practical significance of the theory of boundary-value problems for various applications—theory of nonlinear oscillations [2], [7], [14], [15], [89], [97], [101], [139], [70], theory of stability of motion [50], [51], [98], [97], [108], control theory [158], [128], and numerous problems in radioengineering, mechanics, biology, etc. [121], [152], [81], [89], [103]. As a specific feature of boundary-value problems, we can mention the fact that their linear part is, in most cases, an operator without inverse. This fact makes it impossible to use traditional methods based on the fixed-point principle for the investigation of boundary-value problems of this sort. The uninvertibility of the linear part of the operator is a consequence of the fact that the number m of boundary conditions does not coincide with the number n of unknown variables in the operator system. Problems of this kind for systems of functional differential equations are of Fredholm type (or with Fredholm linear parts). They include extremely complicated and insufficiently studied (both underdetermined and overdetermined) critical and noncritical boundary value-problems. The applicability of the well-known Schmidt lemma [148] to the investigation of boundary-value problems regarded as operator equations with bounded operators in the linear part with an aim to construct a generalized inverse operator resolving the original boundary-value problem is restricted by the requirement that the corresponding boundary-value problem must be of Fredholm type with index zero, i.e., that $m = n$. Therefore, a major part of the works dealing with problems of this sort were carried out under the assumption that these problems are of Fredholm type with index zero (Azbelev, Maksimov, and Rakhmatullina [8], Vejvoda [149], Wexler [153], Grebenikov, Lika, and Ryabov [65], [93], Malkin [101], Mitropol'skii and Martynyuk [107], Samoilenko, Perestyuk, and Ronto [139] and [140]). Moreover, a significant part of these results was, in fact, obtained under the assumption that the operator in the linear part of the original boundary-value problem has the inverse operator (noncritical case). We do not use this assumption.

It is known (Atkinson [6], Vainberg and Trenogin [148], Pyt'yev [123], Turbin [147], Nashed [112]) that the classical Schmidt procedure [141] is applicable to the construction of generalized inverse operators only in the case of Fredholm operators of index zero. Thus, for boundary-value problems regarded as operator systems in abstract spaces [153], we suggest some methods for the construction of the generalized inverse (or pseudo-inverse) operators for the original linear Fredholm operators in Banach (or Hilbert) spaces. As a result of systematic application and development of the theory of generalized inverse operators [123], [112] and matrices [91], [147], [109], [117], new criteria of solvability were obtained and the structure of solutions was determined for linear Fredholm boundary-value problems for various classes of systems of functional differential operators. The methods used for the construction of the generalized Green's operators (and generalized matrices playing the role of kernels of their integral representations) for semihomogeneous boundary-value problems for systems of this sort are presented from the common viewpoint. We also study basic properties of the generalized Green's operator. In particular, it is shown how, using this operator, one can construct the generalized inverse of the operator of the original boundary-value problem.

New efficient methods of perturbation theory were developed in analyzing weakly nonlinear boundary-value problems. These methods, including the Lyapunov–Poincaré method of small parameter [97], [120], asymptotic methods of nonlinear mechanics developed by Krylov, Bogolyubov, Mitropol'skii, and Samoilenko [14], [15], [90], some methods proposed by Tikhonov [144], [145] and the Vishik–Lyusternik method [150], are extensively used for the solution of various problems encountered in different fields of science and engineering, such as radioengineering [121], [101], shipbuilding [89], celestial mechanics [65], [81], biology [152], [103], etc. These methods were developed and used in numerous works (Vainberg and Trenogin [148], Vejvoda [149], Grebenikov and Ryabov [65], Kato [80], Malkin [101], Mishchenko and Rozov [106], and Hayashi [74]). However, the application of the methods of perturbation theory to the analysis of weakly nonlinear boundary-value problems for various classes of differential systems was, for the most part, restricted to the case of ordinary periodic boundary-value problems in the theory of nonlinear oscillations (Grebenikov and Ryabov

[65], Hale [70], Malkin [101], Proskuryakov [122], and Yakubovich and Starzhinskii [154] for systems of ordinary differential equations, Mitropol'skii and Martynyuk [107] and Shimanov [143] for systems with delay, and Samoilenko and Perestyuk [139] and Bainov and Simeonov [10] for systems with pulse action).

We show that the principal results in the theory of weakly nonlinear periodic oscillations remain valid (with necessary refinements, changes, and supplements) for general weakly perturbed (with Fredholm-type linear parts) boundary-value problems for systems of functional differential equations. The boundary-value problems are specified by linear or weakly nonlinear vector functionals such that the number of their components does not coincide with the dimension of the operator system. In our monograph, we develop a general theory of boundary-value problems of this kind, give a natural classification of critical and noncritical cases,[1] establish efficient conditions for the coefficients guaranteeing the existence of solutions, and develop iterative algorithms for the construction of solutions of these problems. Numerous results presented in the monograph were originally obtained and approved in analyzing boundary-value problems for systems of ordinary differential equations (Boichuk [19]). Later, it was discovered that the proposed procedures of investigation and algorithms are applicable to the analysis of much more general objects, including boundary-value problems for ordinary systems with lumped delay [32], [37], [157], systems with pulse action [30], [29], [135], autonomous differential systems [25], [26], [36], and operator equations in functional spaces whose linear part is a normally resolvable operator but they are not everywhere solvable [33], [34], [35].

In the first chapters, to make our presentation more general, we give some results from the theory of generalized inversion of bounded linear operators in abstract spaces, which are then used for the investigation of boundary-value problems for systems of functional differential equations. Some of these results are of independent interest for the theory of linear operators, although our main aim was just to develop the tools required for the analysis of boundary-value problems for systems of functional differential equations. The methods used for the construction of generalized inverse operators in Banach and Hilbert spaces are presented separately because these spaces are characterized by absolutely different geometries. The construction of the generalized inverse operator for a linear Fredholm operator acting in Banach spaces is based on the Atkinson theorem [6] obtained as a generalization of the Nikol'skii theorem [113], which states that any bounded Fredholm operator can be represented in the form of a unilaterally invertible and

[1] In the literature, these cases are sometimes called resonance and nonresonance.

completely continuous (finite-dimensional) operator. By using this fact, we arrive at a construction of the generalized inverse of a Fredholm operator similar to the well-known Schmidt procedure [148] applicable only in the case of generalized inversion of Fredholm operators of index zero in Banach spaces.

The theory of generalized inversion and pseudoinversion of linear Fredholm operators in Banach and Hilbert spaces enabled us to develop a unified procedure for the investigation of Fredholm boundary-value problems for operator equations solvable either everywhere or not everywhere (Chapter 4).

The proposed approach is then improved for the analysis of boundary-value problems for standard operator systems, including systems of ordinary differential equations and equations with delay (Chapter 5) and systems with pulse action (Chapter 6). We obtain necessary and sufficient conditions for the existence of solutions of linear and nonlinear differential and difference systems bounded on the entire axis (Chapter 7).

This enables us to take into account specific features of each analyzed differential system and present necessary examples. The readers interested primarily in the theory of boundary-value problems for specific differential systems may focus their attention on the corresponding chapters and omit the chapters containing preliminary information.

The authors do not even try to present the complete bibliography on the subject, which is quite extensive, and mention only the works required for the completeness of presentation.

In conclusion, the authors wish to express their deep gratitude to all participants of numerous seminars and conferences on the theory of differential equations and nonlinear oscillations, where all principal results included in the book were reported and discussed.

Contents

1. PRELIMINARY INFORMATION

In Chapter 1, we present some well-known definitions and results from functional analysis, the theory of linear operators in Hilbert and Banach spaces, and matrix theory, required for our subsequent presentation. The readers who are not familiar with the theory of linear operators in function spaces can find here an elementary presentation of the facts essentially used in what follows. The other readers can use this material for references. The theorems are presented without proofs, but the reader is referred to the sources for further details. In this chapter, we also introduce necessary notation.

1.1. Metric and Normed Spaces

Definition 1.1. *A metric space is defined as a set* X *equipped with a metric* $\rho(\cdot,\cdot)$, *i.e., a real function defined in the set* X *and such that*

(1) $\rho(x,y) \geq 0 \quad (\rho(x,y) = 0$ *iff* $x = y)$;

(2) $\rho(x,y) = \rho(y,x)$;

(3) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ *(triangle inequality).*

Thus, an arbitrary set equipped with a metric is a metric space.

Example 1. A set X whose points are collections of n-dimensional real vectors $x = (x_1, \ldots, x_n)$ turns into a metric space if we set

$$\rho(x,y) = \Big[\sum_{i=1}^{n}(x_i - y_i)^2\Big]^{1/2}.$$

The same set X can also be equipped with other metrics, e.g.,

$$\rho_1(x,y) = \max_{1\le i\le n} |x_i - y_i|,$$

$$\rho_2(x,y) = \sum_{i=1}^{n} |x_i - y_i|,$$

and, thus, it turns into different metric spaces.

Example 2. Let Y be the set of continuous functions defined on a segment $[a,b]$. If we introduce a metric by setting

$$\rho(x,y) = \max_{a\le t\le b} |x(t) - y(t)| \quad \text{for} \quad x,y \in Y,$$

then Y turns into the well-known metric space $C[a,b]$. The set of continuous functions can be transformed into other metric spaces by introducing different functions, e.g., as follows:

$$\rho_1(x,y) = \Big[\int_a^b |x(t) - y(t)|^p\,dt\Big]^{1/p}, \quad p > 1.$$

The set Z of n $(n \ge 1)$ times continuously differentiable functions defined on the segment $[a,b]$ turns into a metric space $C^m[a,b]$ if we use the following metric:

$$\rho(x,y) = \max_{0\le i\le n} \max_{a\le t\le b} |x^{(i)}(t) - y^{(i)}(t)|,$$

$$x^{(0)}(t) \equiv x(t), \qquad y^{(0)}(t) \equiv y(t), \quad \forall x,y \in Z.$$

Example 3. Consider a set whose points are ordered systems of real numbers $x = (x_1, x_2, \ldots, x_n, \ldots)$ and $y = (y_1, y_2, \ldots, y_n, \ldots)$ such that

$$\sum_{i=1}^{\infty} |x_i|^p < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} |y_i|^p < \infty, \qquad p \ge 1.$$

If the distance is introduced according to the formula

$$\rho(x,y) = \Big[\sum_{i=1}^{\infty} |x_i - y_i|^p\Big]^{1/p},$$

then we get a metric space denoted by l_p, $p \ge 1$.

Definition 1.2. *A sequence* $\{x_n\}_{n=1}^{\infty}$ *of elements of a metric space* X *is called convergent to an element* $x \in X$ *if* $\rho(x, x_n) \to 0$ *as* $n \to \infty$. *The element* x *is called the limit point of the set* X.

Definition 1.3. *A set* $M \subset X$ *is called closed if it contains all its limit points. The empty set is always regarded as closed.*

Definition 1.4. *Let* $M \subset X$ *and let* M' *be a set of limit points of* M. *The set* $\overline{M} = M \cup M'$ *is called the closure of the set* M.

Definition 1.5. *A sequence* $\{x_n\}_{n=1}^{\infty}$ *of elements of a metric space* X *is called a fundamental (Cauchy) sequence if, for any real* $\varepsilon > 0$, *there exists* N *such that* $\rho(x_n, y_m) < \varepsilon$ *whenever* $n, m > N$.

In any convergent sequence, one can select a fundamental subsequence. The example presented below shows that the converse statement is not true.

Example 4. Let Q be a set of rational numbers with metric $\rho(x, y) = |x - y|$ and let x_0 be an irrational number, i.e., $x_0 \in R \setminus Q$, where R is the set of real numbers. We construct a sequence of rational numbers x_n that converges to x_0 in R. Then x_n is a Cauchy sequence in Q, but it does not converge in Q to any rational number y (indeed, if $x_n \to y$ in Q, then $x_n \to y$ also in R, and we set $y = x_0$).

Definition 1.6. *A metric space in which any fundamental sequence is convergent is called complete.*

Thus, the metric space R is complete, but the set Q is not a complete space.

In order to transform an incomplete metric space X into a complete space, it is necessary to expand it by adding the limits of all possible fundamental sequences. The original space X is, in this case, dense in the enveloping space $\overline{X}$ in the following sense:

Definition 1.7. *A set* M *is called dense in a metric space* X *if any element* $x \in X$ *is the limit of a sequence of elements from* M.

As follows from the definition of the operation of closure of a set (Definition 1.4), the statement that a set M is dense in X, $M \subset X$, $M \neq X$, means that $\overline{M} = X$. In other words, the closure of the set M coincides with X.

Definition 1.8. *A normed linear space is defined as a vector space X over the field R equipped with norm $\|\cdot\|$, i.e., a nonnegative function that maps X into R and satisfies the following conditions:*

(1) $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *iff* $x = 0$;

(2) $\|\alpha x\| = |\alpha| \, \|x\|$ *for all* $x \in X$ *and* $\alpha \in R$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.

By using a norm, one can easily introduce a metric in a normed space by setting $\rho(x, y) = \|x - y\|$, and, hence every normed space is a metric space. It is often possible to introduce more than one norm in the same vector space. The normed spaces obtained as a result are regarded as different.

If a normed space X is equipped with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, then any sequence convergent in the norm $\|\cdot\|_1$ converges in the norm $\|\cdot\|_2$ if and only if there exists a positive constant C_1 such that

$$\|x\|_2 \leq C_1 \, \|x\|_1 \quad \forall x \in X.$$

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in a linear space X are called topologically equivalent if convergence in one of these norms implies convergence in the second norm, and vice versa.

The norms are topologically equivalent if and only if there exist positive constants C_1 and C_2 such that

$$C_2 \leq \frac{\|x\|_2}{\|x\|_1} \leq C_1 \qquad \forall x \in X, \quad x \neq 0.$$

Example 5. The set of all n-dimensional vectors x with operations of addition and multiplication by numbers is called the n-dimensional vector space R^n. The number $|x| = \sqrt{\sum_{j=1}^{n} |x_j|^2}$ is called the length (modulus) of a vector x.