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Dorothee D. Haroske

Envelopes and Sharp Embeddings of Function Spaces

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Envelopes and Sharp Embeddings of Function Spaces

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To my family

Preface

We present the new concept of growth envelopes and continuity envelopes in function spaces. This originates from such classical results as the famous Sobolev embedding theorem [Sob38], or, secondly, from the Brézis-Wainger result [BW80] on the almost Lipschitz continuity of functions from a Sobolev space $H_p^{1+n/p}(\mathbb{R}^n)$, $1 < p < \infty$. In the first case questions of growth are studied: what can be said about the unboundedness of functions belonging to, say, $H_p^{n/p}(\mathbb{R}^n)$, $1 < p < \infty$? We introduce the *growth envelope* $\mathfrak{E}_{\mathsf{G}}(X) = (\mathcal{E}_{\mathsf{G}}^X(t), u_{\mathsf{G}}^X)$ of a function space $X \subset L_1^{\mathrm{loc}}$, where

$$\mathcal{E}_{\mathsf{G}}^X(t) \sim \sup\{f^*(t) : \ \|f|X\| \leq 1\}, \quad 0 < t < 1,$$

is the *growth envelope function* of X and $u_{\mathsf{G}}^X \in (0, \infty]$ is some additional index providing an even finer description of the unboundedness of functions belonging to X.

Instead of investigating the growth of functions one can also focus on their smoothness, i.e., for $X \hookrightarrow C$ it makes sense to replace $f^*(t)$ with $\frac{\omega(f,t)}{t}$, where $\omega(f,t)$ is the modulus of continuity. The *continuity envelope function* $\mathcal{E}_{\mathsf{C}}^X$ and the continuity envelope $\mathfrak{E}_{\mathsf{C}}$ are introduced completely parallel to $\mathcal{E}_{\mathsf{G}}^X$ and $\mathfrak{E}_{\mathsf{G}}$, respectively, and similar questions are studied.

These concepts are first explained in detail and demonstrated on some classical and rather obvious examples in Part I; in Part II we deal with these instruments in the context of spaces of Besov and Triebel-Lizorkin type, $B_{p,q}^s$ and $F_{p,q}^s$, respectively.

In the end we turn to some applications, e.g., concerning the asymptotic behaviour of approximation numbers of (corresponding) compact embeddings. Further applications are connected with Hardy-type inequalities and limiting embeddings. We discuss the relation between growth and continuity envelopes of a suitable pair of spaces. Problems of global nature are regarded, and we study situations where the envelope function itself belongs to or can be realised in X, respectively.

I am especially grateful to Professor Hans Triebel; while he was preparing his book [Tri01] (in which Chapter 2 is devoted to related questions) we could discuss the material at various occasions. This led to the preprint version [Har01], and also became part of [Har02]. But for some reason these results –

though extended, improved, used, cited already – were never published elsewhere. In view of the substantial material, the idea appeared later to collect it in a book rather than a number of papers. This gives me the opportunity for special thanks to Professor David E. Edmunds who offered invaluable mathematical and linguistic comments, and to Professor Haïm Brézis who encouraged me in that form of publication. Last but not least I appreciate joint work and exchange of ideas on the subject with many colleagues, in particular Professor António M. Caetano and Dr. Susana D. Moura. Finally, I am indebted to my family for their never-ending patience and support.

Jena, March 2006 *Dorothee D. Haroske*

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Part I

Definition, basic properties, and first examples

Chapter 1

Introduction

We present our recently developed concept of envelopes in function spaces – a relatively simple tool for the study of classical, and also rather complicated spaces, say, of Besov or Triebel-Lizorkin type, respectively, in so-called "*limiting*" situations. This subject is still very new, but in our opinion it has grown to such a degree of maturity that it is now worth a coherent account. The topic is studied in two steps: on a more general level, in Part I, where we do not assume any knowledge of the scales of function spaces mentioned above, and subsequently, in Part II, the results are essentially related to spaces of type $B^s_{p,q}$ and $F^s_{p,q}$. This also explains the main structure of this book.

We first describe the common background for both parts and explicate the programme afterwards. In fact, considerable parts of the outcomes were obtained much earlier and already summarised in the long preprint [Har01]; but for some reason they have not yet been published (apart from [Har02], essentially relying on [Har01]). However, we also complemented and extended the presentation [Har01] quite recently.

The history of such questions starts in the 1930s with Sobolev's famous embedding theorem [Sob38]

$$W^k_p(\Omega) \hookrightarrow L_r(\Omega), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary, L_r, $1 \le r \le \infty$, stands for the usual Lebesgue space, and W^k_p, $k \in \mathbb{N}$, $1 \le p < \infty$, are the classical Sobolev spaces. The latter spaces have been widely accepted as one of the crucial instruments in functional analysis – in particular, in connection with PDEs – and have played a significant role in numerous parts of mathematics for many years. Sobolev's famous result (1.1) holds for $k \in \mathbb{N}$ with $k < \frac{n}{p}$, and r such that $\frac{k}{n} - \frac{1}{p} \ge -\frac{1}{r}$ (strictly speaking, [Sob38] covers the case $\frac{k}{n} - \frac{1}{p} > -\frac{1}{r}$, whereas the extension to $\frac{k}{n} - \frac{1}{p} = -\frac{1}{r}$ was achieved later). In the limiting case, when $k = \frac{n}{p} \in \mathbb{N}$, the inclusion (1.1) does not hold for $r = \infty$, whereas for all $r < \infty$,

$$W^{n/p}_p(\Omega) \hookrightarrow L_r(\Omega). \tag{1.2}$$

The theory of Sobolev-type embeddings originates in classical inequalities from which integrability properties of a real function can be deduced from those of its derivatives. In that sense (1.2) can be understood simply as the impossibility of specifying integrability conditions of a function $f \in W_p^{n/p}(\Omega)$ merely by means of L_r conditions. In order to obtain further refinements of the limiting case of (1.1) it becomes necessary to deal with a wider class of function spaces. In the late 1960s Peetre [Pee66], Trudinger [Tru67], Moser [Mos71], and Pohožaev [Poh65] independently found refinements of (1.1) expressed in terms of Orlicz spaces of exponential type, see also Strichartz [Str72]; this was followed by many contributions in the last decades investigating problems related to (1.1) in detail. In 1979 Hansson [Han79] and Brézis, Wainger [BW80] showed independently that

$$W_p^{n/p}(\Omega) \hookrightarrow L_{\infty,p}\,(\log L)_{-1}\,(\Omega), \tag{1.3}$$

where $1 < p < \infty$, and the spaces $L_{r,u}\,(\log L)_a\,(\Omega)$ appearing in (1.3) provide an even finer tuning, see also Hedberg [Hed72], and sharper results by Maz'ya [Maz72] and [Maz05]. Recently we noticed a revival of interest in limiting embeddings of Sobolev spaces indicated by a considerable number of publications devoted to this subject; let us only mention a series of papers by Edmunds with different co-workers ([EGO96], [EGO97], [EGO00], [EK95], [EKP00]), by Cwikel, Pustylnik [CP98], and – also from the standpoint of applications to spectral theory – the publications [ET95] and [Tri93] by Edmunds and Triebel. This list is by no means complete, but reflects the increased interest in related questions in the last years. There are a lot of different approaches to the modification of (1.1) in order to get – in the adapted framework – appropriately optimal assertions. Apart from the original papers, assertions of this type are indispensable parts in books dealing with Sobolev spaces and related questions, cf. [AF03], [Zie89], [Maz85], [EE87], [EE04], but there is a vast literature devoted to (extensions of) this subject.

Returning to the limiting case $k = \frac{n}{p}$ (1.2), for instance, one can also extend the scale of admitted (source) spaces in another direction, first replacing classical Sobolev spaces $W_p^{n/p}$ by the more general fractional Sobolev spaces $H_p^{n/p}$, or even by spaces of type $B^s_{p,q}$ and $F^s_{p,q}$, respectively. It is well-known, for instance, that $B^{n/p}_{p,q} \hookrightarrow L_\infty$ if, and only if, $0 < p < \infty$, $0 < q \le 1$ – but what can be said about the growth of functions $f \in B^{n/p}_{p,q}$ otherwise, i.e., when $B^{n/p}_{p,q}$ contains essentially unbounded functions ? Edmunds and Triebel proved in [ET99b] that one can characterise such spaces by sharp inequalities involving the non-increasing rearrangement f^* of a function f: Let $\varkappa$ be a bounded, continuous, decreasing function on $(0,1]$ and $1 < p < \infty$. Then there is a constant $c > 0$ such that

$$\left(\int_0^1 \left(\frac{f^*(t)\varkappa(t)}{|\log t|}\right)^p \frac{\mathrm{d}t}{t}\right)^{1/p} \le c\,\left\|f|H_p^{n/p}\right\| \tag{1.4}$$

for all $f \in H_p^{n/p}$ if, and only if, $\varkappa$ is bounded; there are further results related to the case of H_p^s with $0 < s < \frac{n}{p}$ in [Tri99].

Recall that the scale $F_{p,q}^s$ of Triebel-Lizorkin spaces extends the (classical) Sobolev scale further, whereas Besov spaces $B_{p,q}^s$ have been well-known for a long time, either when characterised by differences or – nowadays preferably – in Fourier-analytical terms or via (sub-)atomic decompositions. They appear naturally in signal analysis, in contemporary harmonic analysis, in stochastics, and while studying approximation problems or solving PDEs; thus it is of deep interest to understand these spaces very well – apart from the purely functional analytic purpose. The theory of the scales $B_{p,q}^s$ and $F_{p,q}^s$ has been systematically studied and developed by Triebel in the last decades; his series of monographs [Tri78a], [Tri83], [Tri92], [Tri97], [Tri01], and the forthcoming book [Tri06] can be regarded as the most complete treatment of related questions.

Assertions of type (1.4) are already linked to our concept of envelopes in some sense. We realised that many contributions to the subject of limiting embeddings and sharp inequalities as (partly) mentioned above, share a little disadvantage – beside their unquestioned beauty: as far as characterisations of spaces of type $B_{p,q}^s$ or $F_{p,q}^s$ are concerned, they are usually bound to a certain setting. That is, dealing with such embeddings, one asks, say, for optimality of original or target spaces within a prescribed (fixed) context. We prefer to look for some feature "belonging" to the spaces under consideration only, and, moreover, defined as simply as possible (using classical approaches). This would enable us to gain significantly from the rich history and many forerunners. In view of the above-mentioned papers it was natural to choose the non-increasing rearrangement f^* as the basic concept on which our new tool should be built. This led us to the introduction of the *growth envelope function* of a function space X,

$$\mathcal{E}_{\mathsf{G}}^X(t) := \sup_{\|f|X\| \le 1} f^*(t), \qquad 0 < t < 1. \tag{1.5}$$

It turns out that in rearrangement-invariant spaces there is a connection between $\mathcal{E}_{\mathsf{G}}^X$ and the fundamental function φ_X; we shall derive further properties and give some examples. The pair $\mathfrak{E}_{\mathsf{G}}(X) = \left(\mathcal{E}_{\mathsf{G}}^X(t), u_{\mathsf{G}}^X\right)$ is called the *growth envelope* of X, where u_{G}^X, $0 < u_{\mathsf{G}}^X \le \infty$, is the infimum of all numbers v satisfying

$$\left(\int_0^\varepsilon \left[\frac{f^*(t)}{\mathcal{E}_{\mathsf{G}}^X(t)}\right]^v \mu_{\mathsf{G}}(\mathrm{d}t)\right)^{1/v} \le c\, \|f|X\| \tag{1.6}$$

for some $c > 0$ and all $f \in X$, and μ_{G} is the Borel measure associated with

$-\log \mathcal{E}_{\mathsf{G}}^{X}$. The result reads for the Lorentz spaces $L_{p,q}$ as

$$\mathfrak{E}_{\mathsf{G}}\left(L_{p,q}\right) = \left(t^{-\frac{1}{p}}, q\right),$$

and for Sobolev spaces W_p^k,

$$\mathfrak{E}_{\mathsf{G}}\left(W_p^k\right) = \left(t^{-\frac{1}{p}+\frac{k}{n}},\ p\right), \quad 1 \le p < \infty, \quad k < \frac{n}{p}. \tag{1.7}$$

We also deal with some weighted L_p-spaces, illuminating the difference between locally regular weights like $(1+|x|^2)^{\alpha/2}$, $\alpha > 0$, and corresponding Muckenhoupt $\mathcal{A}_p$ weights like $|x|^\alpha$, $0 < \alpha < n(1-\frac{1}{p})$. In Part II we consider characterisations for spaces of type $B^s_{p,q}$ and $F^s_{p,q}$, where $n(\frac{1}{p}-1)_+ \le s \le \frac{n}{p}$. Returning to our example (1.4) above one proves that

$$\mathfrak{E}_{\mathsf{G}}\left(H_p^{n/p}\right) = \left(|\log t|^{1-\frac{1}{p}}, p\right), \quad 1 < p < \infty, \tag{1.8}$$

even in a more general setting. The counterpart for Besov spaces is given by

$$\mathfrak{E}_{\mathsf{G}}(B^{n/p}_{p,q}) = \left(|\log t|^{1-\frac{1}{q}},\ q\right), \quad 1 < q \le \infty, \quad 0 < p < \infty. \tag{1.9}$$

Unlike [Tri01, Ch. 2] where similar questions have been considered, we also study (some) borderline and weighted cases.

In an appropriately modified context it also makes sense to consider embeddings like (1.1) and (1.2) in "super-critical" situations, that is, when $k > \frac{n}{p}$. Then by simple monotonicity arguments all distributions are essentially bounded; moreover, one even knows that

$$W_p^k \hookrightarrow C \tag{1.10}$$

in this case, where C stands for the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the sup-norm. Parallel to the above question of unboundedness it is natural to consider and qualify the continuity of distributions from W_p^k in dependence upon k and p. As is well-known, the counterparts of (1.1) and (1.2) yield that for $\frac{n}{p} < k < \frac{n}{p}+1$, $1 \le p < \infty$,

$$W_p^k \hookrightarrow \mathrm{Lip}^a, \quad 0 < a \le k - \frac{n}{p} < 1, \tag{1.11}$$

and,

$$W_p^{1+n/p} \hookrightarrow \mathrm{Lip}^a, \quad 0 < a < 1, \tag{1.12}$$

where Lip^a, $0 < a \le 1$, contains all $f \in C$ such that for some $c > 0$ and all $x, h \in \mathbb{R}^n$, $|f(x+h) - f(x)| \le c\,|h|^a$. Similarly to (1.2), the case $a = 1$ in (1.12) is excluded (unless $p = 1$ as some special case), i.e., there are functions from $W_p^{1+n/p}$ that are not Lipschitz-continuous. However, as some compensation, one can consider the celebrated result of Brézis and Wainger [BW80] in which it was shown that every function u in $H_p^{1+n/p}$, $1 < p < \infty$, is "almost" Lipschitz-continuous, in the sense that for all $x, y \in \mathbb{R}^n$, $0 < |x-y| < \frac{1}{2}$,

$$|u(x) - u(y)| \leq c\,|x-y|\,\Big|\log|x-y|\Big|^{1-\frac{1}{p}}\,\|u|H_p^{1+n/p}\|\,. \tag{1.13}$$

Here c is a constant independent of x, y and u. In [EH99] we investigated the "sharpness" of this result (concerning the exponent of the log-term), as well as possible extensions to the wider scale of F-spaces and parallel results for B-spaces. Using the classical concept of the modulus of continuity $\omega(f,t)$, (1.13) can be reformulated as

$$\sup_{0<t<1/2} \frac{\omega(f,t)}{t\,|\log t|^{1-\frac{1}{p}}} \leq c\,\Big\|f|H_p^{1+n/p}\Big\|, \tag{1.14}$$

which will be strengthened to

$$\left(\int_0^{\frac{1}{2}} \left[\frac{\omega(f,t)}{t\,|\log t|}\right]^p \frac{\mathrm{d}t}{t}\right)^{1/p} \leq c\,\Big\|f|H_p^{1+n/p}\Big\|,$$

and an assertion similar to (1.4).

Consequently, based on observations like (1.14) we shall focus on the smoothness of functions instead of their growth; i.e., when $X \hookrightarrow C$ it makes sense to replace $f^*(t)$ by $\frac{\omega(f,t)}{t}$ in (1.5) and (1.6), and to introduce the *continuity envelope function*

$$\mathcal{E}_{\mathsf{C}}^X(t) := \sup_{\|f|X\|\leq 1} \frac{\omega(f,t)}{t}, \qquad 0 < t < 1,$$

and the *continuity envelope* $\mathfrak{E}_{\mathsf{C}}$ in a way completely parallel to that of $\mathcal{E}_{\mathsf{G}}^X$ and $\mathfrak{E}_{\mathsf{G}}$, respectively. The famous Brézis-Wainger result [BW80] then appears as

$$\mathfrak{E}_{\mathsf{C}}(H_p^{1+n/p}) = \left(|\log t|^{1-\frac{1}{p}}, p\right), \quad 1 < p < \infty,$$

whereas we obtain for Lipschitz spaces Lip^a of order $0 < a < 1$,

$$\mathfrak{E}_{\mathsf{C}}(\mathrm{Lip}^a) = \left(t^{-(1-a)}, \infty\right).$$

The counterpart of (1.7) reads as

$$\mathfrak{E}_{\mathsf{C}}\left(W_p^k\right) = \left(t^{-\left(\frac{n}{p}+1-k\right)},\ p\right), \tag{1.15}$$