TABLES OF BESSEL FUNCTIONS

Of the true argument and of integrals derived from them

by E. A. Chistova

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E.A.CHISTOVA
The Computing Centre of the U.S.S.R. Academy of Sciences

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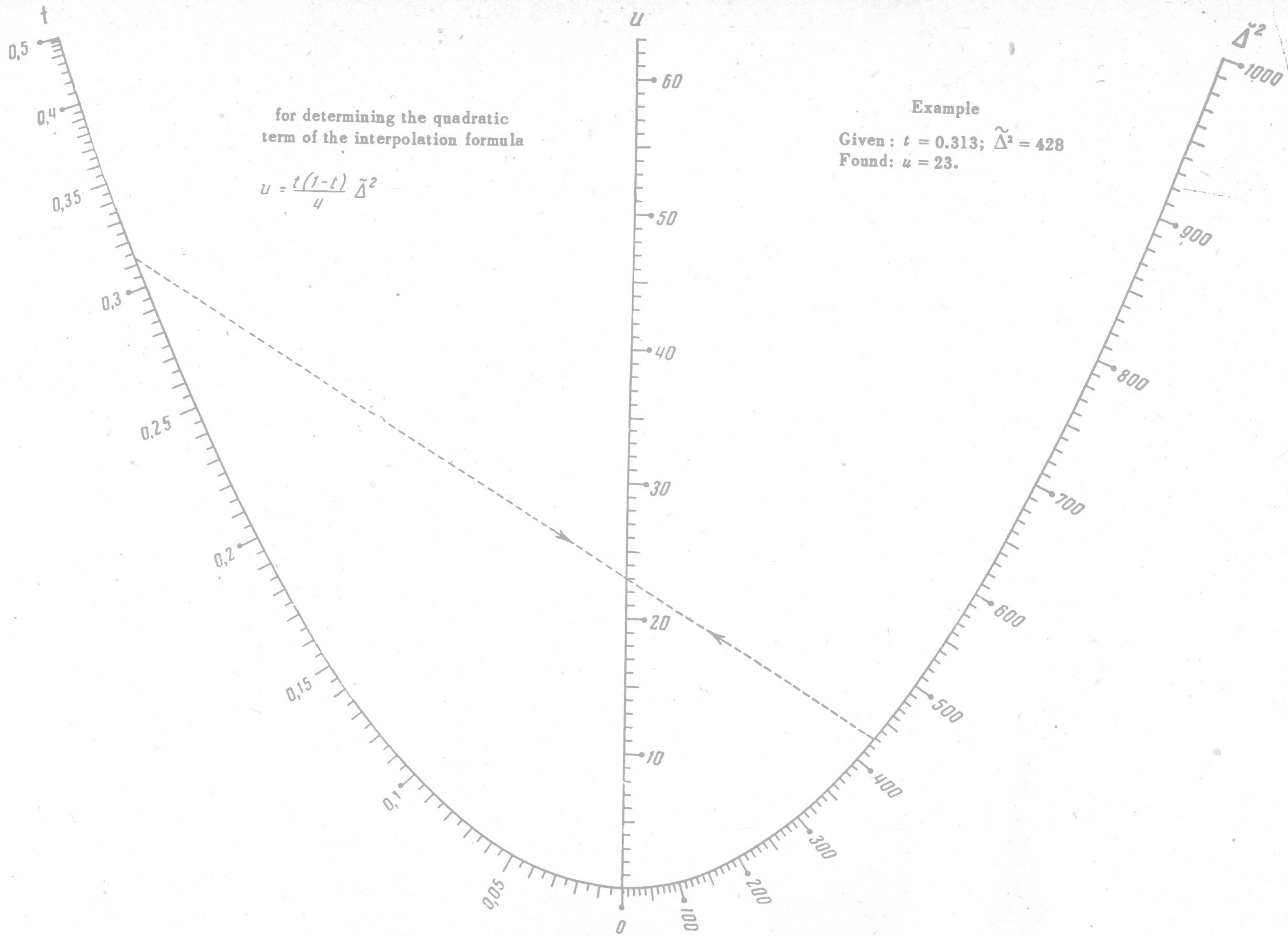
TABLE OF VALUES FOR $\frac{t(1-t)}{4}$

t	0	1	2	3	4	5	6	7	8	9	10	t
0.00	0.00000	0.00025	0.00050	0.00075	0.00100	0.00124	0.00149	0.00174	0.00198	0.00223	0.00248	0.99
01	00248	00272	00296	00321	00345	00369	00394	00418	00442	00466	00490	98
02	00490	00514	00538	00562	00586	00609	00633	00657	00680	00704	00728	97
03	00728	00751	00774	00798	00821	00844	00868	00891	00914	00937	00960	96
04	00960	00983	01006	01029	01052	01074	01097	01120	01142	01165	01188	95
0.05	0.01188	0.01210	0.01232	0.01255	0.01277	0.01299	0.01322	0.01344	0.01366	0.01388	0.01410	0.94
06	01410	01432	01454	01476	01498	01519	01541	01563	01584	01606	0162	93
07	01628	01649	01670	01692	01713	01734	01756	01777	01798	01819	0184	92
08	01840	01861	01882	01903	01924	01944	01965	01986	02006	02027	0204	91
09	02048	02068	02088	02109	02129	02149	02170	02190	02210	02230	0225	90
0.10	0.02250	0.02270	0.02290	0.02310	0.02330	0.02349	0.02369	0.02389	0.02408	0.02428	0.0244	0.89
11	02448	02467	02486	02506	02525	02544	02564	02583	02602	02621	0264	88
12	02640	02659	02678	02697	02716	02734	02753	02772	02790	02809	0282	87
13	02828	02846	02864	02883	02901	02919	02938	02956	02974	02992	0301	86
14	03010	03028	03046	03064	03082	03099	03117	03135	03152	03170	0318	85
0.15	0.03188	0.03205	0.03222	0.03240	0.03257	0.03274	0.03292	0.03309	0.03326	0.03343	0.03360	0.84
16	03360	03377	03394	03411	03428	03444	03461	03478	03494	03511	03528	83
17	03528	03544	03560	03577	03593	03609	03626	03642	03658	03674	03690	82
18	03690	03706	03722	03738	03754	03769	03785	03801	03816	03832	03848	81
19	03848	03863	03878	03894	03909	03924	03940	03955	03970	03985	04000	80
0.20	0.04000	0.04015	0.04030	0.04045	0.04060	0.04074	0.04089	0.04104	0.04118	0.04133	0.04148	0.79
21	04148	04162	04176	04191	04205	04219	04234	04248	04262	04276	04290	78
22	04290	04304	04318	04332	04346	04359	04373	04387	04400	04414	04428	77
23	04428	04441	04454	04468	04481	04494	04508	04521	04534	04547	04560	76
24	04560	04573	04586	04599	04612	04624	04637	04650	04662	04675	04688	75
0.25	0.04688	0.04700	0.04712	0.04725	0.04737	0.04749	0.04762	0.04774	0.04786	0.04798	0.04810	0.74
26	04810	04822	04834	04846	04858	04869	04881	04893	04904	04916	04928	73
27	04928	04939	04950	04962	04973	04984	04996	05007	05018	05029	05040	72
28	05040	05051	05062	05073	05084	05094	05105	05116	05126	05137	05148	71
29	05148	05158	05168	05179	05189	05199	05210	05220	05230	05240	05250	70
0.30	0.05250	0.05260	0.05270	0.05280	0.05290	0.05299	0.05309	0.05319	0.05328	0.05338	0.05348	0.69
31	05348	05357	05366	05376	05385	05394	05404	05413	05422	05431	05440	68
32	05440	05449	05458	05467	05476	05484	05493	05502	05510	05519	05528	67
33	05528	05536	05544	05553	05561	05569	05578	05586	05594	05602	05610	66
34	05610	05618	05626	05634	05642	05649	05657	05665	05672	05680	05688	65
0.35	0.05688	0.05695	0.05702	0.05710	0.05717	0.05724	0.05732	0.05739	0.05746	0.05753	0.05769	0.64
36	05760	05767	05774	05781	05788	05794	05801	05808	05814	05821	05828	63
37	05828	05834	05840	05847	05853	05859	05866	05872	05878	05884	05890	62
38	05890	05896	05902	05908	05914	05919	05925	05931	05936	05942	05948	61
39	05948	05953	05958	05964	05969	05974	05980	05985	05990	05995	06000	60
0.40	0.06000	0.06005	0.06010	0.06015	0.06020	0.06024	0.06029	0.06034	0.06038	0.06043	0.06048	0.59
41	06048	06052	06056	06061	06065	06069	06074	06078	06082	06086	06090	58
42	06090	06094	06098	06102	06106	06109	06113	06117	06120	06124	06128	57
43	06128	06131	06134	06138	06141	06144	06148	06151	06154	06157	06160	56
44	06160	06163	06166	06169	06172	06174	06177	06180	06182	06185	06188	55
0.45	0.06188	0.06190	0.06192	0.06195	0.06197	0.06199	0.06202	0.06204	0.06206	0.06208	0.06210	0.54
46	06210	06212	06214	06216	06218	06219	06221	06223	06224	06226	06228	53
47	06228	06229	06230	06232	06233	06234	06236	06237	06238	06239	06240	52
48	06240	06241	06242	06243	06244	06244	06245	06246	06246	06247	06248	51
49	06248	06248	06248	06249	06249	06249	06250	06250	06250	06250	06250	50
t	10	9	8	7	6	5	4	3	2	1	0	t



for determining the quadratic term of the interpolation formula

$$u = \frac{t(1-t)}{4}\tilde{\Delta}^2$$

Example

Given: $t = 0.313$; $\tilde{\Delta}^2 = 428$

Found: $u = 23$.

FOREWORD

These tables have been prepared at the instigation of V.A. Diktin. The values of the functions $J_0(x)$, $J_1(x)$ were taken from [10] and rounded off to seven places of decimals. The remaining values in the tables have been calculated anew. The calculation of the integrals $Ji_0(x)$ and $Ji_1(x)$ was carried out by Simpson's quadrature formula on an electronic calculator and analytical computers. The functions $Y_0(x)$, $Y_1(x)$, $Yi_0(x)$, $Yi_1(x)$ were calculated on the electronic computer BESM.* Taylor's series and asymptotic expansions were used in the calculations. All values in the tables have been checked by differences.

Calculation of the integrals $Ji_0(x)$, $Ji_1(x)$, checking of the tables and their preparation for the press were carried out under the supervision of E.A. Maurit. The illustrative nomogram was worked out by S.N. Borisov.

The author offers her heartfelt thanks to V.A. Diktin and L.N. Karamzin for their great help in the work on the tables.

* *Translator's note:* BESM – the high-speed electronic computer of the Academy of Sciences of the U.S.S.R.

INTRODUCTION

1. THE CONTENTS OF THE TABLES, AND RULES FOR THEIR EMPLOYMENT

THE Bessel functions $J_n(x)$ and $Y_n(x)$ are linear-independent solutions of the differential equation of the second order

$$\frac{d^2u}{dx^2} + \frac{1}{x}\cdot\frac{du}{dx} + \left(1 - \frac{n^2}{x^2}\right)u = 0. \qquad (1)$$

This book contains tables of these functions for $n = 0$ and 1

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k}}{(k!)^2}, \qquad (2)$$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+1}}{k!\,(k+1)!}, \qquad (3)$$

$$Y_0(x) = \frac{2}{\pi}\left\{\ln\frac{x}{2} + C + \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k}}{(k!)^2}\left(\ln\frac{x}{2} + C - \sum_{m=1}^{k}\frac{1}{m}\right)\right\}, \qquad (4)$$

$$Y_1(x) = -\frac{2}{\pi}\left\{\frac{1}{x} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\left(\frac{x}{2}\right)^{2k-1}}{(k-1)!\,k!}\left(\ln\frac{x}{2} + C - \frac{1}{2k} - \sum_{m=1}^{k-1}\frac{1}{m}\right)\right\} \qquad (5)$$

and of the integrals connected with these functions,

$$Ji_0(x) = \int_x^{\infty} \frac{J_0(u)}{u}\,du = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\left(\frac{x}{2}\right)^{2k}}{2k\,(k!)^2} - \ln\frac{x}{2} - C,$$

$$Ji_1(x) = \int_x^{\infty} \frac{J_1(u)}{u}\,du = 1 - \sum_{k=0}^{\infty} \frac{(-1)^k\left(\frac{x}{2}\right)^{2k+1}}{(2k+1)\,k!\,(k+1)!},$$

$$Yi_0(x) = \int_x^{\infty} \frac{Y_0(u)}{u}\,du = K - \frac{2}{\pi}\left\{C\cdot\ln x + \frac{1}{2}\ln^2\frac{x}{2} + \right.$$

$$+\sum_{k=1}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k}}{2k\,(k!)^2}\left(\ln\frac{x}{2}+C-\frac{1}{2k}-\sum_{m=1}^{k}\frac{1}{m}\right)\Big\},$$

$$Yi_1(x)=\int_x^{\infty}\frac{Y_1(u)}{u}\,du=-\frac{2}{\pi}\left\{\frac{1}{x}+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\left(\frac{x}{2}\right)^{2k-1}}{(2k-1)\,(k-1)!\,k!}\left[\ln\frac{x}{2}+\right.\right.$$

$$\left.\left.+C-\frac{4k-1}{2k\,(2k-1)}-\sum_{m=1}^{k-1}\frac{1}{m}\right]\right\}. \tag{9}$$

Here

$$\frac{2}{\pi}=0.63661\ 97724\ldots,$$

$$C=0.57721\ 56649\ldots,$$

$$K=0.67225\ 35977\ldots$$

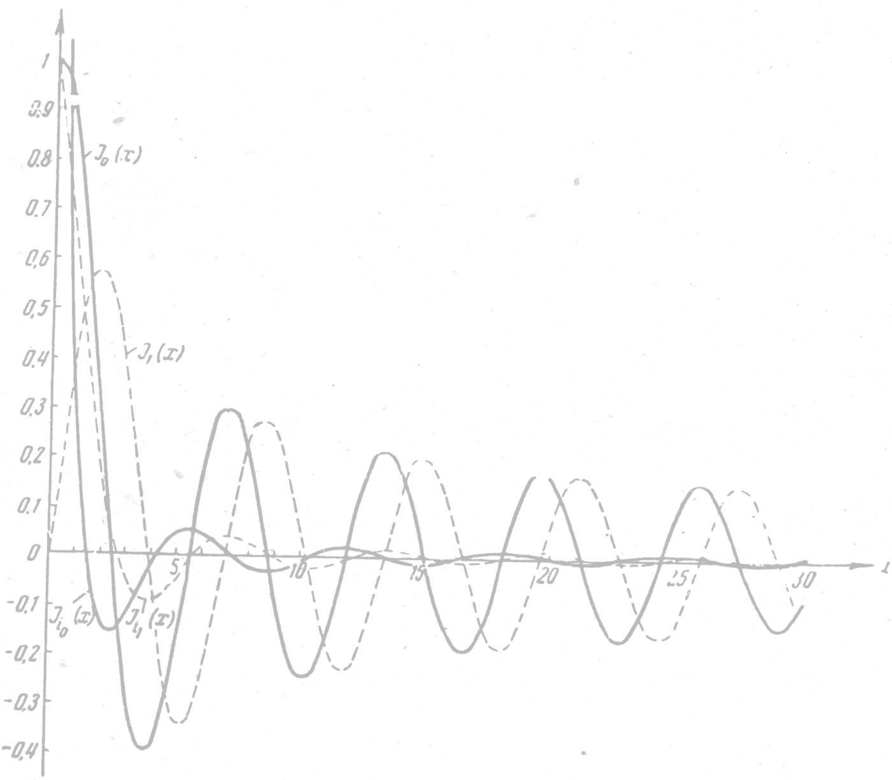


FIG. 1

We already have sufficiently detailed tables for the functions $J_0(x)$ and $J_1(x)$ (for instance, [10], [11]); they are included in this publication for convenience in obtaining the values of functions closely connected with those tabulated (see formulae (47)-(53)). The existing tables of Bessel functions of the second kind $Y_0(x)$ and $Y_1(x)$[11, 12, 15] have steps of a larger scale or a smaller region of change of the argument than those in this work. Tables of integrals derived from Bessel functions, with an exception of no great significance [13, 2] are lacking. A detailed bibliography of tables of Bessel functions and of integrals derived from them is to be found in [5].

The tables in this work contain the functions indicated above, carried to seven places of decimals or seven significant figures, for $0 < x < 100$. In the interval (0.15) the argument x changes by 0.001, and in the interval (15.100) by 0.01. The step in the tables has been so selected that in most cases the intermediate values of the functions, with an error not exceeding two units of the last value in the table, can be obtained by linear interpolation according to the formula

$$f(x) = f(x_0 + th) = f(x_0) + t[f(x_0 + h) - f(x_0)]. \tag{10}$$

Where linear interpolation does not give the accuracy indicated, we recommend quadratic interpolation with the wing use of Bessel's formula

$$f(x) = f(x_0 + th) = f(x_0) + t\Delta f(x_0) - \frac{t(1-t)}{4}\widetilde{\Delta}^2 f(x_0), \tag{11}$$

where

$$\Delta f(x_0) = f(x_0 + h) - f(x_0), \tag{12}$$

$$\widetilde{\Delta}^2 f(x_0) = \Delta^2 f(x_{-1}) + \Delta^2 f(x_0) = $$
$$= f(x_0 + 2h) - f(x_0 + h) - f(x_0) + f(x_0 - h). \tag{13}$$

In this case the second differences $\widetilde{\Delta}^2 f(x_n)$*, expressed in terms of the last class of tabular values, are to be found immediately after the column of values of the function. At the end of the book and on the inserted loose leaf we introduce the values for the quantity $t(1-t)/4$ with t changing by 0.001, and we also give a nomogram for the determination of the absolute quantity of the quadratic term of this formula.

Example 1. To find Yi_1 (11.03814).

$$Yi_1(11.93814) = Yi_1(11.938 + 0.14 \cdot 0.001).$$

* If in the tables $|\Delta^2 f(x_n)| < 16$, the intermediate values of a function can be obtained by linear interpolation.

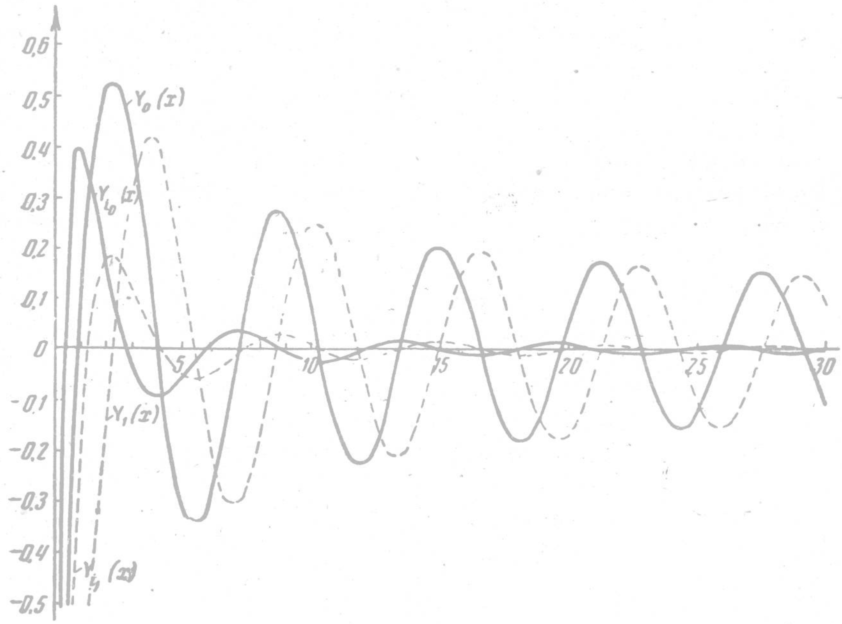


FIG. 2

In the given case $t = 0.14$. A second difference is lacking, so we use the formula of linear interpolation and find

Yi_1 (11.93814) = Yi_1 (11.938) + 0.14 [Yi_1 (11.939) − Yi_1 (11.938)] = − 0.0190470 (see page 278).

Example 2. To find Y_1 (0.180313).

$$Y_1\ (0.180313) = Y_1\ (0.180 + 0.313.0.001).$$

The values Y_1 (0.180) = − 3.669604 and Y_1 (0.181) = − 3.650476 are to be found on page 27. To the right of Y_1 (0.180) on the same line we find a second difference $\tilde{\Delta}^2\ Y_1$ (0.180) = − 0.000428, so we shall use formula (11) to obtain Y_1 (0.180313). The value of the quantity $t\ (1-t)/4 = 0.05376$ is to be found on the loose leaf.

Then $Y_1\ (0.180313) = Y_1\ (0.180) + 0.313\ [Y_1\ (0.181) - Y_1\ (0.180)] -$
$- \frac{1}{4}\ 0.313\ (1 - 0.313)\ \tilde{\Delta}^2\ Y_1\ (0.180) = -3.663594.$

Since for the functions Ji_0 (x), Y_0 (x), Y_1 (x), Yi_0 (x) and Yi_1 (x) $x = 0$ is a singular point, in the vicinity of zero ($0 < x \leqslant 0.150$) the intermediate values of these functions can be obtained by means of linear interpolation of the supplementary functions $Li_0(x)$, $C_0(x)$, $C_1(x)$, $D_0(x)$, $D_0(x)$, $D_1(x)$, $E_0(x)$ $E_1(x)$, $F_0(x)$ and of quadratic interpolation of the function $F_1(x)$ for which $\tilde{\Delta}^2 F_1(x_r)$ is everywhere in the interval indicated equal to 20.

The functions $Ji_0(x)$, $Y_0(x)$, $Y_1(x)$, $Yi_0(x)$, $Yi_1(x)$ are connected with the following supplementary correlations:

$$Ji_0(x) = Li_0(x) - \ln \frac{x}{2}, \tag{14}$$

$$Y_0(x) = C_0(x) + D_0(x) \ln x, \tag{15}$$

$$Y_1(x) = \frac{C_1(x)}{x} + D_1(x) \ln x, \tag{16}$$

$$Yi_0(x) = F_0(x) + \left[E_0(x) - \frac{1}{\pi} \ln x\right] \ln x, \tag{17}$$

$$Yi_1(x) = \frac{F_1(x)}{x} + E_1(x) \ln x. \tag{18}$$

The values of the supplementary functions are laid out on pages 17–19.

Example 3. To find $Y_1(0.08991)$

By linear interpolation of the functions $C_1(x)$ and $D_1(x)$ for the value $x = 0.08991$ and using formula (16), we find:

$$C_1(0.08991) = C_1(0.089 + 0.91 \cdot 0.001) =$$
$$= C_1(0.089) + 0.91\,[C_1(0.090) - C_1(0.089)] = -0.6382011,$$

$$D_1(0.08991) = D_1(0.089 + 0.91 \cdot 0.001) =$$
$$= D_1(0.089) + 0.91\,[D_1(0.090) - D_1(0.089)] = 0.0285903$$ (see page 18),

$$Y_1(0.08991) = \frac{C_1(0.08991)}{0.08991} + D_1(0.08991) \ln 0.08991^{*} = -7.167094.$$

In the values of the functions given in the tables the error does not exceed 0.6 of a unit of the last value, with the exception of the functions $C_0(x)$, $C_1(x)$, $F_0(x)$ and $F_1(x)$, in which the error attains the unit of the last value.

2. CERTAIN PROPERTIES OF THE FUNCTIONS

For calculation of values of the functions tabulated here outside the region in which the arguments are given the following formulae may be useful.

1) Asymptotic presentations:

* The value of $\ln x$ is taken from [17].

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\left\{P_0(x)\cos\left(x - \frac{\pi}{4}\right) - Q_0(x)\sin\left(x - \frac{\pi}{4}\right)\right\}, \qquad (19)$$

$$J_1(x) = \sqrt{\frac{2}{\pi x}}\left\{P_1(x)\cos\left(x - \frac{3\pi}{4}\right) - Q_1(x)\sin\left(x - \frac{3\pi}{4}\right)\right\}, \qquad (20)$$

$$Y_0(x) = \sqrt{\frac{2}{\pi x}}\left\{P_0(x)\sin\left(x - \frac{\pi}{4}\right) + Q_0(x)\cos\left(x - \frac{\pi}{4}\right)\right\}, \qquad (21)$$

$$Y_1(x) = \sqrt{\frac{2}{\pi x}}\left\{P_1(x)\sin\left(x - \frac{3\pi}{4}\right) + Q_1(x)\cos\left(x - \frac{3\pi}{4}\right)\right\}, \qquad (22)$$

$$Ji_0(x) = \sqrt{\frac{2}{\pi x}}\left\{A_0(x)\cos\left(x - \frac{\pi}{4}\right) + B_0(x)\sin\left(x - \frac{\pi}{4}\right)\right\}, \qquad (23)$$

$$Ji_1(x) = \sqrt{\frac{2}{\pi x}}\left\{A_1(x)\cos\left(x - \frac{3\pi}{4}\right) + B_1(x)\sin\left(x - \frac{3\pi}{4}\right)\right\}, \qquad (24)$$

$$Yi_0(x) = \sqrt{\frac{2}{\pi x}}\left\{A_0(x)\sin\left(x - \frac{\pi}{4}\right) - B_0(x)\cos\left(x - \frac{\pi}{4}\right)\right\}, \qquad (25)$$

$$Yi_1(x) = \sqrt{\frac{2}{\pi x}}\left\{A_1(x)\sin\left(x - \frac{3\pi}{4}\right) - B_1(x)\cos\left(x - \frac{3\pi}{4}\right)\right\}, \qquad (26)$$

where

$$P_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\,[(4k-1)!!]^2}{(2k)!\,(8x)^{2k}} = 1 - \frac{0.0703\,125}{x^2} + \cdots, \qquad (27)$$

$$P_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,[(4k-3)!!]\,[(4k+1)!!]}{(2k)!\,(8x)^{2k}} = 1 + \frac{0.1171\,875}{x^2} - \cdots, \qquad (28)$$

$$Q_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}[(4k+1)!!]^2}{(2k+1)!\,(8x)^{2k+1}} = -\frac{0.125}{x} + \frac{0.0732\,4219}{x^3} - \cdots, \qquad (29)$$

$$Q_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\,[(4k-1)!!]\,[(4k+3)!!]}{(2k+1)!\,(8x)^{2k+1}} = \frac{0.375}{x} - \frac{0.1025\,3906}{x^3} + \cdots, \qquad (30)$$

$$\left.\begin{aligned} A_0(x) = \sum_{k=1}^{\infty} \frac{a_k^0}{x^{2k}} = \frac{1.625}{x^2} - \frac{14.53\,8086}{x^4} + \cdots, \\ a_k^0 = c_{k-1}^0 - \frac{(4k-1)(4k-3)}{4} a_{k-1}^0, \quad a_0^0 = 0, \\ c_k^0 = \frac{(-1)^k [(4k-1)!!!]^2 (48k^2 + 48k + 13)}{(2k+1)!\, 8^{2k+1}}. \end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned} A_1(x) = \sum_{k=1}^{\infty} \frac{a_k^1}{x^{2k}} = \frac{1.125}{x^2} - \frac{9.331\,0547}{x^4} + \cdots, \\ a_k^1 = c_{k-1}^1 - \frac{(4k-1)(4k-3)}{4} a_{k-1}^1, \quad a_0^1 = 0, \\ c_k^1 = \frac{(-1)^{k+1} [(4k-3)!!!]\, [(4k+1)!!!]\, (48k^2 + 48k + 9)}{(2k+1)!\, 8^{2k+1}}. \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} B_0(x) = \sum_{k=0}^{\infty} \frac{b_k^0}{x^{2k+1}} = -\frac{1}{x} + \frac{4.132\,8125}{x^3} - \cdots \\ b_k^0 = d_k^0 - \frac{16k^2 - 1}{4} b_{k-1}^0, \quad b_0^0 = -1, \\ d_k^0 = \frac{(-1)^{k+1} [(4k-3)!!!]^2 (48k^2 + 1)}{(2k)!\, 8^{2k}}. \end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} B_1(x) = \sum_{k=0}^{\infty} \frac{b_k^1}{x^{2k+1}} = -\frac{1}{x} + \frac{2.695\,3123}{x^3} - \cdots \\ b_k^1 = d_k^1 - \frac{16k^2 - 1}{4} b_{k-1}^1, \quad b_0^1 = -1, \\ d_k^1 = \frac{(-1)^k [(4k-5)!!!]\, [(4k-1)!!!]\, (48k^2 - 3)}{(2k)!\, 8^{2k}}. \end{aligned}\right\} \tag{34}$$

When $x > 100$, two terms of each of the series (27) – (34) are sufficient for obtaining seven places of decimals.

2) Correlations for functions with negative values of the argument:

$$J_0(e^{\pi i} x) = J_0(x), \tag{35}$$

$$J_1(e^{\pi i}x) = -J_1(x), \tag{36}$$

$$Y_0(e^{\pi i}x) = Y_0(x) + 2iJ_0(x), \tag{37}$$

$$Y_1(e^{\pi i}x) = -Y_1(x) - 2iJ_1(x), \tag{38}$$

$$Ji_0(e^{\pi i}x) = Ji_0(x) - \pi i, \tag{39}$$

$$Ji_1(e^{\pi i}x) = -Ji_1(x) + 2, \tag{40}$$

$$Yi_0(e^{\pi i}x) = Yi_0(x) + \pi + 2iJi_0(x), \tag{41}$$

$$Yi_1(e^{\pi i}x) = -Yi_1(x) + 2i\,[1 - Ji_1(x)]. \tag{42}$$

To extend the table by n and obtain derivatives from the Bessel functions we can use the recurrent formulae:

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x), \tag{43}$$

$$Y_{n+1}(x) = \frac{2n}{x} Y_n(x) - Y_{n-1}(x), \tag{44}$$

$$J'_n(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] = J_{n-1}(x) - \frac{n}{x} J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x), \tag{45}$$

$$Y'_n(x) = \frac{1}{2}[Y_{n-1}(x) - Y_{n+1}(x)] = Y_{n-1}(x) - \frac{n}{x} Y_n(x) = \frac{n}{x} Y_n(x) - Y_{n+1}(x). \tag{46}$$

With the help of the tables it is easy to obtain the values of certain other integrals derived from Bessel functions.
For example:

$$\int_x^\infty J_0(u)\,du = \int_x^\infty \frac{J_1(u)}{u}\,du - J_1(x) = Ji_1(x) - J_1(x), \tag{47}$$

$$\int_x^\infty J_1(u)\,du = J_0(x), \tag{48}$$

$$\int_x^\infty Y_0(u)\,du = \int_x^\infty \frac{Y_1(u)}{u}\,du - Y_1(x) = Yi_1(x) - Y_1(x), \tag{49}$$

$$\int_x^\infty Y_1(u)\,du = Y_0(x), \tag{50}$$

$$\int_0^x J_0(u)\,du = 1 + J_1(x) - \int_x^\infty \frac{J_1(u)}{u}\,du = 1 + J_1(x) - Ji_1(x), \tag{51}$$

$$\int_0^x J_1(u)\,du = 1 - J_0(x), \tag{52}$$

$$\int_0^x Y_0(u)\,du = \ln 2 - C - \int_x^\infty Y_0(u)\,du = Y_1(x) - Yi_1(x) + \ln 2 - C \tag{53}$$

and others [7].

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$$\left[\begin{array}{ll} J_0(x),\ J_1(x),\ Y_0(x),\ Y_1(x) & \text{7 places of decimals } x = 0\,(0.02)\,16.00. \\ \frac{1}{2}\int_0^x J_0(u)\,du, \quad \frac{1}{2}\int_0^x Y_0(u)\,du & \text{7 places of decimals } x = 0\,(0.02)\,1.00. \end{array}\right]$$

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