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Michel Chipot

Elliptic Equations: An Introductory Course

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Author:

Michel Chipot
Institute of Mathematics
University of Zürich
Winterthurerstr. 190
8057 Zürich
Switzerland
e-mail: m.m.chipot@math.uzh.ch

2000 Mathematics Subject Classification: 35Jxx, 35J15, 35J25, 35J67, 35J85

Library of Congress Control Number: 2008939515

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN 978-3-7643-9981-8 Birkhäuser Verlag AG, Basel · Boston · Berlin

Basel · Boston · Berlin
P.O. Box 133, CH-4010 Basel, Switzerland
Part of Springer Science+Business Media
Printed on acid-free paper produced of chlorine-free pulp. TCF ∞
Printed in Germany

ISBN 978-3-7643-9981-8

e-ISBN 978-3-7643-9982-5

9 8 7 6 5 4 3 2 1

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Preface

The goal of this book is to introduce the reader to different topics of the theory of elliptic partial differential equations avoiding technicalities and refinements. The material of the first part is written in such a way it could be taught as an introductory course. Most of the chapters – except the four first ones – are independent and some material can be dropped in a short course. The four first chapters are fundamental, the next ones are devoted to teach or present a larger spectrum of the techniques of this topics showing some qualitative properties of the solutions to these problems. Everywhere just a minimum on Sobolev spaces has been introduced, work or integration on the boundary has been carefully avoided in order not to crowd the mind of the reader with technicalities but to attract his attention to the beauty and variety of these issues. Also very often the ideas in mathematics are very simple and the discovery of them is a powerful engine to learn quickly and get further involved with a theory. We have kept this in mind all along Part 1.

Part II contains more advanced material like nonlinear problems, systems, regularity... Again each chapter is relatively independent of the others and can be read or taught separately.

We would also like to point that numerous results presented here are original and have not been published elsewhere.

This book grew out of lectures given at the summer school of Druskininkai (Lithuania), in Tokyo (Waseda University) and in Rome (La Sapienza). It is my pleasure to acknowledge the rôle of these different places and to thank K. Pileckas, Y. Yamada, D. Giachetti for inviting me to deliver these courses.

I would also like to thank Senoussi Guesmia, Sorin Mardare and Karen Yeressian for their careful reading of the manuscript and Mrs. Gerda Schacher for her constant help in preparing and typing this book. I am also very grateful to H. Amann and T. Hempfling for their support.

Finally I would like to thank the Swiss National Science Foundation who supported this project under the contracts #20-113287/1, 20-117614/1.

Zürich, December 2008

Contents

Part I

Basic Techniques

Chapter 1

Hilbert Space Techniques

The goal of this chapter is to collect the main features of the Hilbert spaces and to introduce the Lax–Milgram theorem which is a key tool for solving elliptic partial differential equations.

1.1 The projection on a closed convex set

Definition 1.1. A Hilbert space H over $\mathbb{R}$ is a vector space equipped with a scalar product $(\cdot, \cdot)$ which is complete for the associated norm

$$|u| = (u, u)^{\frac{1}{2}}, \tag{1.1}$$

i.e., such that every Cauchy sequence admits a limit in H.

Examples.

1. $\mathbb{R}^n$ equipped with the Euclidean scalar product

$$(x, y) = x \cdot y = \sum_{i=1}^{n} x_i y_i \quad \forall\, x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n.$$

   (We will prefer the notation with a dot for this scalar product.)
2. $L^2(A) = \{\, v : A \to \mathbb{R}, v \text{ measurable} \mid \int_A v^2(x)\, dx < +\infty \,\}$ with A a measurable subset of $\mathbb{R}^n$. Recall that $L^2(A)$ is in fact a set of "class" of functions. $L^2(A)$ is a Hilbert space when equipped with the scalar product

$$(u, v) = \int_A u(x)v(x)\, dx. \tag{1.2}$$

Remark 1.1. Let us recall the important "Cauchy–Schwarz inequality" which asserts that

$$|(u, v)| \le |u|\, |v| \quad \forall\, u, v \in H. \tag{1.3}$$

One of the important results is the following theorem.

Theorem 1.1 (Projection on a convex set). *Let $K \neq \emptyset$ be a closed convex subset of a Hilbert space H. For every $h \in H$ there exists a unique u such that*

$$\begin{cases} u \in K, \\ |h-u| \leq |h-v|, \quad \forall v \in K, \end{cases} \tag{1.4}$$

(i.e., u realizes the minimum of the distance between h and K). Moreover u is the unique point satisfying (see Figure 1.1)

$$\begin{cases} u \in K, \\ (u-h, v-u) \geq 0, \quad \forall v \in K. \end{cases} \tag{1.5}$$

u is called the orthogonal projection of h on K and will be denoted by $P_K(h)$.

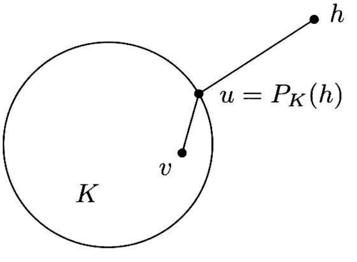

Figure 1.1: Projection on a convex set

Proof. Consider a sequence $u_n \in K$ such that when $n \to +\infty$

$$|h-u_n| \to \operatorname*{Inf}_{v \in K} |h-v| := d.$$

The infimum above clearly exists and thus such a minimizing sequence also. Note now the identities

$$|u_n - u_m|^2 = |u_n - h + h - u_m|^2 = |h-u_n|^2 + |h-u_m|^2 + 2(u_n - h, h - u_m)$$
$$|2h - u_n - u_m|^2 = |h - u_n + h - u_m|^2 = |h-u_n|^2 + |h-u_m|^2 + 2(h - u_n, h - u_m).$$

Adding up we get the so-called parallelogram identity

$$|u_n - u_m|^2 + |2h - u_n - u_m|^2 = 2|h-u_n|^2 + 2|h-u_m|^2. \tag{1.6}$$

Recall now that a convex set is a subset of H such that

$$\alpha u + (1-\alpha)v \in K \quad \forall u, v \in K, \quad \forall \alpha \in [0,1]. \tag{1.7}$$

From (1.6) we derive

$$
\begin{aligned}
|u_n - u_m|^2 &= 2|h-u_n|^2 + 2|h-u_m|^2 - 4\left|h - \frac{u_n+u_m}{2}\right|^2 \\
&\leq 2|h-u_n|^2 + 2|h-u_m|^2 - 4d^2 \qquad (1.8)
\end{aligned}
$$

since $\frac{u_n+u_m}{2} \in K$ (take $\alpha = \frac{1}{2}$ in (1.7)). Since the right-hand side of (1.8) goes to 0 when $n, m \to +\infty$, u_n is a Cauchy sequence. It converges toward a point $u \in K$ – since K is closed – such that

$$|h-u| = \operatorname*{Inf}_{v\in K} |h-v|. \qquad (1.9)$$

This shows the existence of u satisfying (1.4). To prove the uniqueness of such a u one goes back to (1.8) which is valid for any u_n, u_m in K. Taking u, u' two solutions of (1.4), (1.8) becomes

$$|u-u'|^2 \leq 2|h-u|^2 + 2|h-u'|^2 - 4d^2 = 0,$$

i.e., $u = u'$. This completes the proof of the existence and uniqueness of a solution to (1.4). We show now the equivalence of (1.4) and (1.5). Suppose first that u is solution to (1.5). Then we have

$$|h-v|^2 = |h-u+u-v|^2 = |h-u|^2 + |u-v|^2 + 2(h-u, u-v) \geq |h-u|^2 \quad \forall v \in K.$$

Conversely suppose that (1.4) holds. Then for any $\alpha \in (0,1)$ – see (1.7) – we have for $v \in K$

$$
\begin{aligned}
|h-u|^2 \leq |h - [\alpha v + (1-\alpha)u]|^2 &= |h - u - \alpha(v-u)|^2 \\
&= |h-u|^2 + 2\alpha(u-h, v-u) + \alpha^2|v-u|^2.
\end{aligned}
$$

This implies clearly

$$2\alpha(u-h, v-u) + \alpha^2|v-u|^2 \geq 0. \qquad (1.10)$$

Dividing by α and letting $\alpha \to 0$ we derive that (1.5) holds. This completes the proof of the theorem. □

Remark 1.2. If $h \in K$, $P_K(h) = h$. (1.5) is an example of variational inequality.

In the case where K is a closed subspace of H (this is a special convex set) Theorem 1.1 takes a special form.

Corollary 1.2. *Let V be a closed subspace of H. Then for every $h \in H$ there exists a unique u such that*

$$\begin{cases} u \in V, \\ |h-u| \leq |h-v|, \quad \forall v \in V. \end{cases} \qquad (1.11)$$

Moreover u is the unique solution to

$$\begin{cases} u \in V, \\ (h-u, v) = 0, \quad \forall v \in V. \end{cases} \qquad (1.12)$$

Proof. It is enough to show that (1.5) is equivalent to (1.12). Note that if (1.12) holds then (1.5) holds. Conversely if (1.5) holds then for any $w \in V$ one has

$$v = u \pm w \in V$$

since V is a vector space. One deduces

$$\pm(u - h, w) \geq 0 \quad \forall w \in V$$

which is precisely (1.12). $u = P_V(h)$ is described in Figure 1.2 below. It is the unique vector of V such that $h - u$ is orthogonal to V. □

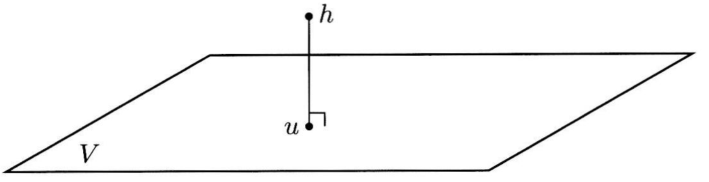

Figure 1.2: Projection on a vector space

1.2 The Riesz representation theorem

If H is a real Hilbert space we denote by H^* its dual – i.e., H^* is the set of continuous linear forms on H. If $h \in H$ then the mapping

$$v \mapsto (h, v) \tag{1.13}$$

is an element of H^*. Indeed this is a linear form that is to say a linear mapping from H into $\mathbb{R}$ and the continuity follows from the Cauchy–Schwarz inequality

$$|(h, v)| \leq |h|\,|v|. \tag{1.14}$$

The Riesz representation theorem states that all the elements of H^* are of the type (1.13) which means can be represented by a scalar product. This fact is easy to see on $\mathbb{R}^n$. We will see that it extends to infinite-dimensional Hilbert spaces. First let us analyze the structure of the kernel of the elements of H^*.

Proposition 1.3. *Let $h^* \in H^*$. If $h^* \neq 0$ the set*

$$V = \{\, v \in H \mid \langle h^*, v \rangle = 0 \,\} \tag{1.15}$$

is a closed subspace of H of codimension 1, i.e., a hyperplane of H. (We denote with brackets the duality writing $\langle h^*, v \rangle = h^*(v)$.)

Proof. Since h^* is continuous V is a closed subspace of H. Let $h \notin V$. Such an h exists since $h^* \neq 0$. Then set

$$v_0 = h - P_V(h) \neq 0. \tag{1.16}$$

Any element of $v \in H$ can be decomposed in a unique way as

$$v = \lambda v_0 + w \tag{1.17}$$

where $w \in V$. Indeed if $w \in V$ one has necessarily

$$\langle h^*, v\rangle = \lambda \langle h^*, v_0\rangle,$$

i.e., $\lambda = \langle h^*, v\rangle / \langle h^*, v_0\rangle$ and then

$$v = \frac{\langle h^*, v\rangle}{\langle h^*, v_0\rangle} v_0 + v - \frac{\langle h^*, v\rangle}{\langle h^*, v_0\rangle} v_0.$$

This completes the proof of the proposition. □

We can now show

Theorem 1.4 (Riesz representation theorem). *For any $h^* \in H^*$ there exists a unique $h \in H$ such that*

$$(h, v) = \langle h^*, v\rangle \quad \forall v \in H. \tag{1.18}$$

Moreover

$$|h| = |h^*|_* = \operatorname*{Sup}_{\substack{v \in H \\ v \neq 0}} \frac{\langle h^*, v\rangle}{|v|}. \tag{1.19}$$

(*This last quantity is called the strong dual norm of* h^*.)

Proof. If $h^* = 0$, $h = 0$ is the only solution of (1.18). We can assume then that $h^* \neq 0$. Let $v_0 \neq 0$ be a vector orthogonal to the hyperplane

$$V = \{\, v \in H \mid \langle h^*, v\rangle = 0 \,\},$$

(see (1.16), (1.17)). We set

$$h = \frac{\langle h^*, v_0\rangle}{|v_0|^2} v_0. \tag{1.20}$$

Due to the decomposition (1.17) we have

$$(h, v) = (h, \lambda v_0 + w) = \lambda (h, v_0) = \lambda \langle h^*, v_0\rangle = \langle h^*, \lambda v_0 + w\rangle = \langle h^*, v\rangle$$

for every $v \in H$. Thus h satisfies (1.18). The uniqueness of h is clear since

$$(h - h', v) = 0 \quad \forall v \in H \Longrightarrow h = h'$$

(take $v = h - h'$).

Now from (1.20) we have

$$|h| = \frac{|\langle h^*, v_0\rangle|}{|v_0|} \le |h^*|_* \tag{1.21}$$

and from (1.18)

$$|h^*|_* = \operatorname*{Sup}_{v\neq 0} \frac{(h,v)}{|v|} \le \operatorname*{Sup}_{v\neq 0} \frac{|h|\,|v|}{|v|} = |h|.$$

This completes the proof of the theorem. □

1.3 The Lax–Milgram theorem

Instead of a scalar product one can consider more generally a continuous bilinear form. That is to say if $a(u,v)$ is a continuous bilinear form on H, then for every $u \in H$

$$v \mapsto a(u,v) \tag{1.22}$$

is an element of H^*. As for the Riesz representation theorem one can ask if every element of H^* is of this type. This can be achieved with some assumptions on a which reproduce the properties of the scalar product, namely:

Theorem 1.5 (Lax–Milgram). *Let $a(u,v)$ be a bilinear form on H such that*

- *a is continuous, i.e., $\exists \Lambda > 0$ such that $|a(u,v)| \le \Lambda |u|\,|v| \quad \forall\, u,v \in H$,* (1.23)
- *a is coercive, i.e., $\exists \lambda > 0$ such that $a(u,u) \ge \lambda |u|^2 \quad \forall\, u \in H$.* (1.24)

Then for every $f \in H^*$ there exists a unique $u \in H$ such that

$$a(u,v) = \langle f, v\rangle \quad \forall\, v \in H. \tag{1.25}$$

In the case where a is symmetric that is to say

$$a(u,v) = a(v,u) \quad \forall\, u,v \in H \tag{1.26}$$

then u is the unique minimizer on H of the functional

$$J(v) = \frac{1}{2}a(v,v) - \langle f, v\rangle. \tag{1.27}$$

Proof. For every $u \in H$, by (1.23), $v \mapsto a(u,v)$ is an element of H^*. By the Riesz representation theorem there exists a unique element in H that will be denoted by Au such that

$$a(u,v) = (Au, v) \quad \forall\, v \in H.$$

We will be done if we can show that A is a bijective mapping from H into H. (Indeed one will then have $\langle f, v\rangle = (h,v) = (Au,v) = a(u,v)\ \forall\, v \in H$ for a unique u in H.)