UNIVERSITY

MATHEMATICAL TEXTS

VOLUME AND INTEGRAL

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PREFACE

It is now half a century since *H. Lebesgue* created his theory of the integral which has widely superseded in modern analysis the classical conception due to *B. Riemann*. It is, I think, regrettable that knowledge of the Lebesgue integral seems to be still largely confined to the research worker. There is nothing unduly abstract or unnatural in this theory, nor anything in the proofs which would be too difficult for a good honours student to grasp. If the aim of university education be the teaching of general ideas rather than that of technicalities, then the modern notion of the integral should not be omitted from the mathematical honours syllabus.

The main object of this book is to provide an introduction to the theory of the so-called *absolute* integral. It is not an introduction to the "calculus": it is assumed that the reader is familiar with this. But the book should give the student a deeper understanding of the ideas underlying the calculus. It is also hoped that he will appreciate the aesthetic side of a purely mathematical theory, quite apart from its practical implications. Such an appreciation is quite as essential as technical skill.

As the title indicates, I have tried to bring out consistently the *geometrical* aspect of integration: the integral of a (positive) function is the volume of the ordinate set of the function. This seems to me, both historically and intrinsically, the natural approach and that which is likely to suit the student best.

The first part of the book deals with the problem of volume in a space of n dimensions. First the older definition of *content* (*Peano*, *Jordan*) is discussed. It is followed by the theory of the modern and more satisfactory definition of *measure* (*Lebesgue*).

The second part begins with the theory of *Riemann*'s integral of a function of n variables. This is defined geometrically, using content as the underlying notion of volume. *Lebesgue*'s integral is then obtained in a similar way on replacing content by measure. The relation between the two definitions, and the striking advantages of the new integral, are thus clearly set out. The book ends with the theory of the *indefinite* integral of a function of one variable : the discussion of the familiar feature of the calculus, that differentiation and integration are inverse operations.

It has been necessary, for reasons of space economy, to restrict this account to the essentials of the theory : the properties of the spaces L^p and such important applications as length of arc and surface area had to be omitted. Nor are the notions of the *Stieltjes* integral and of a non-absolute integral (*Denjoy-Perron*) included. A list of books, suitable for comparison or further study, is given at the end. Of these, the book by *H. Kestelman* and the recent Cambridge Tract by *J. C. Burkill* proceed on lines similar to ours. In particular, it is hoped, that our geometrical account of the absolute integral may serve as a stimulating introduction to the standard work on integration, the book by *St. Saks*, which, in the two different editions, presents the modern more abstract approach to the subject.

It is somewhat difficult to provide exercises in a subject which is essentially theoretical. I have given a few : the solutions are at the end of each chapter.

Many friends have helped in preparing this text by suggestions, criticism, and proof reading, and all deserve my thanks : my Newcastle-Durham colleagues *F. F. Bonsall*, *Professor A. C. Offord* (now in London), and *Dr. J. V. Whitworth* must be specially mentioned. My main thanks, however, are due to *Dr. D. E. Rutherford* who as editor suggested the book and helped it along in many ways. Finally, I wish to express my gratitude to the Publishers and Printers for their patient and fine work under somewhat difficult circumstances.

W. W. ROGOSINSKI

DURHAM UNIVERSITY, KING'S COLLEGE
NEWCASTLE UPON TYNE
January 1952

PREFACE TO SECOND EDITION

ONLY a few minor inaccuracies in the text have come to my notice. These, together with some misprints, have now been amended. Otherwise, apart from occasional added remarks, the new edition is unchanged.

W. W. R.

THE UNIVERSITY OF AARHUS
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PART I

VOLUME

CHAPTER I

SETS OF POINTS

1.1. The Euclidean space. The aim of the theory of measure is to give a precise meaning, in as general form as possible, to the intuitive but vague geometrical concepts of length, area, and volume. The Euclidean space of three dimensions, an ideal image of the " intuitive " space of primitive sense experience, is a logical system of abstract entities called points, lines, and planes which are inter-related to form a " geometrical " pattern according to certain rules called axioms

It is shown in co-ordinate geometry how to establish an arithmetical model of this space. On introducing a system of Cartesian co-ordinates a one-one correspondence between all points P of the space and all ordered triplets (x, y, z) of real numbers is obtained. To the planes correspond linear equations between the co-ordinates, and to the lines pairs of simultaneous linear equations. The Euclidean axioms are the equivalent of the ordinary arithmetical axioms of this co-ordinate algebra.

In a similar way the system of all real numbers x can be interpreted as an arithmetical equivalent of the line (a space of one dimension), and the system of all ordered pairs (x, y) of real numbers as an arithmetical equivalent of the plane (a space of two dimensions). All this is familiar.

1.2. The space of n dimensions. More generally, we consider complexes of n real numbers,

$$P = (x_1, x_2, \ldots x_n) = (x_i) \quad 1 \leq i \leq n, \quad . \quad (1.2.1)$$

where n is a fixed positive integer. These complexes are ordered : that is, $P=(x_i)$ and $Q=(y_i)$ are the same complex if, and only if, $x_i=y_i$ for all i. Thus, when $n=2$, the two complexes (1, 2) and (2, 1) are different.

We use geometrical language and call each complex P a *point in the space of* n *dimensions* ; the numbers x_i are the *co-ordinates* of P. It should be noted that these are mere names so far, at least when $n>3$.† We shall have to attach some " geometrical " significance to them : the analogy with the true geometrical cases $n\leq 3$ will serve as a guide.

Thus the obvious definition of the *distance* between two points $P=(x_i)$ and $Q=(y_i)$ will be

$$PQ=QP=[(x_1-y_1)^2+(x_2-y_2)^2+\dots(x_n-y_n)^2]^{\frac{1}{2}} \quad . \quad (1.2.2)$$

We shall then have, for any three points P, Q, R, the triangle relation

$$PQ\leq PR+RQ \; ; \quad . \quad . \quad . \quad (1.2.3)$$

that is, *one side of a " triangle " is at most equal to the sum of the two other sides.*

The proof is elementary. We use the inequality

$$(a-b)^2\leq a^2+b^2+2|\,ab\,|$$

and Cauchy's inequality

$$\left(\sum_1^n a_ib_i\right)^2\leq \sum_1^n a_i^2 \,.\, \sum_1^n b_i^2 \quad . \quad . \quad . \quad (1.2.4)$$

First, if R is the origin $O=(0, 0, \dots 0)$, then

$$\begin{aligned} PQ^2=\Sigma(x_i-y_i)^2 &\leq \Sigma x_i^2+\Sigma y_i^2+2\Sigma|\,x_iy_i\,| \\ &\leq \Sigma x_i^2+\Sigma y_i^2+2(\Sigma x_i^2\,.\,\Sigma y_i^2)^{\frac{1}{2}} \\ &=[(\Sigma x_i^2)^{\frac{1}{2}}+(\Sigma y_i^2)^{\frac{1}{2}}]^2=[PO+OQ]^2. \end{aligned}$$

In the general case, when $R=(z_i)$, consider the points $P'=(x_i-z_i)$ and $Q'=(y_i-z_i)$. Clearly, $PQ=P'Q'$, $PR=P'O$, and $RQ=OQ'$, so that the general case is reduced to the previous one.

1.3. Sets. Any prescription of points P in a given space of n dimensions defines a *set* (of points) in this space. We

† When $n=4$, a complex (x, y, z, t) can be interpreted as an " event ", that is as a point (x, y, z) at the time t.

denote a set, usually, by E†; and we shall often write $P \varepsilon E$ for "P belongs to E".

The most comprehensive set is the given space itself. We denote it by E, or by E_n, if we wish to place in evidence the number of dimensions. Thus E_1 is the set of all real numbers x (the line), E_2 is the set of all ordered pairs (x, y) (the plane), and E_3 is the set of all ordered triplets (x, y, z) of real numbers (the "ordinary" space).

The following simple sets will frequently occur.

If C is a fixed point and ρ is a given positive number, then the set of all points P in E_n for which $PC < \rho$ is called an *open sphere*, of *centre* C and *radius* ρ. It is denoted by $K_\rho(C)$, or simply by K.‡ Thus, if $n=1$, a linear "sphere" is an open interval of centre C and length 2ρ; if $n=2$, a "sphere" is a circle. In these cases we shall, of course, retain the usual words. The set of points for which $PC \leq \rho$ is called a *closed sphere*.

A *closed interval* $I = \langle a_i, b_i \rangle$ is defined as the set of all points $P = (x_i)$ for which

$$a_i \leq x_i \leq b_i, \quad 1 \leq i \leq n. \qquad (1.3.1)$$

Thus, if $n=2$, a closed interval is a rectangle; if $n=3$, it is a cuboid. Note that the sides, or edges, are, by definition, parallel to the co-ordinate axes. In these cases we shall retain the usual words.

The *edges* of an interval I (more precisely, the lengths of the edges) are the numbers $b_i - a_i$; and the *volume* of I is defined as

$$| I | = \prod_1^n (b_i - a_i). \qquad (1.3.2)$$

If all edges are equal we speak of a *cube* (or a square, if $n=2$). We allow some, or all, edges to be zero: in the extreme case, an interval may reduce to a point. In this chapter, however, the edges will usually be positive.

An *open interval*, denoted by $(I) = (a_i, b_i)$, is the set of

† E indicates the French word *ensemble*; we reserve S for another use.

‡ K indicates the German word *Kugel*.

all points for which $a_i < x_i < b_i$. Its edges and volume are defined as above. The point $(\frac{1}{2}(a_i + b_i))$ is called the *centre* of the interval I, or (I).

A set E is said to be *finite* if it contains only a finite number of points. It will be convenient to admit as finite also a "set" which contains *no* point. This set is called the *null set* (or *empty set*) and is denoted by O. A non-finite set is said to be *infinite*.

1.4. Subsets. We consider, throughout this book, a given space $\mathsf{E} = \mathsf{E}_n$.

A set E_1 is said to be a *subset* of the set E_2, if every point of E_1 also belongs to E_2: $P \varepsilon E_1$ implies $P \varepsilon E_2$. We then say that E_1 is contained in E_2, or that E_2 contains E_1, and write this as

$$E_1 \subset E_2, \quad \text{or} \quad E_2 \supset E_1 \quad . \quad . \quad . \quad (1.4.1)$$

Clearly, this relation is transitive: if $E_1 \subset E_2$ and $E_2 \subset E_3$, then $E_1 \subset E_3$. Every set is a subset of itself: $E \subset E$.

If $P \varepsilon E$, then $\{P\} \subset E$ where $\{P\}$ is the set consisting of the point P only.

Clearly, $\mathsf{E} \supset E$ whatever E may be. At the other extreme we regard the null set as a subset of every set: $O \subset E$.

If $E_1 \subset E_2$ but $E_1 \neq E_2$, then E_1 is called a *proper subset* of E_2. Thus O is a proper subset of any non-empty set. An open sphere is a proper subset of the corresponding closed sphere; and similarly for intervals.

The set of all points (x, y, c), where c is fixed, is a two-dimensional subset (a plane) of E_3; the set of all points (a, y, c), where a and c are fixed, represents a line in E_3. Similarly, fixing k $(<n)$ of the co-ordinates, we obtain a $(n-k)$-dimensional *subspace* of E_n. More generally, we can define such a subspace by k independent linear equations between the co-ordinates.

A set E is called *bounded* if there is a closed interval I such that $E \subset I$. Any finite set, interval, or sphere is bounded. Neither the space itself nor any of its subspaces is bounded.