

Second Edition / Teacher's Edition

Algebra and Trigonometry

Functions and Applications

Paul A. Foerster



Addison-Wesley Publishing Company

Second Edition

Algebra and Trigonometry

Functions and Applications

Paul A. Foerster

Teacher's Edition



Addison-Wesley Publishing Company
Menlo Park, California • Reading, Massachusetts
New York • Don Mills, Ontario • Wokingham, England
Amsterdam • Bonn • Sydney • Singapore • Tokyo
Madrid • San Juan

Paul A. Foerster has taught mathematics at Alamo Heights High School in San Antonio, Texas since 1961. In that same year he received his teaching certificate from Texas A&M University. His B.S. degree in Chemical Engineering and M.A. degree in Mathematics are from the University of Texas. Among many honors, he was awarded the Presidential Award for Excellence in Mathematics in 1983.

To A. W. Foerster, who helped me to understand the real world,
to Admiral Rickover, who taught me how to write about it, and
to Jo Ann, who helps me to live in it.

Copyright © 1990, 1984, 1980 by Addison-Wesley Publishing Company, Inc. All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior written permission of the publisher. Printed in the United States of America. Published simultaneously in Canada.

ISBN-0-201-25086-1

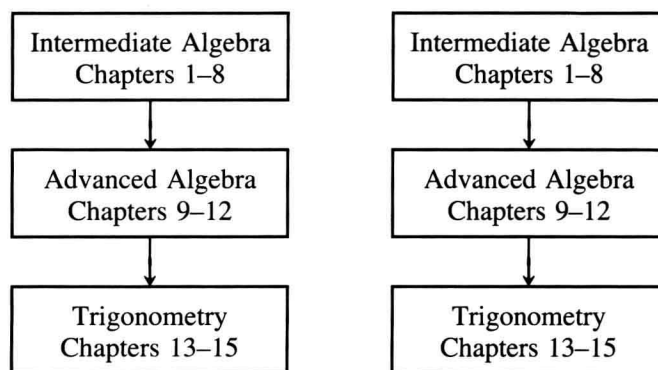
ABCDEFGHIJKL-VH-892109.

Foreword

Algebra and Trigonometry—Functions and Applications is designed for a course in intermediate algebra, advanced algebra, and trigonometry. The book can be used in two different ways:

1. As an algebra and trigonometry book, with applications,
2. As an applications book, with supporting algebra and trigonometry.

In either case, there are two possible sequences of presentation:



Applications are handled by creating mathematical models of phenomena in the real world. Students must select a kind of function that fits a given situation, and derive an equation that suits the information in the problem. The equation is then used to predict values of y when x is given or values of x when y is given. Sometimes students must use the results of their work to make interpretations about the real world, such as what “slope” means, or why there cannot be people as small as in *Gulliver’s Travels*. The problems require the students to use *many* mathematical concepts in the *same* problem. This is in contrast to the traditional “word problems”

of elementary algebra, in which the same one concept is used in many problems.

The second edition differs from the first and revised editions in two significant ways—technology and pedagogy. The computer is assumed to be a normal part of students' classroom experience rather than simply a novelty. For the most part, students are expected to use existing computer programs, notably computer graphic programs, rather than write their own. A disk accompanying the Teacher's Resource Book contains programs written by the author that are sufficient for the purposes of this text. You are, however, urged to seek out commercially-available software that is faster and more user-friendly.

Ideas for using calculators have caused substantial differences in presentation of certain topics. For instance, exponential equations are solved at the beginning of Chapter 6 by iterative methods on the calculators. Logarithms then arise naturally as a quicker way to get the unknown exponent. Presented this way, there is no doubt in the students' minds that a logarithm is an exponent! The calculator thus leads to a better understanding of theoretical concepts and is not simply a way to work old problems quicker.

Pedagogically, there are far more opportunities for review of previous concepts. One major way this review is accomplished is through the "Do These Quickly" problems at the beginning of each problem set. Once the students have learned to work a particular kind of problem, they develop speed as these problems reappear in five-minute exercises that concentrate only on answers. Mathematical-model problems have been spread out so the students must recall how to use linear and quadratic functions while they are working exponential model problems.

The topics in the text itself are largely the same as in the previous editions. Work on data analysis has been added in some problem sets to show students how to decide which kind of function is an appropriate model. Material on statistics, including the normal distribution, appears in the probability chapter.

The text still starts with a brief review of the basic axioms and properties. It moves quickly to topics the students have probably not seen before, at least in the method of presentation. The purpose is two-fold. First, students should not feel as if most of the course is spent reviewing elementary algebra. Second, the presentation emphasizes the role of algebra and trigonometry as the foundation for calculus, rather than as the completion of elementary algebra. By presenting both algebra and trigonometry as the study of classes of functions, students learn the essential unity of the two subjects.

Henry Pollack of Bell Telephone Labs claims that there are just two kinds of numbers: “real” numbers such as encountered in everyday life, and “fake” numbers such as encountered in most mathematics classes! Since this book has many problems involving untidy decimals (“real” numbers), a calculator or computer are called for where appropriate. There are also problems that have small-integer answers (“fake” numbers) so that students may gain confidence in their work when they are just learning a new technique.

Since all educators share the responsibility of teaching students to read and write, there are discovery exercises so that students may wrestle with a new concept before it is reinforced by classroom discussion. The students are helped with this reading by the fact that much of the wording came from the mouths of my own students. Special thanks go to Susan Cook, Brad Foster, and Nancy Carnes, whose good class notes supplied input for certain sections. Students Lewis Donzis and David Frey wrote computer programs for some of the problems.

Thanks go to instructors in Florida, Illinois, Pennsylvania, South Dakota, Texas, and Virginia for pilot testing the original materials. Special thanks go to Bob Enenstein and his instructors in California for pilot testing the second edition. The text reflects comments from review and classroom testing by Charley Brown, Sharon Sasch Button, Pat Causey, Loyce Collenback, Bob Davies, Walter DeBill, Rich Dubsky, Michelle Edge, Sandra Frasier, Byron Gill, Pat Johnson, Michael Keeton, Carol Kipps, Bill McNabb, Shirley Scheiner, Ann Singleton, Chuck Straley, Rhett Tatsch, Joel Teller, Susan Thomas, Kay Thompson, Zalman Usiskin, Jim Wieboldt, Marv Wielard, Mercille Wisakowsky, Martha Zelinka, and Isabel Zsohar. Calvin Butterball and Phoebe Small appear with the kind permission of their parents, Richard and Josephine Andree.

Paul A. Foerster

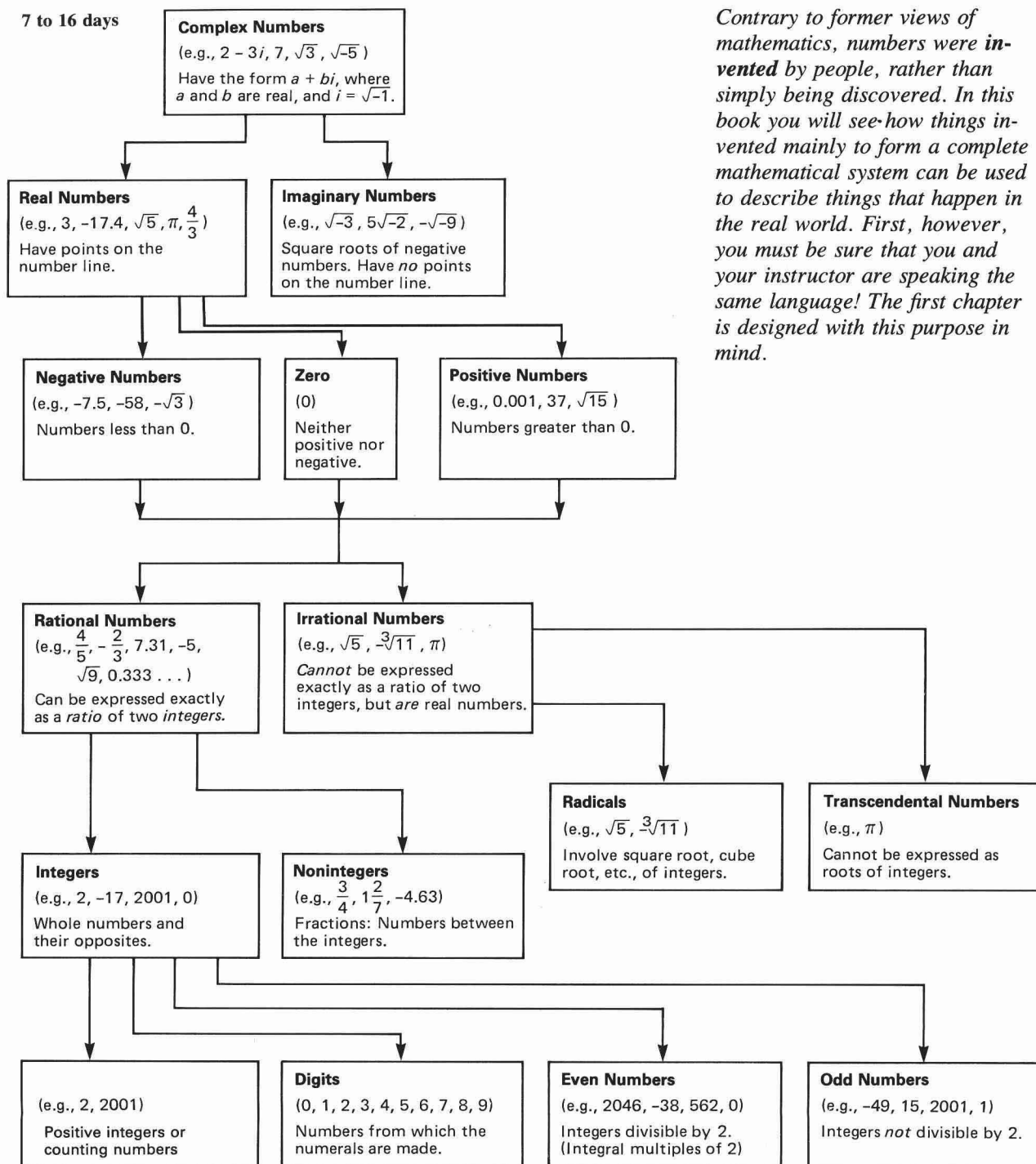
Photograph Acknowledgements

Australian Information Service: 165
Joe Baraban: 176
The Bettmann Archive: 219
Daniel S. Brody/Stock, Boston: 378
John Colwell/Grant Heilman Photography: 64, 123, 338
Jose Cuervo: 42
E. R. Degginger/Bruce Coleman Inc.: 233
Diamond Information Center, N.W. Ayer ABH International: 216
Richard H. Dohrmann*: 130; 300 left; 300 right, courtesy Fidelity Savings & Loan Association
Howard Hall/Tom Stack & Associates: 218
Grant Heilman: 168, 213, 235
Ira Kirschenbaum/Stock, Boston: 167
Keith Murakami/Tom Stack & Associates: 359
(c) Bill Pierce/Rainbow: 358
Bil Plummer*: 430
F. Roe/Camera 5: 278
Joe Scherschel, *Life* Magazine, (c) 1957 Time Inc.: 224
Peter Southwick/Stock, Boston: 28
U.S. Department of the Interior, Bureau of Reclamation: 98
Yerkes Observatory: 250

*Photographs provided expressly for the publisher. All other photographs by Addison-Wesley staff.



7 to 16 days



*Contrary to former views of mathematics, numbers were **invented** by people, rather than simply being discovered. In this book you will see how things invented mainly to form a complete mathematical system can be used to describe things that happen in the real world. First, however, you must be sure that you and your instructor are speaking the same language! The first chapter is designed with this purpose in mind.*

Contents

1 PRELIMINARY INFORMATION 1

- 1-1 Sets of Numbers 1
- 1-2 The Field Axioms 3
- 1-3 Variables and Expressions 9
- 1-4 Polynomials 16
- 1-5 Equations 20
- 1-6 Inequalities 27
- 1-7 Properties Provable from the Axioms 32
- 1-8 Chapter Review and Test 44

2 FUNCTIONS AND RELATIONS 50

- 2-1 Graphs of Equations with Two Variables 51
- 2-2 Graphs of Functions 53
- 2-3 Functions in the Real World 58
- 2-4 Functions and Relations 64
- 2-5 Chapter Review and Test 69

3 LINEAR FUNCTIONS 72

- 3-1 Introduction to Linear Functions 73
- 3-2 Graphs of Linear Functions from their Equations 74
- 3-3 Other Forms of the Linear Function Equation 82
- 3-4 Equations of Linear Functions from their Graphs 86
- 3-5 Linear Functions as Mathematical Models 93
- 3-6 Chapter Review and Test 105

4 SYSTEMS OF LINEAR EQUATIONS AND INEQUALITIES 110

- 4-1 Introduction to Linear Systems 111
- 4-2 Solution of Systems in Linear Equations 112
- 4-3 Second-Order Determinants 120
- 4-4 $f(x)$ Terminology, and Systems as Models 126
- 4-5 Linear Equations with Three or More Variables 133
- 4-6 Systems of Linear Equations with Three or More Variables 138
- 4-7 Solutions of Systems by Augmented Matrices 142
- 4-8 Systems of Linear Inequalities with Two Variables 146

4-9	Linear Programming	150
4-10	Systems of Linear Inequalities	152
4-11	Linear Programming	157
4-12	Chapter Review and Test	169

5 QUADRATIC FUNCTIONS AND COMPLEX NUMBERS 173

5-1	Introduction to Quadratic Functions	174
5-2	Graphs of Quadratic Functions	175
5-3	x -Intercepts, and the Quadratic Formula	181
5-4	Imaginary and Complex Numbers	188
5-5	Evaluating Quadratic Functions	195
5-6	Equations of Quadratic Functions from their Graphs	200
5-7	Quadratic and Linear Functions as Mathematical Models	204
5-8	Chapter Review and Test	221
5-9	Cumulative Review, Chapters 1, 2, 3, 4, 5	224

6 EXPONENTIAL AND LOGARITHMIC FUNCTIONS 228

6-1	Introduction to Exponential Functions	229
6-2	Exponentiation for Positive Integer Exponents	230
6-3	Properties of Exponentiation	236
6-4	Exponentiation for Rational Exponents	241
6-5	Powers of Radicals without Calculators	248
6-6	Scientific Notation	253
6-7	Exponential Equations Solved by Brute Force	260
6-8	Exponential Equations Solved by Logarithms	263
6-9	Logarithms with Other Bases	269
6-10	Properties of Logarithms	274
6-11	Proofs of Properties of Logarithms	282
6-12	Inverses of Functions—The Logarithmic Function	289
6-13	The Add-Multiply Property of Exponential Functions	295
6-14	Exponential and Other Functions as Mathematical Models	300
6-15	Chapter Review and Test	317

7 RATIONAL ALGEBRAIC FUNCTIONS 322

7-1	Introduction to Rational Algebraic Functions	323
7-2	Rational Function Graphs—Discontinuities and Asymptotes	324
7-3	Special Products and Factoring	328
7-4	More Factoring and Graphing	336
7-5	Long Division of Polynomials	343

7-6	Factoring Higher Degree Polynomials—The Factor Theorem	347
7-7	Products and Quotients of Rational Expressions	356
7-8	Sums and Differences of Rational Expressions	364
7-9	Graphs of Rational Algebraic Functions, Again	370
7-10	Fractional Equations and Extraneous Solutions	378
7-11	Variation Functions	383
7-12	Chapter Review and Test	405

8 IRRATIONAL ALGEBRAIC FUNCTIONS 412

8-1	Introduction to Irrational Algebraic Functions	413
8-2	Graphs of Irrational Numbers	414
8-3	Radicals, and Simple Radical Form	416
8-4	Radical Equations	425
8-5	Variation Functions with Non-Integer Exponents	432
8-6	Functions of More than One Independent Variable	442
8-7	Chapter Review and Test	452
8-8	Cumulative Review, Chapters 6, 7, 8	455

9 QUADRATIC RELATIONS AND SYSTEMS 460

9-1	Introduction to Quadratic Relations	461
9-2	Circles	462
9-3	Ellipses	468
9-4	Hyperbolas	477
9-5	Parabolas	486
9-6	Equations from Geometrical Definitions	490
9-7	Quadratic Relations— xy -Term	495
9-8	Systems of Quadratics	497
9-9	Chapter Review and Test	507

10 HIGHER DEGREE FUNCTIONS AND COMPLEX NUMBERS 513

10-1	Introduction to Higher Degree Functions	514
10-2	Complex Number Review	515
10-3	Quadratic Equations from their Solutions—Complex Number Factors	523
10-4	Graphs of Higher Degree Functions—Synthetic Substitution	530
10-5	Descartes' Rule of Signs, and the Upper Bound Theorem	544
10-6	Higher Degree Functions as Mathematical Models	549
10-7	Chapter Review and Test	554

11 SEQUENCES AND SERIES 559

- 11-1 Introduction to Sequences 560
- 11-2 Arithmetic and Geometric Sequences 564
- 11-3 Arithmetic and Geometric Means 572
- 11-4 Introduction to Series 577
- 11-5 Arithmetic and Geometric Series 581
- 11-6 Convergent Geometric Series 589
- 11-7 Sequences and Series as Mathematical Models 597
- 11-8 Factorials 612
- 11-9 Introduction to Binomial Series 617
- 11-10 The Binomial Formula 619
- 11-11 Chapter Review and Test 625

12 PROBABILITY AND FUNCTIONS OF A RANDOM VARIABLE 631

- 12-1 Introduction to Probability 632
- 12-2 Words Associated with Probability 633
- 12-3 Two Counting Principles 636
- 12-4 Probabilities of Various Permutations 642
- 12-5 Probabilities of Various Combinations 651
- 12-6 Properties of Probability 661
- 12-7 Functions of a Random Variable 671
- 12-8 Mathematical Expectation 681
- 12-9 Statistics, and Data Analysis 690
- 12-10 Chapter Review and Test 700
- 12-11 Cumulative Review, Chapters 9, 10, 11, 12 705

13 TRIGONOMETRIC AND CIRCULAR FUNCTIONS 709

- 13-1 Introduction to Periodic Functions 710
- 13-2 Measurement of Arcs and Rotation 711
- 13-3 Definitions of Trigonometric and Circular Functions 717
- 13-4 Approximate Values of Trigonometric and Circular Functions 729
- 13-5 Graphs of Trigonometric and Circular Functions 735
- 13-6 General Sinusoidal Graphs 742
- 13-7 Equations of Sinusoids from their Graphs 750
- 13-8 Sinusoidal Functions as Mathematical Models 756
- 13-9 Inverse Circular Functions 767
- 13-10 Evaluation of Inverse Relations 781
- 13-11 Inverse Circular Relations as Mathematical Models 786
- 13-12 Chapter Review and Test 793

14 PROPERTIES OF TRIGONOMETRIC AND CIRCULAR FUNCTIONS 799

- 14-1 Three Properties of Trigonometric Functions 800
- 14-2 Trigonometric Identities 807
- 14-3 Properties Involving Functions of More than One Argument 814
- 14-4 Multiple Argument Properties 825
- 14-5 Half-Argument Properties 831
- 14-6 Sum and Product Properties 837
- 14-7 Linear Combination of Cosine and Sine with Equal Arguments 841
- 14-8 Simplification of Trigonometric Expressions 845
- 14-9 Trigonometric Equations 850
- 14-10 Chapter Review and Test 858

15 TRIANGLE PROBLEMS 862

- 15-1 Right Triangle Problems 863
- 15-2 Oblique Triangles—Law of Cosines 873
- 15-3 Area of a Triangle 880
- 15-4 Oblique Triangles—Law of Sines 883
- 15-5 The Ambiguous Case 888
- 15-6 General Solution of Triangles 894
- 15-7 Vectors 898
- 15-8 Vectors—Resolution into Components 906
- 15-9 Real-World Triangle Problems 915
- 15-10 Chapter Review and Test 923
- 15-11 Cumulative Review, Chapters 13, 14, 15 927

FINAL EXAMINATION 930

APPENDIX A OPERATIONS WITH MATRICES 937

APPENDIX B MATHEMATICAL INDUCTION 944

TABLES 957

- I Squares and Square Roots, Cubes and Cube Roots 957
- II Four-Place Logarithms 959
- III Trigonometric Functions and Degrees-to-Radians 961
- IV Circular Functions and Radians-to-Degrees 968

GLOSSARY 971

INDEX OF PROBLEM TITLES 985

GENERAL INDEX 989

ANSWERS TO SELECTED PROBLEMS 999

From previous work in mathematics you should recall the names of different kinds of numbers (positive, even, irrational, etc.). In this section you will refresh your memory so that you will know the exact meaning of these names.

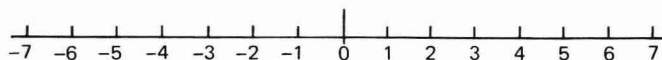
Objective:

Given the name of a set of numbers, provide an example; or given a number, name the sets to which it belongs.

There are two major sets of numbers you will deal with in this course, the *real* numbers and the *imaginary* numbers. The real numbers are given this name because they are used for “real” things such as measuring and counting. The imaginary numbers are square roots of negative numbers. They are useful, too, but you must learn more mathematics to see why.

The real numbers are all numbers which you can plot on a number line (see Figure 1-1). They can be broken into subsets in several ways. For instance, there are positive and negative real numbers, integers and non-integers, rational and irrational real numbers, and so forth. The diagram facing this page shows some subsets of the set of real numbers.

The numbers in the diagram were invented in *reverse* order. The natural (or “counting”) numbers came first because mathematics was first used for counting. The negative numbers (those less than zero) were invented so that there would always be answers to subtraction problems. The rational numbers were invented to provide answers to division problems, and the irrational ones came when it was shown that numbers such as $\sqrt{2}$ could not be expressed as a ratio of two integers.



The real number line

Time: 1 day

Chalkboard Examples:

The key to success seems to be clear understanding of the word “rational” number.

Two features are important:

1. *Rational* means *ratio* of two *integers*.
2. The numeral need not *be* a ratio of two integers, just as long as it *can* be expressed that way. For example, 5, $-\sqrt{49}$, $1\frac{2}{3}$, etc., are rational numbers. Also, not every ratio of two integers is a rational number, for example $\frac{2}{5}$.

Irrational—Can’t be expressed as a ratio of two integers.

Transcendental—Can’t be expressed even with radicals.

In both cases it is important for the students to realize that these are *real* numbers. For example, one does not even *consider* whether $\sqrt{-49}$ is rational or irrational since it is not a real number to begin with!

Assignment:

Assign all 9 problems.

Problem Notes:

Problem 3 (the table) is tedious to grade. However, it makes an excellent teaching tool if a copy of the table is projected on the screen. After the students have worked the problem, you can quickly check the boxes, a row at a time. You call out the *kind* of number, and they respond with “Yes,” or “No. When you hear wrong answers, stop and talk!

Problem 8—Most students readily accept 2.333... as equaling $2\frac{1}{3}$, a rational number. If they want a more convincing proof, try the following:



Let $x = 2.333 \dots$
 Then $10x = 23.333 \dots$
 $\therefore 10x - x = 23.333 \dots - 2.333 \dots$
 $9x = 21$
 $x = \frac{21}{9} = \frac{7}{3} = 2\frac{1}{3}$, a rational number.

In Section 11-6 they will use convergent geometric series to prove that *any* repeating decimal represents a rational number.

Other operations you will invent, such as taking logarithms and cosines, lead to irrational numbers which go beyond even extracting roots. These are called “transcendental” numbers, meaning “going beyond.” When all of these various kinds of numbers are put together, you get the set of *real* numbers. The imaginary numbers were invented because no real number squared equals a negative number. Later, you will see that the real and imaginary numbers are themselves simply subsets of a larger set, called the “complex numbers.”

The following exercise is designed to help you accomplish the objectives of this section.

EXERCISE 1-1

1. a. whole numbers, positive, negative, and 0
 b. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
 c. integers divisible by 2
 d. numbers greater than 0
 e. numbers less than 0
 f. numbers expressible as a ratio of two integers
 g. numbers not expressible as a ratio of two integers
 h. square roots of negative numbers
 i. numbers on the number line
 j. positive integers
 k. positive integers
 l. numbers not expressible using only a finite number of the operations $+$, $-$, \times , \div , or $\sqrt{}$ on integers

2. Examples may vary.

- | | |
|--------------------|----------------|
| a. 71 | g. $\sqrt{3}$ |
| b. 3 | h. $\sqrt{-9}$ |
| c. -58 | i. 56.732 |
| d. 17.23 | j. 1027 |
| e. $-9\frac{2}{3}$ | k. 19 |
| f. $5\frac{3}{8}$ | l. π |

- 3.
4. {counting numbers}
5. {natural numbers} counting was probably the first thing done with numbers
6. a. {real numbers}
 b. {imaginary numbers}
7. rational
8. rational
9. 0

1. Write a definition for each of the following sets of numbers. Try to do this *without* referring to the diagram opposite page 1. Then look to make sure you are correct.

a. {integers}	b. {digits}
c. {even numbers}	d. {positive numbers}
e. {negative numbers}	f. {rational numbers}
g. {irrational numbers}	h. {imaginary numbers}
i. {real numbers}	j. {natural numbers}
k. {counting numbers}	l. {transcendental nos.}
2. Write an example of each type of number mentioned in Problem 1.
3. Copy the chart at right. Put a check mark in each box for which the number on the left of the chart belongs to the set across the top.
4. Write another name for {natural numbers}.
5. Which of the sets of numbers in Problem 1 do you suppose was the *first* to be invented? Why?
6. One of the sets of numbers in Problem 1 contains all but one of the others as subsets.
 - a. Which one *contains* the others?
 - b. Which one is left out?
7. Do decimals such as 2.718 represent *rational* numbers or *irrational* numbers? Explain.
8. Do repeating decimals such as 2.3333 \dots represent *rational* numbers or *irrational* numbers? Explain.
9. What real number is neither positive nor negative?

	Integers	Digits	Even Numbers	Positive Numbers	Negative Numbers	Rational Numbers	Irrational Numbers	Imaginary Numbers	Real Numbers	Natural Numbers	Counting Numbers	Transcendental Numbers
a. 5	✓	✓		✓		✓			✓	✓	✓	
b. 2/3				✓		✓			✓			
c. -7	✓				✓	✓			✓			
d. $\sqrt{3}$				✓			✓		✓			
e. $\sqrt{16}$	✓	✓	✓	✓		✓			✓	✓	✓	
f. $\sqrt{-16}$								✓				
g. $\sqrt{-15}$								✓				
h. 44	✓		✓	✓		✓			✓	✓	✓	
i. π				✓			✓		✓			✓
j. 1.765				✓		✓			✓			
k. -10000	✓				✓	✓			✓			
l. $-1\frac{1}{2}$					✓	✓			✓			
m. $-\sqrt{6}$					✓		✓		✓			
n. 0	✓	✓	✓			✓		*	✓			
o. 1	✓	✓		✓		✓			✓	✓	✓	
p. 1/9				✓		✓			✓			

*0 is also an imaginary number.

1-2 THE FIELD AXIOMS

From previous mathematics courses you probably remember names such as, “Distributive Property,” “Reflexive Property,” and “Multiplication Property of Zero.” Some of these properties, called *axioms*, are accepted without proof and are used as a starting point for working with numbers.

Time: 1 day

Chalkboard Examples:

In this section students must recall the meanings “distributive,” “commutative,” and so forth. One way to introduce the topic is to have the students call out names of all the properties they can remember. let stu-